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A fine microlocalization and hypoellipticity

By

Yoshitsugu TAKEI

§0. Introduction

In the present article, we shall consider microhypoellipticity of the pseudo-differential operator $P = p(x, D)$ in $\mathbf{R}^d$. Recall that P is called microhypoelliptic if

$$WF(Pu) = WF(u) \qquad \text{for} \quad u \in \mathscr{E}'(\mathbf{R}^d)\,. \tag{0-1}$$

(See Hörmander [7], Definition 13.4.3). Microhypoellipticity of P implies hypoellipticity of P. In particular, we are mainly concerned with microhypoellipticity of P at a point (x_0, ξ^0) of $\mathbf{R}^d \times S^{d-1}$, which means that (0-1) holds at (x_0, ξ^0). (For the precise meaning, see Definition 3 in §1 below). As is well known, if an operator is elliptic at (x_0, ξ^0), then it is microhypoelliptic at the same point. This important theorem is based on the fact that ellipticity at (x_0, ξ^0) implies ellipticity on a conic neighborhood of (x_0, ξ^0). To study an operator on a conic neighborhood of a given point is a key idea of the so-called microlocal analysis. On the other hand, there are cases where an operator P is not elliptic on any conic neighborhood of a point (x_0, ξ^0), but if we divide a conic neighborhood into finer pieces, P can be regarded as 'elliptic' on each of them. Such finer pieces will be named "Γ-parabolic neighborhoods" of (x_0, ξ^0). And we would like to call this subdivision a "fine microlocalization". In this paper we discuss microhypoellipticity from the viewpoint of the fine microlocalization.

Let us explain our idea briefly by giving an example. We take up the heat operator $P = \partial^2/\partial x_1^2 - \partial/\partial x_2$, which is not elliptic but microhypoelliptic at $(x_0, \xi^0) = (0, (0, 1))$. We consider this operator on the following subset of $\mathbf{R}^2 \times \mathbf{R}^2$.

$$W_{a,b} = \{(x, \xi) \in \mathbf{R}^2 \times \mathbf{R}^2 \,|\, \xi_2^a \leqslant |\xi_1| \leqslant \xi_2^b,\ \xi_2 > 0,\ |x| \leqslant 1\} \qquad (0 < a \leqslant b \leqslant 1)\,,$$

or

$$W_{-\infty,b} = \{(x, \xi) \in \mathbf{R}^2 \times \mathbf{R}^2 \,|\, |\xi_1| \leqslant \xi_2^b,\ \xi_2 > 0,\ |x| \leqslant 1\} \qquad (0 < b \leqslant 1)\,.$$

We call such a subset a Γ-parabolic neighborhood of $(0, (0, 1))$. In our terminology, P is elliptic on $W_{-\infty,1/2}$, because P can be regarded as an operator of the first order on $W_{-\infty\,1/2}$ and the term $\partial/\partial x_2$ guarantees the ellipticity of P there. On the other hand, P can also be called elliptic on $W_{a,b}$ if we choose $1 \geqslant b \geqslant 1/2$ and a

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($\leqslant b$) close enough to b. On $W_{a,b}$, the term $\partial^2/\partial x_1^2$ dominates $\partial/\partial x_2$. The union of $W_{-\infty,1/2}$ and two of $W_{a,b}$'s fills a conic neighborhood of (0, (0, 1)). Therefore we can conclude, by our Theorem 1, that P is microhypoelliptic at (0, (0, 1)). (See for details §1 and §5).

Concerning the anisotropic wave front set, there are some works, for example Lascar [9], Parenti-Rodino [13], Rodino [14], etc. They considered such wave front sets in the study of quasi-homogeneous or, more generally, spatially inhomogeneous pseudo-differential operators. In particular, in [13] Parenti and Rodino investigated the relations between the usual microhypoellipticity and the microhypoellipticity with respect to such wave front sets. Our idea is to consider Γ-parabolic neighborhoods which are similar to such wave front sets, and to divide a conic neighborhood by making use of them.

Now it is the microlocal energy method of Mizohata that we use practically in considering the fine microlocalization. Mizohata initiated it for the study of the Caucy problem, the characterizations of the analytic and the Gevrey wave front sets, and so on. (See Mizohata [10], [11], [12]). We believe that his method is quite elementary and straight-forward. In this article we use this method in a little modified manner, which is more suitable in some examples for discussing the regularity in the C^∞ class and in Sobolev spaces.

