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C^0-sufficiency of jets via blowing-up

Dedicated to Professor Hirosi Toda on his 60th birthday

By

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Let $\mathcal{E}_{[q]}(n, 1)$ denote the set of C^q function germs: $(\boldsymbol{R}^n, 0) \to (\boldsymbol{R}, 0)$. For a function $f \in \mathcal{E}_{[q]}(n, 1)$, $j^s f$ denotes the s-jet of f at $0 \in \boldsymbol{R}^n$ ($q \geqq s$). $J^s(n, 1)$ is the set of s-jets of function germs in $\mathcal{E}_{[q]}(n, 1)$. An s-jet $z \in J^s(n, 1)$ is called *C^0-sufficient* in $\mathcal{E}_{[q]}(n, 1)$, if for any functions f, g in $\mathcal{E}_{[q]}(n, 1)$ such that $j^s f = j^s g = z$, there exists a local homeomorphism $\sigma: (\boldsymbol{R}^n, 0) \to (\boldsymbol{R}^n, 0)$ such that $f \circ \sigma = g$. Thus C^0-sufficiency of jets amounts to saying that all terms of degree $> s$ can be omitted without changing the local topological behavior of the realizations. Concerning a characterization of C^0-sufficiency in $\mathcal{E}_{[s]}(n, 1)$ or $\mathcal{E}_{[s+1]}(n, 1)$, the "Kuiper-Kuo theorem" is well-known (see § 3).

In this note, we shall give another characterization of C^0-sufficiency of jets by using the "after blowing-up functions". In practice, this criteria is often easier to check than the above one. Here we describe the results about C^0-sufficiency in $\mathcal{E}_{[s]}(n, 1)$ only. Of course, similar results hold also for C^0-sufficiency in $\mathcal{E}_{[s+1]}(n, 1)$.

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1. Observation.

First consider the case $n=2$. Then, due to Lu Theorems ([8]), we can assume the given jet is in Weierstrass form:

$$(*) \qquad w(x, y) = x^k + H_{k+1}(x, y) + \cdots + H_{k+r}(x, y),$$

where $H_j(x, y)$ is a homogeneous polynomial of degree j ($k+1 \leqq j \leqq k+r$). Then

$$w(XY, Y) = Y^k H(X, Y),$$

where $H(X, Y) = X^k + YH_{k+1}(X, 1) + \cdots + Y^r H_{k+r}(X, 1)$.

Observation. Consider the level surfaces of the following three functions around $0 \in \boldsymbol{R}^2$:

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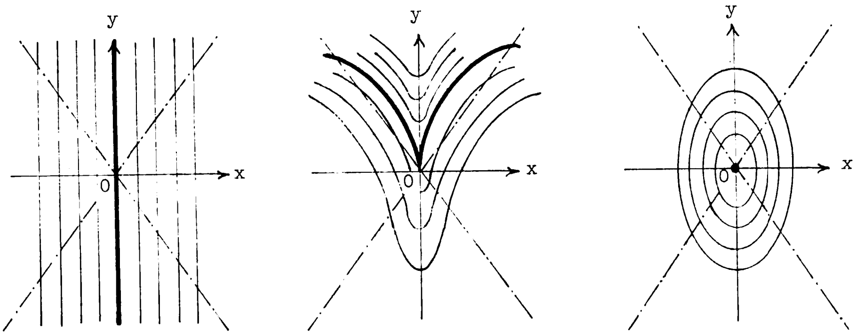

$f_0(x, y)=x^2$, $f_1(x, y)=x^2-y^3$, $f_2(x, y)=x^2+y^4$.

Notice that although the paterns of their level surfaces are different from one another, they are almost (intrinsically) the same outside the sector of $\boldsymbol{R}^2$ containing the singular part of their "tangent cone (the variety of the initial form)" $\{x=0\}$. That is to say, the initial form controls the behavior of a function outside the sector of $\boldsymbol{R}^2$ containing the singular part of its tangent cone.

Let us consider the blowing-up of $\boldsymbol{R}^2$ at 0, $\pi: \mathcal{M}_2 \to \boldsymbol{R}^2$. The sector of $\boldsymbol{R}^2$ containing $\{x=0\}$ is the image by π of an open set containing the intersection of the exceptional variety and the strict transform of $\{x=0\}$ in $\mathcal{M}_2$.

Thus the above observation gives rise to the following problem naturally:

Can we find certain conditions on H which control the behavior of w in the sector of $\boldsymbol{R}^2$ containing the singular part of the tangent cone?