The fine microlocalization seems to be very useful in the study of microhypoellipticity. As a matter of fact, we can show the microhypoellipticity of the operator which is more degenerate than the elliptic-like operator as the heat operator. As an example we are going to deal with a differential operator $P_m = (\partial/\partial x)^{2m} + x(\partial/\partial y)$ where m is a positive integer. When $m = 1$, this is the simplest one that satisfies the criterion of Hörmander [4] on the commutator of vector fields. In §6 we shall prove the hypoellipticity of P_m by full use of Γ-parabolic neighborhoods. We expect that in future the method studied here will produce plenty of results, say the hypoellipticity of a class of operators including P_m.

The plan of this paper is as follows: In §1, we define the Γ-parabolic neighborhood and state our main result (Theorem 1). In §2 and §3, we explain the basic calculus of the fine microlocalization, making use of the microlocal energy method. The proof of the main result will be given in §4, and some examples will be studied in §5 and §6. Finally in §7, we will prove the fundamental propositions stated in §2.

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§1. Main Result

Let $x = (x_1, \cdots, x_d)$ and $\xi = (\xi_1, \cdots, \xi_d)$ be the independent variables running over $\mathbf{R}^d$ respectively. Fix a point $(x_0, \xi^0) \in \mathbf{R}^d \times (\mathbf{R}^d \backslash 0)$, $|\xi^0| = 1$. We shall define the notion of the Γ-parabolic neighborhoods around (x_0, ξ^0) in general.

For simplicity we assume $x_0 = 0$ and $\xi^0 = (0, \cdots, 0, 1)$. If it is not so, we take an orthogonal matrix T such that $(0, \cdots, 0, 1) = T\xi^0$, and introduce new coordinates y and η as follows:

$$y = T(x - x_0), \qquad \eta = T\xi.$$

Then (x_0, ξ^0) corresponds to $(y_0, \eta^0) = (0, (0, \cdots, 0, 1))$ in the new coordinates. So the argument is reduced to the case $(0, (0, \cdots, 0, 1))$ by using (y, η) instead of (x, ξ).

In accordance with the operators (or the distributions) and (x_0, ξ^0) we want to treat later, we associate some weights K and L to $\xi' = (\xi_1, \cdots, \xi_{d-1})$ and x. That is, let $K = (K_1, \cdots, K_{d-1})$ and $L = (L_1, \cdots, L_d)$ be vectors whose elements are non-negative integers, and set

$$[\xi']_K = (\textstyle\sum' \xi_j^{2K_j})^{1/(2 \max K_j)},$$

$$[x]_L = (\textstyle\sum'' x_j^{2L_j})^{1/(2 \min L_j)}.$$

Here $\sum'$ (resp. $\sum''$) denotes the summation which runs over j satisfying $K_j \neq 0$ (resp. $L_j \neq 0$). With this notation, the Euclidean norm $|x|$ in $\mathbf{R}^d$ is equal to $[x]_{(1, \cdots, 1)}$. We fix K and L as well as x_0 and ξ^0 until we deal with examples.

Definition 1. A subset W_Γ (or $W_{(x_0, \xi^0; \Gamma)}$) of $\mathbf{R}^d \times \mathbf{R}^d$ given by

$$\begin{aligned} W_\Gamma = \{(x, \xi) \in \mathbf{R}^d \times \mathbf{R}^d \mid & A\xi_d^a \leqslant [\xi']_K \leqslant B\xi_d^b, \\ & G\xi_d^{-g} \leqslant [x]_L \leqslant H\xi_d^{-h}, \ \xi_d > 0, \ |\xi'| \leqslant \varepsilon\xi_d, \ |x| \leqslant \varepsilon\} \end{aligned} \tag{1-1}$$

is called a *Γ-parabolic neighborhood* of $(x_0, \xi^0) = (0, (0, \cdots, 0, 1))$.

Here Γ stands for the set of parameters x_0, ξ^0, K, L, a, A, b, B, g, G, h, H and ε. As x_0, ξ^0, K and L are fixed, we denote it abbreviatedly in the following manner:

$$\Gamma = \begin{pmatrix} a & b & g & h & \\ A & B & G & H & \end{pmatrix}\!\!\!\!\!\!\begin{matrix} \\ \varepsilon \\ \end{matrix}$$

where $0 < a \leqslant b \leqslant 1$, $0 \leqslant h \leqslant g < 1$ and A, B, G, H, ε are positive.