We shall formulate such conditions for more general cases in §2.

2. Results.

Let $w: (\boldsymbol{R}^n, 0) \to (\boldsymbol{R}, 0)$ be a polynomial of degree $\leqq k+r$, and w be written as follows:

$$w(x) = Z_k(x)+G(x) \quad \text{with} \quad Z_k \not\equiv 0 \text{ and } j^k G = 0\,.$$

$\Pi: S^{n-1} \to \boldsymbol{R}P^{n-1}$ is a projection, and put $\Pi(a)=\tilde{a}$ for $a \in S^{n-1}$. We define $A=\Pi(\Sigma Z_k^{-1}(0) \cap S^{n-1})$ where $\Sigma Z_k^{-1}(0)=\{p \in \boldsymbol{R}^n \mid \frac{\partial Z_k}{\partial x_1}(p)=\cdots=\frac{\partial Z_k}{\partial x_n}(p)=0\}$ and $B[\tilde{a}]=\{\sigma \in O(n) \mid \sigma(a)=e_n \text{ or } \sigma(-a)=e_n \ (e_n=(0, \cdots, 0, 1))\}$ for $\tilde{a} \in A$. For $\sigma_a \in B[\tilde{a}]$, we write $w_{\sigma(a)}=w \circ \sigma_a^{-1}$. Here we put

$$w_{\sigma(a)}(X_1 X_n, \cdots, X_{n-1} X_n, X_n) = X_n{}^k H_{\sigma(a)}(X_1, \cdots, X_n)\,.$$

Then $Z_k \circ \sigma_a^{-1}$ is a homogeneous polynomial of degree k and $Z_k \circ \sigma_a^{-1}(e_n)=0$. Therefore $Z_k \circ \sigma_a^{-1}$ does not contain the term $\alpha x_n{}^k$ $(\alpha \neq 0)$. Thus $H_{\sigma(a)}$ is a polynomial with $H_{\sigma(a)}(0)=0$. Then we have the following characterization of C^0-sufficiency of $(k+r)$-jets by using the "after blowing-up functions" $H_{\sigma(a)}$.

Theorem 2.1. *For $w \in J^{k+r}(n, 1)$, the following conditions are equivalent.*

1) *w is C^0-sufficient in $\mathcal{E}_{[k+r]}(n, 1)$.*

2) *For any $\tilde{a}\in A$, there exist $\sigma_a\in B[\tilde{a}]$, $C_a>0$, and a neighborhood W_a of 0 in $\boldsymbol{R}^n$ such that*

$$(**)\qquad \left|\left(\frac{\partial H_{\sigma(a)}}{\partial X_1}, \cdots, \frac{\partial H_{\sigma(a)}}{\partial X_{n-1}}, X_n\frac{\partial H_{\sigma(a)}}{\partial X_n}\right)\right| \geqq C_a |X_n|^r$$

in W_a.

Remark. If for some $\sigma_a\in B[\tilde{a}]$, there exists $C_a>0$ such that (**) holds, then for any $\sigma_a\in B[\tilde{a}]$, there exists $C_a>0$ such that (**) holds. In other words, the property (**) does not depend on σ_a; We merely use σ_a for the formulation.

Concerning the proof of the theorem, we have the following inequality on analytic functions.

Proposition 2.2. *Let f: $(\boldsymbol{R}^n, 0)\to(\boldsymbol{R}, 0)$ be an analytic function. For $\varepsilon_i=0$ or 1 $(i=1,\cdots, n)$, we put $k=\#\{i\mid\varepsilon_i=1\}$. If $0<\theta<\frac{1}{\sqrt{k}}$, then there exists a neighborhood U of 0 in $\boldsymbol{R}^n$ such that*

$$(***)\qquad \left|\left(X_1^{\varepsilon_1}\frac{\partial f}{\partial X_1}, \cdots, X_n^{\varepsilon_n}\frac{\partial f}{\partial X_n}\right)\right| \geq \theta |f(X)|$$

in U.