Remark 1. In the above definition of W_Γ we admit $a = -\infty$. In this case we replace the condition $A\xi_d^a \leqslant [\xi']_K \leqslant B\xi_d^b$ in (1-1) by $[\xi']_K \leqslant B\xi_d^b$ and write

$$\Gamma = \begin{pmatrix} & b & g & h & \\ -\infty & & & & \varepsilon \\ & B & G & H & \end{pmatrix}.$$

We also admit $g = +\infty$ in the similar way, denoting

$$\Gamma = \begin{pmatrix} a & b & & h & \\ & & +\infty & & \varepsilon \\ A & B & & H & \end{pmatrix}.$$

Remark 2. When $K = (0, \cdots, 0)$, the subset we consider is only the following:

$$W_\Gamma = \{(x, \xi) \in \mathbf{R}^d \times \mathbf{R}^d \mid G\xi_d^{-g} \leqslant [x]_L \leqslant H\xi_d^{-h}, \ \xi_d > 0, \ |\xi'| \leqslant \varepsilon\xi_d, \ |x| \leqslant \varepsilon\}.$$

In this case we write

$$\Gamma=\begin{pmatrix}\phi & \phi & \begin{matrix} g \\ G\end{matrix} & \begin{matrix} h \\ H\end{matrix} & \varepsilon\end{pmatrix}.$$

We do the same when $L=(0,\cdots,0)$.

Remark 3. A Γ-parabolic neighborhood is not a neighborhood of (x_0,ξ^0) in the proper sense of the word. But it is really a neighborhood when $a=-\infty$ (or $K=(0,\ldots,0)$) and $g=+\infty$ (or $L=(0,\cdots,0)$). Particularly, if Γ is one of the followings:

$$\begin{pmatrix}-\infty & \begin{matrix} 1 \\ B\end{matrix} & +\infty & \begin{matrix} 0 \\ H\end{matrix} & \varepsilon\end{pmatrix},\quad \begin{pmatrix}\phi & \phi & +\infty & \begin{matrix} 0 \\ H\end{matrix} & \varepsilon\end{pmatrix},$$

$$\begin{pmatrix}-\infty & \begin{matrix} 1 \\ B\end{matrix} & \phi & \phi & \varepsilon\end{pmatrix},\quad (\phi \quad \phi \quad \phi \quad \phi \quad \varepsilon),$$

then a Γ-parabolic neighborhood is nothing but a conic neighborhood of (x_0,ξ^0). In the sequel, such a Γ will be denoted by Γ^0. (When all of K_j's and L_j's are equal to zero, we consider only Γ^0).

Remark 4. In [9] and [13], Lascar and Parenti-Rodino also deal with such subsets of $\mathbf{R}^d\times\mathbf{R}^d$. The subsets they consider are the following Γ-parabolic neighborhoods in our terminology.

$$\begin{pmatrix}\begin{matrix} b \\ A\end{matrix} & \begin{matrix} b \\ B\end{matrix} & \phi & \phi & \varepsilon\end{pmatrix},\quad \begin{pmatrix}-\infty & \begin{matrix} b \\ B\end{matrix} & \phi & \phi & \varepsilon\end{pmatrix}.$$

Definition 1′. For a Γ-parabolic neighborhood W_Γ we put

$$\rho_\Gamma=a \quad\text{and}\quad \delta_\Gamma=g\,.$$

In case $a=-\infty$ (resp. $g=+\infty$), we define $\rho_\Gamma=b$ (resp. $\delta_\Gamma=h$). When all of K_j's (resp. L_j's) are equal to zero, we define $\rho_\Gamma=1$ (resp. $\delta_\Gamma=0$).

ρ_Γ and δ_Γ are often abbreviated to ρ and δ when no confusion arises. In order to make calculations meaningful, we impose the essential restriction $\rho_\Gamma>\delta_\Gamma$.

For a Γ-parabolic neighborhood W_Γ we often call another $\tilde{W}_\Gamma$, given by substituting $A-r_0$, $B+r_0$, $G-r_0$, $H+r_0$, $\varepsilon+r_0$ $(r_0>0)$ for A, B, G, H, ε respectively, a neighborhood of W_Γ. It is obvious that $\tilde{W}_\Gamma\supset W_\Gamma$.