As corollaries of the theorem, we have

Corollary 2.3. *If for any $\tilde{a}\in A$, there exists $\sigma_a\in B[\tilde{a}]$ such that $j^r H_{\sigma(a)}$ is C^0-sufficient in $\mathcal{E}_{[r]}(n, 1)$, then $w\in J^{k+r}(n, 1)$ is C^0-sufficient in $\mathcal{E}_{[k+r]}(n, 1)$.*

Corollary 2.4. *For the Weierstrass jet (*) $w\in J^{k+r}(2, 1)$ in § 1, the following conditions are equivalent.*

1) *w is C^0-sufficient in $\mathcal{E}_{[k+r]}(2, 1)$.*

2) *There exist $C>0$ and a neighborhood U of 0 in $\boldsymbol{R}^2$ such that*

$$\left|\left(\frac{\partial H}{\partial X}, Y\frac{\partial H}{\partial Y}\right)\right| \geqq C |Y|^r \qquad in \quad U.$$

Corollary 2.5. *Let w be in the Weierstrass form (*). If $j^r H$ is C^0-sufficient in $\mathcal{E}_{[r]}(2, 1)$, then $w\in J^{k+r}(2, 1)$ is C^0-sufficient in $\mathcal{E}_{[k+r]}(2, 1)$.*

3. Proof of the theorem and corollaries.

We start by recalling the Kuiper-Kuo theorem:

Kuiper-Kuo theorem ([1]–[3]). *For $z\in J^s(n, 1)$, the following conditions are equivalent.*

1) *z is C^0-sufficient in $\mathcal{E}_{[s]}(n, 1)$.*

(*resp. z is C^0-sufficient in $\mathcal{E}_{[s+1]}(n, 1)$.*)

2) *There exist $C>0$ (resp. $\delta>0$) and a neighborhood U of 0 in $\boldsymbol{R}^n$ such that*

$$|\operatorname{grad} z(x)| \geqq C|x|^{s-1}$$
$$(resp. |\operatorname{grad} z(x)| \geqq C|x|^{s-\delta}) \quad in \quad U.$$

We shall give the proof of the theorem. Therefore we consider a $(k+r)$-jet $w \in J^{k+r}(n, 1)$. For $\delta>0$, we put $S_\delta(\tilde{a}) = \{x \in S^{n-1} | d(x, \pm a) < \delta\}$ and $V_\delta = \bigcup_{\tilde{a} \in A} S_\delta(\tilde{a})$. We denote by $C_0(D)$ a cone of D with $0 \in \boldsymbol{R}^n$ as a vertex for a subset D of S^{n-1}.

Lemma 3.1. *For any $\delta>0$, there exists $b>0$ such that*

$$|\operatorname{grad} Z_k(x)| \geqq b|x|^{k-1} \quad in \quad C_0(S^{n-1}-V_\delta) \cap B_1(0),$$

where $B_1(0) = \{x \in \boldsymbol{R}^n \mid |x| \leqq 1\}$.

Proof. Put $b = \min_{x \in S^{n-1}-V_\delta} |\operatorname{grad} Z_k(x)| > 0$. Then it is easy to see the statement of this lemma.

By Lemma 3.1 and the Kuiper-Kuo theorem, we have the following:

(1) *$j^{k+r}w$ is C^0-sufficient in $\mathcal{E}_{[k+r]}(n, 1)$, if and only if there exist $C, \delta>0$ and a neighborhood U of 0 in $\boldsymbol{R}^n$ such that $|\operatorname{grad} w(x)| \geqq C|x|^{k+r-1}$ in $\overline{C_0(V_\delta)} \cap U$, where $\overline{C_0(V_\delta)}$ denotes a closure of $C_0(V_\delta)$ in $\boldsymbol{R}^n$.*

Let $\mathcal{H}_{k+r}(w; \varepsilon)$ $(\varepsilon>0)$ denote a horn-neighborhood $\{x \in \boldsymbol{R}^n \mid |w(x)| \leqq \varepsilon |x|^{k+r}\}$ (see [4]).

Lemma 3.2. *For $w \in J^{k+r}(n, 1)$, the following conditions are equivalent.*

1) *There exist $C, \delta>0$ and a neighborhood U of 0 in $\boldsymbol{R}^n$ such that*

$$|\operatorname{grad} w(x)| \geqq C|x|^{k+r-1} \quad in \quad \overline{C_0(V_\delta)} \cap U.$$

2) *For any $\tilde{a} \in A$, there exist $C_a, \varepsilon_a>0$ and a neighborhood U_a of 0 in $\boldsymbol{R}^n$ such that*

$$|\operatorname{grad} w(x)| \geqq C_a|x|^{k+r-1} \quad in \quad \overline{C_0(S_{\varepsilon_a}(\tilde{a}))} \cap U_a.$$

3) *For any $\tilde{a} \in A$, there exist $\sigma_a \in B[\tilde{a}]$ and $C_a, \varepsilon_a, \delta_a>0$ and a neighborhood U'_a of 0 in $\boldsymbol{R}^n$ such that*

$$|\operatorname{grad} w_{\sigma(a)}(x)| \geqq C_a|x|^{k+r-1} \quad in \quad \overline{C_0(S_{\varepsilon_a}(\tilde{e}_n))} \cap \mathcal{H}_{k+r}(w_{\sigma(a)}; \delta_a) \cap U'_a.$$

Proof. As A is compact, conditions 1) and 2) are equivalent. It is easy to see that conditions 2) and 3) are equivalent, by using Bochnak-Lojasiewicz inequality ([1] Lemma 2) and the fact $\sigma_a \in B[\tilde{a}]$.

Here we consider the mapping (blowing-up) $\pi: (\boldsymbol{R}^n, 0) \to (\boldsymbol{R}^n, 0)$ defined by $(x_1, \cdots, x_n) = \pi(X_1, \cdots, X_n)$:

$$x_1 = X_1 X_n, \cdots, x_{n-1} = X_{n-1} X_n\,, \; x_n = X_n\,.$$

For $\varepsilon > 0$, put $D_\varepsilon = \{(X_1, \cdots, X_n) \in \boldsymbol{R}^n \mid |X_j| \leqq \varepsilon \ (1 \leqq j \leqq n)\}$. Let $\mathcal{R}_r^n(H_{\sigma(a)}; \varepsilon_a)$ denote the set $\{X \in \boldsymbol{R}^n \mid |H_{\sigma(a)}(X)| \leqq \varepsilon_a |X_n|^r\}$ for $r \in \boldsymbol{N}$ and $\varepsilon_a > 0$.

Lemma 3.3. *For $\sigma_a \in B[\tilde{a}]$ ($\tilde{a} \in A$), the following conditions are equivalent.*

1) *There exist C_a, ε_a, $\delta_a > 0$ and a neighborhood U_a' of 0 in $\boldsymbol{R}^n$ such that*

$$|\operatorname{grad} w_{\sigma(a)}(x)| \geqq C_a |x|^{k+r-1} \quad in \quad \overline{C_0(S_{\varepsilon_a}(\tilde{e}_n))} \cap \mathcal{H}_{k+r}(w_{\sigma(a)}; \delta_a) \cap U_a'\,.$$

2) *There exist C_a', $\varepsilon_a' > 0$ and a neighborhood W_a' of 0 in $\boldsymbol{R}^n$ such that*

$$\left|\left(\frac{\partial H_{\sigma(a)}}{\partial X_1}, \cdots, \frac{\partial H_{\sigma(a)}}{\partial X_{n-1}}, kH_{\sigma(a)} + X_n \frac{\partial H_{\sigma(a)}}{\partial X_n} - \sum_{j=1}^{n-1} X_j \frac{\partial H_{\sigma(a)}}{\partial X_j}\right)\right| \geqq C_a' |X_n|^r$$

in $\mathcal{R}_r^n(H_{\sigma(a)}; \varepsilon_a') \cap W_a'$.

Proof. For an arbitarily small $\varepsilon_a > 0$ (resp. $\varepsilon_0 > 0$), there exists $\varepsilon_1' > 0$ (resp. $\varepsilon_1 > 0$) and a neighborhood U of 0 in $\boldsymbol{R}^n$ such that

$$(2) \qquad \begin{cases} \pi(D_{\varepsilon_1'}) \subset \overline{C_0(S_{\varepsilon_a}(\tilde{e}_n))} \\ (\text{resp. } \overline{C_0(S_{\varepsilon_1}(\tilde{e}_n))} \subset \pi(D_{\varepsilon_0})) \end{cases} \quad \text{in} \quad U\,.$$

For $x \in \pi(D_\varepsilon)$, $|x|^2 = (1 + \sum_{j=1}^{n-1} X_j^2) X_n^2$. Therefore we have

$$(3) \qquad |X_n|^2 \leqq |x|^2 \leqq 2|X_n|^2 \qquad \text{for} \quad x \in \pi(D_\varepsilon) \text{ near } 0 \in \boldsymbol{R}^n\,.$$

By an easy calculation, we have

$$(4) \qquad \begin{aligned} &\operatorname{grad} w_{\sigma(a)}(X_1 X_n, \cdots, X_{n-1} X_n, X_n) \\ &= X_n^{k-1} \left(\frac{\partial H_{\sigma(a)}}{\partial X_1}, \cdots, \frac{\partial H_{\sigma(a)}}{\partial X_{n-1}}, kH_{\sigma(a)} + X_n \frac{\partial H_{\sigma(a)}}{\partial X_n} - \sum_{j=1}^{n-1} X_j \frac{\partial H_{\sigma(a)}}{\partial X_j}\right). \end{aligned}$$