Hereafter we are going to study the pseudo-differential operators by full use of Γ-parabolic neighborhoods. So the behavior of a given operator on W_Γ comes into question. Let $S^m_{\rho,\delta}$ $(0\leqslant\delta\leqslant\rho\leqslant 1)$ be the class of symbols of order m defined by Hörmander [3]. That is, $p(x,\xi)\in C^\infty(\mathbf{R}^d\times\mathbf{R}^d)$ belongs to $S^m_{\rho,\delta}$ if for every multi-index μ, ν there exists a constant $C_{\mu,\nu}$ such that

$$|p^{(\mu)}_{(\nu)}(x,\xi)|\leqslant C_{\mu,\nu}\langle\xi\rangle^{m-\rho|\mu|+\delta|\nu|},\qquad (x,\xi)\in\mathbf{R}^d\times\mathbf{R}^d,$$

where $p^{(\mu)}_{(\nu)}(x,\xi)=\partial^\mu_\xi(-i\partial_x)^\nu p(x,\xi)$, $\langle\xi\rangle=(1+|\xi|^2)^{1/2}$. We say that $p(x,\xi)\in S^m_{1,0}$

is a symbol of order m' on W_Γ or $p(x, \xi)$ belongs to $S_\Gamma^{m'}$ if there exists a neighborhood $\tilde{W}_\Gamma$ of W_Γ such that for any multi-index μ, ν we have with a constant $\tilde{C}_{\mu,\nu}$

$$|p_{(\nu)}^{(\mu)}(x, \xi)| \leqslant \tilde{C}_{\mu,\nu}\langle\xi\rangle^{m'-\rho_\Gamma|\mu|+\delta_\Gamma|\nu|}, \qquad (x, \xi) \in \tilde{W}_\Gamma.$$

Furthermore let us define the ellipticity of an operator on W_Γ.

Definition 2. We say that $p(x, \xi) \in S_{1,0}^m$ is *Γ-elliptic* if there exists a neighborhood $\tilde{W}_\Gamma$ of W_Γ and positive constants C_0, R such that on $\tilde{W}_\Gamma \cap \{|\xi| \geqslant R\}$ one of the following estimates holds.

$$\operatorname{Re}(zp(x, \xi)) \geqslant C_0\langle\xi\rangle^{m'-\sigma} \tag{1-2}$$

for some $z \in \mathbf{C}\backslash 0$ and σ, $0 \leqslant \sigma < \rho_\Gamma - \delta_\Gamma$, or

$$|p(x, \xi)| \geqslant C_0\langle\xi\rangle^{m'-\sigma} \tag{1-3}$$

for some σ, $0 \leqslant \sigma < (\rho_\Gamma - \delta_\Gamma)/2$. Here z and σ are independent of (x, ξ) and we are assuming that $p(x, \xi)$ is a symbol of order m' on W_Γ.

Properly speaking, we may as well say "Γ-subelliptic" instead of "Γ-elliptic". But we use the above terminology for the sake of simplicity.

Now we consider microhypoellipticity of a given operator $P = p(x, D)$ ($p(x, \xi) \in S_{1,0}^m$) at a given point (x_0, ξ^0). We define the microhypoellipticity of P at (x_0, ξ^0) as follows:

Definition 3. Let $p(x, \xi) \in S_{1,0}^m$. We say that $P = p(x, D)$ is microhypoelliptic at a point (x_0, ξ^0) of $\mathbf{R}^d \times S^{d-1}$ when P satisfies the following condition:

$$(x_0, \xi^0) \in WF(Pu) \quad \text{if and only if} \quad (x_0, \xi^0) \in WF(u) \qquad \text{for} \quad u \in \mathscr{E}'(\mathbf{R}^d).$$

Here $WF(u)$ denotes the wave front set of a distribution u defined in Hörmander [5], II. §5. For $u \in \mathscr{D}'(\mathbf{R}^d)$ and $(x_0, \xi^0) \in \mathbf{R}^d \times S^{d-1}$, $(x_0, \xi^0) \notin WF(u)$ if there exists a function $\zeta(x) \in C_0^\infty(\mathbf{R}^d)$ which is equal to 1 in a neighborhood of x_0 and a conic neighborhood V of ξ^0 such that for every positive M we have with a constant C_M

$$|(\zeta u)\hat{\ }(\xi)| \leqslant C_M\langle\xi\rangle^{-M}, \qquad \xi \in V.$$

Here and in what follows, $\hat{v}$ denotes the Fourier transform of $v \in \mathscr{S}'$. Note that, if P is microhypoelliptic at every point of $\mathbf{R}^d \times S^{d-1}$, then P is (micro)hypoelliptic.