On the other hand,

$$w_{\sigma(a)}(X_1 X_n, \cdots, X_{n-1} X_n, X_n) = X_n^k H_{\sigma(a)}(X)\,.$$

Thus it follows from (2) and (3) that for $\varepsilon_a' > 0$, there exist ε_a, $\delta_a > 0$ (resp. for ε_a, $\delta_a > 0$, there exists $\varepsilon_a' > 0$) and a neighborhood U' of 0 in $\boldsymbol{R}^n$ such that

$$(5) \qquad \begin{cases} \overline{C_0(S_{\varepsilon_a}(\tilde{e}_n))} \cap \mathcal{H}_{k+r}(w_{\sigma(a)}; \delta_a) \subset \pi(\mathcal{R}_r^n(H_{\sigma(a)}; \varepsilon_a')) \\ (\text{resp. } \pi(\mathcal{R}_r^n(H_{\sigma(a)}; \varepsilon_a')) \subset \overline{C_0(S_{\varepsilon_a}(\tilde{e}_n))} \cap \mathcal{H}_{k+r}(w_{\sigma(a)}; \delta_a)) \end{cases}$$

in U'. Lemma 3.3 follows immediately from (2), (3), (4), and (5).

Lemma 3.4. *For $\sigma_a \in B[\tilde{a}]$ ($\tilde{a} \in A$), the following conditions are equivalent.*

1) *There exist C_a', $\varepsilon_a' > 0$ and a neighborhood W_a' of 0 in $\boldsymbol{R}^n$ such that*

$$\left|\left(\frac{\partial H_{\sigma(a)}}{\partial X_1}, \cdots, \frac{\partial H_{\sigma(a)}}{\partial X_{n-1}}, kH_{\sigma(a)} + X_n \frac{\partial H_{\sigma(a)}}{\partial X_n} - \sum_{j=1}^{n-1} X_j \frac{\partial H_{\sigma(a)}}{\partial X_j}\right)\right| \geqq C_a' |X_n|^r$$

in $\mathcal{R}_r^n(H_{\sigma(a)}; \varepsilon_a') \cap W_a'$.

2) *There exist C_a'', $\varepsilon_a''>0$ and a neighborhood W_a'' of* 0 *in* $\boldsymbol{R}^n$ *such that*

$$\left|\left(\frac{\partial H_{\sigma(a)}}{\partial X_1}, \cdots, \frac{\partial H_{\sigma(a)}}{\partial X_{n-1}}, X_n \frac{\partial H_{\sigma(a)}}{\partial X_n}\right)\right| \geqq C_a'' |X_n|^r \quad \text{in} \quad \mathcal{R}_r^n(H_{\sigma(a)}; \varepsilon_a'') \cap W_a''.$$

Proof. We show the implication 2)$\Rightarrow$1), by separating a neighborhood of $0 \in \boldsymbol{R}^n$ into the following two domains:

$$\text{(i)} \qquad \left|\left(\frac{\partial H_{\sigma(a)}}{\partial X_1}, \cdots, \frac{\partial H_{\sigma(a)}}{\partial X_{n-1}}\right)\right| \geqq \left| X_n \frac{\partial H_{\sigma(a)}}{\partial X_n}\right|,$$

$$\text{(ii)} \qquad \left|\left(\frac{\partial H_{\sigma(a)}}{\partial X_1}, \cdots, \frac{\partial H_{\sigma(a)}}{\partial X_{n-1}}\right)\right| \leqq \left| X_n \frac{\partial H_{\sigma(a)}}{\partial X_n}\right|.$$

In the case (i), for $X \in \mathcal{R}_r^n(H_{\sigma(a)}; \varepsilon_a'') \cap W_a''$, we have

$$2\left|\left(\frac{\partial H_{\sigma(a)}}{\partial X_1}, \cdots, \frac{\partial H_{\sigma(a)}}{\partial X_{n-1}}\right)\right| \geqq \left|\left(\frac{\partial H_{\sigma(a)}}{\partial X_1}, \cdots, \frac{\partial H_{\sigma(a)}}{\partial X_{n-1}}, X_n \frac{\partial H_{\sigma(a)}}{\partial X_n}\right)\right| \geqq C_a'' |X_n|^r.$$

Thus we have

$$\left|\left(\frac{\partial H_{\sigma(a)}}{\partial X_1}, \cdots, \frac{\partial H_{\sigma(a)}}{\partial X_{n-1}}, kH_{\sigma(a)} + X_n \frac{\partial H_{\sigma(a)}}{\partial X_n} - \sum_{j=1}^{n-1} X_j \frac{\partial H_{\sigma(a)}}{\partial X_j}\right)\right| \geqq \frac{C_a''}{2} |X_n|^r.$$

In the case (ii), $\left|X_n \frac{\partial H_{\sigma(a)}}{\partial X_n}\right| \geqq \left|\frac{\partial H_{\sigma(a)}}{\partial X_j}\right|$ $(1 \leqq j \leqq n-1)$. Similarly as (i), we have $\left|X_n \frac{\partial H_{\sigma(a)}}{\partial X_n}\right| \geqq \frac{C_a''}{2} |X_n|^r$ for $X \in \mathcal{R}_r^n(H_{\sigma(a)}; \varepsilon_a'') \cap W_a''$. Thus, for $X \in \mathcal{R}_r^n(H_{\sigma(a)}; \varepsilon_a') \cap W_a'$ where $W_a'' \supset W_a'$; sufficiently small and $0 < \varepsilon_a' \leqq \min\left\{\frac{C_a''}{8k}, \varepsilon_a''\right\}$, we have

$$\begin{aligned}
&\left|\left(\frac{\partial H_{\sigma(a)}}{\partial X_1}, \cdots, \frac{\partial H_{\sigma(a)}}{\partial X_{n-1}}, kH_{\sigma(a)} + X_n \frac{\partial H_{\sigma(a)}}{\partial X_n} - \sum_{j=1}^{n-1} X_j \frac{\partial H_{\sigma(a)}}{\partial X_j}\right)\right| \\
&\geqq \left|X_n \frac{\partial H_{\sigma(a)}}{\partial X_n}\right| - \sum_{j=1}^{n-1} |X_j| \left|\frac{\partial H_{\sigma(a)}}{\partial X_j}\right| - k|H_{\sigma(a)}| \\
&\geqq \frac{1}{2}\left|X_n \frac{\partial H_{\sigma(a)}}{\partial X_n}\right| - k|H_{\sigma(a)}| \\
&\geqq \frac{C_a''}{4} |X_n|^r - k\varepsilon_a' |X_n|^r \geqq \frac{C_a''}{8} |X_n|^r.
\end{aligned}$$

We can show the implication 1)$\Rightarrow$2) similarly as the above, by separating a neighborhood of $0 \in \boldsymbol{R}^n$ into the following two domains:

$$\text{(i)} \qquad \left|\left(\frac{\partial H_{\sigma(a)}}{\partial X_1}, \cdots, \frac{\partial H_{\sigma(a)}}{\partial X_{n-1}}\right)\right| \leqq \left|kH_{\sigma(a)} + X_n \frac{\partial H_{\sigma(a)}}{\partial X_n} - \sum_{j=1}^{n-1} X_j \frac{\partial H_{\sigma(a)}}{\partial X_j}\right|,$$

$$\text{(ii)} \qquad \left|\left(\frac{\partial H_{\sigma(a)}}{\partial X_1}, \cdots, \frac{\partial H_{\sigma(a)}}{\partial X_{n-1}}\right)\right| \geqq \left|kH_{\sigma(a)} + X_n \frac{\partial H_{\sigma(a)}}{\partial X_n} - \sum_{j=1}^{n-1} X_j \frac{\partial H_{\sigma(a)}}{\partial X_j}\right|.$$

By (1) and Lemmas 3.2, 3.3, 3.4, it suffices to show Proposition 2.2 (in particular, the case $k=1$). Here we consider the case $k=n$ only. The other cases follow similarly.

Assume that the inequality (***) fails. Then, by the curve selection lemma ([7]), there exists an analytic curve λ: $[0, \rho)\to \boldsymbol{R}^n$ ($\rho>0$) with $\lambda(0)=0$ such that

$$\left|\left(X_1\frac{\partial f}{\partial X_1}, \cdots, X_n\frac{\partial f}{\partial X_n}\right)\right|<\theta|f(X)| \tag{6}$$

along $\lambda-\{0\}$. Let λ be written as follows:

$$X_i(t) = k_1^{(i)}t^{\varepsilon_1(i)}+k_2^{(i)}t^{\varepsilon_2(i)}+\cdots,$$

where $1\leqq\varepsilon_1(i)<\varepsilon_2(i)<\cdots$

and
$$\begin{cases} k_1^{(i)}\neq 0 & \text{if } X_i(t)\not\equiv 0 \\ \varepsilon_1(i)=\infty & \text{if } X_i(t)\equiv 0, \quad (1\leqq i\leqq n). \end{cases}$$