The main theorem of the present article is:

Theorem 1. *Let $p(x, \xi) \in S_{1,0}^m$. Suppose that there exists a family $\{W_{\Gamma_j}\}_{1 \leqslant j \leqslant J}$ of Γ-parabolic neighborhoods of (x_0, ξ^0) satisfying*

(a) *$p(x, \xi)$ is Γ_j-elliptic for every j,*

(b) *$\bigcup_{j=1}^{J} W_{\Gamma_j}$ contains a conic neighborhood of (x_0, ξ^0),*

(c) *$\max_j \delta_{\Gamma_j} < \min_j \rho_{\Gamma_j}$.*

Then $P = p(x, D)$ is microhypoelliptic at (x_0, ξ^0).

The proof of this theorem will be given in §4.

Example. Let us consider the heat operator $P = \partial^2/\partial x_1^2 - \partial/\partial x_2$ *at* $(0, (0, 1))$. We define Γ_j and W_j $(j = 1, 2, 3)$ as follows:

$$\Gamma_1 = \begin{pmatrix} -\infty & \begin{matrix} 1/2 \\ 1 \end{matrix} & \phi & \phi & 1 \end{pmatrix}, \qquad W_1 = \{(x, \xi) \,|\, |\xi_1| \leqslant \xi_2^{1/2},\ \xi_2 > 0,\ |x| \leqslant 1\},$$

$$\Gamma_2 = \begin{pmatrix} 1/2 & \rho & \phi & \phi & 1 \\ 1 & 1 & & & \end{pmatrix}, \qquad W_2 = \{(x, \xi) \,|\, \xi_2^{1/2} \leqslant |\xi_1| \leqslant \xi_2^{\rho},\ \xi_2 > 0,\ |x| \leqslant 1\},$$

$$\Gamma_3 = \begin{pmatrix} \rho & 1 & \phi & \phi & 1 \\ 1 & 1 & & & \end{pmatrix}, \qquad W_3 = \{(x, \xi) \,|\, \xi_2^{\rho} \leqslant |\xi_1| \leqslant \xi_2,\ \xi_2 > 0,\ |x| \leqslant 1\},$$

where ρ is a real number satisfying $2/3 < \rho < 3/4$. Then P is Γ_j-elliptic for each j with

$$\begin{aligned} m' &= 1, & \sigma &= 0 & &\text{on } W_1, \\ &= 2\rho, & &= 2\rho - 1 & &\text{on } W_2, \\ &= 2, & &= 2 - 2\rho & &\text{on } W_3. \end{aligned}$$

It is also clear that this $\{\Gamma_j\}_{j=1,2,3}$ satisfies the conditions (b) and (c). So Theorem 1 implies that P is microhypoelliptic at $(0, (0, 1))$. Again in §5, we are going to consider a class of operators including this operator P and to show its (micro) hypoellipticity.

§2. Preliminaries

In this section we explain some terminologies which will be used throughout this paper. First we construct cut-off functions. Let $\varphi(t)$ be an element of $C^\infty(\mathbf{R}^1)$ satisfying $0 \leqslant \varphi(t) \leqslant 1$, $\varphi(t) = 1$ on $(-\infty, r_0/2]$ and $\varphi(t) = 0$ on $[r_0, +\infty)$. Moreover, given positive numbers S_1, S_2, θ and real numbers s_1, s_2, we put

$$\psi_{s_1, S_1}^{s_2, S_2}(t; \theta) = \varphi\left(1 - \frac{\theta^{s_1} t}{S_1}\right) + \varphi\left(\frac{\theta^{s_2} t}{S_2} - 1\right) - 1.$$

Using this function, we define cut-off functions as follows:

$$\alpha_n^\Gamma(\xi) = \psi_{-a, A}^{-b, B}([\xi']_K; n)\varphi\left(\frac{|\xi'|}{\varepsilon n} - 1\right)\psi_{-1, 1}^{-1, 1}(\xi_d; n),$$

$$\beta_n^\Gamma(x) = \psi_{g, G}^{h, H}([x]_L; n)\varphi\left(\frac{|x|}{\varepsilon} - 1\right),$$

where n is a large parameter. Therefore, our cut-off functions depend on n, r_0 and Γ. In case $a = -\infty$ we define simply

$$\alpha_n^\Gamma(\xi) = \varphi\left(\frac{n^{-b}[\xi']_K}{B} - 1\right)\varphi\left(\frac{|\xi'|}{\varepsilon n} - 1\right)\psi_{-1, 1}^{-1, 1}(\xi_d; n).$$

We define $\beta_n^\Gamma(x)$ in the similar way when $g = +\infty$.

Definition 4. A pseudo-differential operator $\Lambda_n^\Gamma(x, D)$ defined by

$$\Lambda_n^\Gamma(x, D)u = \alpha_n^\Gamma(D)(\beta_n^\Gamma(x)u) \qquad \text{for} \quad u \in \mathscr{D}'(\mathbf{R}^d)$$

is said to be a *microlocalizer attached to* W_Γ. We call r_0 in the definition of α_n^Γ and β_n^Γ the size of microlocalizer. (Usually we choose it small).

Notice that $\Lambda_n^\Gamma(x, D) \in S^{-\infty}$ for each n, and so $\Lambda_n^\Gamma(x, D)u \in \mathscr{S}(\mathbf{R}^d)$ for every $u \in \mathscr{D}'(\mathbf{R}^d)$.

By definition there exist positive constants C_1 and C_2 independent of ξ, n and Γ such that we have

$$C_1 n \leqslant \langle \xi \rangle \leqslant C_2 n \qquad \text{on } \operatorname{supp} \alpha_n^\Gamma(\xi)\,. \tag{2-1}$$

Furthermore we see that for every μ, ν there exist constants $C_{1,\mu}$, $C_{2,\nu}$ independent of n such that

$$\begin{aligned} |\partial_\xi^\mu \alpha_n^\Gamma(\xi)| &\leqslant C_{1,\mu} n^{-\rho|\mu|}\,, \\ |\partial_x^\nu \beta_n^\Gamma(x)| &\leqslant C_{2,\nu} n^{\delta|\nu|}\,. \end{aligned} \tag{2-2}$$

From these estimates we can derive some fundamental properties of microlocalizers. Before stating them, let us introduce a few of notations.

Definition 5. 1) $\Gamma \subset \Gamma'$ (or $W_\Gamma \subset W_{\Gamma'}$) means that $W_\Gamma(R) \subset W_{\Gamma'}(R)$ when R is large, where $W_\Gamma(R) = W_\Gamma \cap \{\xi_d = R\}$. Similarly $\Gamma \subset\subset \Gamma'$ (or $W_\Gamma \subset\subset W_{\Gamma'}$) means that $W_\Gamma(R)$ is relatively compact in $W_{\Gamma'}(R)$ when R is large.

2) $\alpha_n^\Gamma(\xi) \subset\subset \alpha_n^{\Gamma'}(\xi)$ means that $\alpha_n^{\Gamma'}(\xi) = 1$ in a neighborhood of the support of $\alpha_n^\Gamma(\xi)$ when n is large. It is the same for $\beta_n^\Gamma(x) \subset\subset \beta_n^{\Gamma'}(x)$.

3) $\Lambda_n^\Gamma(x, D) \subset\subset \Lambda_n^{\Gamma'}(x, D)$ means that α_n^Γ, β_n^Γ and $\alpha_n^{\Gamma'}$, $\beta_n^{\Gamma'}$, corresponding to Λ_n^Γ and $\Lambda_n^{\Gamma'}$ respectively, satisfy $\alpha_n^\Gamma \subset\subset \alpha_n^{\Gamma'}$ and $\beta_n^\Gamma \subset\subset \beta_n^{\Gamma'}$. When $\Lambda_n^\Gamma(x, D) \subset\subset \Lambda_n^{\Gamma'}(x, D)$, we say that $\Lambda_n^\Gamma(x, D)$ is subordinate to $\Lambda_n^{\Gamma'}(x, D)$.

Remark. It may happen that a microlocalizer is subordinate to another even if they are both attached to the same Γ-parabolic neighborhood, because of the difference in the size of them. If it is so, we are going to use the notation $\Lambda_n^\Gamma \subset\subset \tilde{\Lambda}_n^\Gamma$ in the sequel.

Definition 6. 1) We say that a sequence of real numbers $\{a_n\}$ depending on parameter n is rapidly decreasing or negligible if $n^M|a_n| \to 0$ for any positive number M when n tends to infinity.