Define

$$P(t)=f(X(t))=a_1t^{\alpha_1}+a_2t^{\alpha_2}+\cdots, \tag{7}$$

where $1\leqq\alpha_1<\alpha_2<\cdots$. Then P is analytic. Furthermore, by (6), we can assume that $a_1\neq 0$. By (7), we have

$$\frac{\partial P}{\partial t}(t)=a_1\alpha_1t^{\alpha_1-1}+a_2\alpha_2t^{\alpha_2-1}+\cdots. \tag{I}$$

On the other hand,

$$\frac{\partial P}{\partial t}(t)=\sum_{i=1}^n\frac{\partial f}{\partial X_i}(X(t))\frac{\partial X_i}{\partial t}(t). \tag{II}$$

We write

$$\frac{\partial f}{\partial X_i}(X(t))=b_1^{(i)}t^{\beta_1(i)}+b_2^{(i)}t^{\beta_2(i)}+\cdots,$$

where
$$\begin{cases} b_1^{(i)}\neq 0 & \text{if } \dfrac{\partial f}{\partial X_i}(X(t))\not\equiv 0 \\ \beta_1(i)=\infty & \text{if } \dfrac{\partial f}{\partial X_i}(X(t))\equiv 0, \quad (1\leqq i\leqq n). \end{cases}$$

Then it follows from (6) that

$$\alpha_1\leqq\varepsilon_1(i)+\beta_1(i) \qquad (1\leqq i\leqq n).$$

Without loss of generality, we may assume that

$$\begin{cases} \alpha_1=\varepsilon_1(i)+\beta_1(i) & (1\leqq i\leqq i_0) \\ \alpha_1<\varepsilon_1(i)+\beta_1(i) & (i_0+1\leqq i\leqq n), \end{cases}$$

where $0\leqq i_0\leqq n$.

Remark. For $1\leqq i\leqq i_0$, the following facts hold.
1) Since $a_1\neq 0$, we have $1\leqq\alpha_1<\infty$. It follows that $\varepsilon_1(i)$, $\beta_1(i)<\infty$, $k_1(i)\neq 0$ and $b_1(i)\neq 0$.
2) $\varepsilon_1(i)\leqq\alpha_1$.

By (6), we have

$$(8)\qquad \sqrt{\sum_{i=1}^{i_0}|k_1^{(i)}b_1^{(i)}|^2}\leqq\theta|a_1|.$$

Finally we compare the absolute value C_{I} of the coefficient of t^{α_1-1} in (I) with the absolute value C_{II} of that in (II).

$$\begin{aligned}
C_{\mathrm{II}} &= |\sum_{i=1}^{i_0} k_1^{(i)}\varepsilon_1(i)b_1^{(i)}| \\
&\leqq\sqrt{i_0}\sqrt{\sum_{i=1}^{i_0}|k_1^{(i)}\varepsilon_1(i)b_1^{(i)}|^2} \\
&\leqq\sqrt{i_0}\,|\alpha_1|\sqrt{\sum_{i=1}^{i_0}|k_1^{(i)}b_1^{(i)}|^2} &&\text{(by Remark 2))} \\
&\leqq\sqrt{n}\,|\alpha_1|\theta|a_1| &&\text{(by (8))} \\
&<|a_1\alpha_1|=C_{\mathrm{I}}\,.
\end{aligned}$$

This is a contradiction.

This completes the proof of Theorem 2.1.

It is easy to see Corollary 2.3 by using the Kuiper-Kuo theorem and Theorem 2.1. Corollary 2.4 (resp. Corollary 2.5) follows immediately from Theorem 2.1 (resp. Corollary 2.3).

4. Examples.

We shall give first examples to show that it is easier to check the hypothesis of corollaries than that of the Kuiper-Kuo theorem. Example 5.3. (which will be stated in § 5) is one of such examples too.

Example 4.1. Let $w(x, y)=x^{10}-10xy^{11}$. Then we have $k=10$ and $H(X, Y)=X^{10}-10XY^2$. It is easy to see that the condition 2) in Corollary 2.4 is satisfied for $r=3$. Thus it follows that $w\in J^{13}(2, 1)$ is C^0-sufficient in $\mathcal{E}_{[13]}(2, 1)$.

Example 4.2. Let $w(x, y)=x^2-2xy^{50}$. We put

$$\begin{aligned}
w(XY, Y) &= Y^2H_1(X, Y)\,, \\
H_k(XY, Y) &= Y^2H_{k+1}(X, Y) \qquad (1\leqq k\leqq 48)\,.
\end{aligned}$$

Then $j^2H_{49}=X^2-2XY\in J^2(2, 1)$ is C^0-sufficient in $\mathcal{E}_{[2]}(2, 1)$. By applying Corollary 2.5 49 times, we can see that $w\in J^{100}(2, 1)$ is C^0-sufficient in $\mathcal{E}_{[100]}(2, 1)$.

Next we give an example which is related to results in this note and some other facts in "Singularity Theory".

Example 4.3. Let $P_t(x, y)=x^3+3xy^8+ty^{10}$. We put $P_t(XY, Y)=Y^3H_t(X, Y)$ and $H_t(XY, Y)=Y^3G_t(X, Y)$. Then we have $H_t(X, Y)=X^3+3XY^6+tY^7$ and $G_t(X, Y)=X^3+3XY^4+tY^4$.

For $t\neq 0$, $j^4G_t(X, Y)=X^3+tY^4$ is C^0-sufficient in $\mathcal{E}_{[4]}(2, 1)$. Therefore X^3+tY^4 is $\mathcal{R}$-C^0 equivalent to $X^3+3XY^4+tY^4$ for $t\neq 0$. Hence it follows from the proof of the theorem in this note that x^3+ty^{10} is $\mathcal{R}$-C^0 equivalent to $x^3+3xy^8+ty^{10}$ for $t\neq 0$. Thus we can omit the lower ordered term $3xy^8$ from $x^3+3xy^8+ty^{10}$, using the notion of sufficiency of jets.

On the other hand, by the Kuiper-Kuo theorem, $j^9P_t=x^3+3xy^8$ is C^0-sufficient in $\mathcal{E}_{[9]}(2, 1)$. Thus x^3+3xy^8 is $\mathcal{R}$-C^0 equivalent to $x^3+3xy^8+ty^{10}$ for any t. (T.C. Kuo has pointed out to me that x^3+3xy^8 is not "blow-analytic equivalent" (in his sense [5], [6]) to $x^3+3xy^8+y^{10}$.) Note that x^3+3xy^8 is a weighted homogeneous polynomial of type $\left(\frac{1}{3}, \frac{1}{12}\right)$ with an isolated singularity. The weight of y^{10} equals $\frac{5}{6}<1$. In this case, x^3+3xy^8 controls the behavior of the lower weight term ty^{10} (Compare [9]).

5. The converse problem.

When we consider the problem of omitting lower ordered terms by using the notion of sufficiency of jets, the converse problem of Corollary 2.5 is important too. Consider a polynomial $w(x, y)=x^k+G(x, y)$ with an isolated singularity and $j^kG=0$. Then there exists $r\geqq 1$ such that $j^{k+r}w$ is C^0-sufficient in $\mathcal{E}_{[k+r]}(2, 1)$. Let $k+r(w)$ denote "the degree of C^0-sufficiency of w", namely the smallest integer having the above property. Then we put $w(XY, Y)=Y^kH(X, Y)$. It follows that H also has an isolated singularity at 0. Similarly there exists $l\geqq 1$ such that j^lH is C^0-sufficient in $\mathcal{E}_{[l]}(2, 1)$. Let $l(w)$ be the degree of C^0-sufficiency of H. Here we put

$$A_r=\left\{x^k+G \,\middle|\, \begin{array}{l} G\text{: polynomial of deg.}\leqq k+r,\ j^kG=0 \\ j^{k+r}(x^k+G)\text{: } C^0\text{-sufficient in } \mathcal{E}_{[k+r]}(2, 1) \end{array}\right\}.$$

Then the following fact follows easily from Corollary 2.5.

$$l(w)\geqq r \qquad \text{for any} \quad w\in A_r-A_{r-1}.$$

We now make some remarks on the converse problem without proof.

Proposition 5.1. *For* $w\in A_r$,

$$l(w)\leqq\max\{k, r+1, r(k-2)+1\}.$$

Remark. In the above proposition,

1) We can replace r by $r(w)$;
2) If $k \leqq 3$, then $\ell(w) \leqq \max\{k, r(w)+1\}$.

Problem 5.2. *More generally, is it true that*

$$\ell(w) \leqq \max\{k, r(w)+1\}\,?$$

Example 5.3. Let $w(x, y)=x^8+x^4y^5+xy^9$. Then we have $k=8$ and $r(w)=3$. On the other hand, $H(X, Y)=X^8+X^4Y+XY^2$. Then we have $\ell(w)=7$.

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