2) We say that a sequence of functions $\{u_n\}$ depending on n is negligible if the sequence of numbers $\{\|u_n\|\}$ is rapidly decreasing when $n \to \infty$.

3) We say that a sequence of pseudo-differential operators $\{p_n(x, D)\}$ depending on n is negligible if $p_n(x, D) \in S^{-\infty}$ for every n and $\{|p_n|_l^{(m)}\}$ is rapidly decreasing for any m and l when $n \to \infty$. In this case we write $\{p_n(x, D)\} \in S_{neg}$. Moreover for two sequences of operators $\{p_n(x, D)\}$ and $\{q_n(x, D)\}$ we use the following notation:

$$p_n(x, D) \equiv q_n(x, D) \qquad (\mathrm{mod}\ S_{neg})$$

when $\{p_n(x, D) - q_n(x, D)\} \in S_{neg}$.

Here $|p|_l^{(m)} = \max_{|\mu+\nu| \leq l} \sup_{(x,\xi)} \{|p_{(\nu)}^{(\mu)}(x, \xi)| \langle\xi\rangle^{-(m-|\mu|)}\}$ and $\|\ \|$ stands for the L^2-norm in the x-space $\mathbf{R}^d$. We should remark that, as is easily seen, for any $u \in \mathscr{E}'(\mathbf{R}^d)$ $\{p_n(x, D)u\}$ is negligible when $\{p_n(x, D)\} \in S_{neg}$.

Now one of the most important properties of microlocalizers is the following:

Proposition 1. 1) *$\{\Lambda_n^\Gamma(x, D)\}$ is a bounded subset of $S_{\rho,\delta}^0$ when n tends to infinity.*

2) *Suppose that $\{p_n(x, \xi)\}$ is a bounded subset of $S_{1,0}^m$. If $\Lambda_n^\Gamma \subset\subset \Lambda_n^{\Gamma'}$ and* $\max(\delta_\Gamma, \delta_{\Gamma'}) < \min(\rho_\Gamma, \rho_{\Gamma'})$, *then*

$$\{\Lambda_n^\Gamma(x, D) p_n(x, D)(1 - \Lambda_n^{\Gamma'}(x, D))\} \in S_{neg}\,,$$

$$\{(1 - \Lambda_n^{\Gamma'}(x, D)) p_n(x, D) \Lambda_n^\Gamma(x, D)\} \in S_{neg}\,.$$

The same conclusions hold if $\Lambda_n^\Gamma \subset\subset \Lambda_n^{\Gamma'}$ and $\{p_n(x, \xi)\}$ is a bounded subset of S_{ρ_1,δ_1}^m where $\max(\delta_\Gamma, \delta_{\Gamma'}, \delta_1) < \min(\rho_\Gamma, \rho_{\Gamma'}, \rho_1)$.

As is seen below, this proposition is very useful in applications. We will prove it in §7.

Next we consider the microlocalization of symbols.

Definition 7. For $p(x, \xi) \in S_{1,0}^m$ we define the microlocalized symbol to W_Γ by

$$p_{n,loc}^\Gamma(x, \xi) = \beta_n^\Gamma(x) p(x, \xi) \alpha_n^\Gamma(\xi)\,.$$

Remark. Because the microlocalized symbol is defined by using cut-off functions, the microlocalized symbol depends on r_0 as well as n and Γ, though it is not explicitly written in the above notation. We often call r_0 the size of microlocalization (or the size of $\{p_{n,loc}^\Gamma(x, \xi)\}$).

Under this notation we say also that $p(x, \xi) \in S_\Gamma^{m'}$ if and only if $\{p_{n,loc}^\Gamma(x, \xi)\}$ is a bounded subset of $S_{\rho,\delta}^{m'}$ when n tends to infinity.

Proposition 2. *For $p(x, \xi) \in S_{1,0}^m$ we have*

$$p(x, D)\Lambda_n^\Gamma \equiv p_{n,loc}^\Gamma(x, D)\Lambda_n^\Gamma \qquad (\mathrm{mod}\ S_{neg})\,,$$

$$\Lambda_n^\Gamma p(x, D) \equiv \Lambda_n^\Gamma p_{n,loc}^\Gamma(x, D) \qquad (\mathrm{mod}\ S_{neg})\,,$$

if the size of Λ_n^Γ is smaller than a half of the size of $\{p_{n,loc}^\Gamma(x, \xi)\}$.

We will prove this proposition in §7 together with Proposition 1. This proposition shows the importance of the microlocalized symbol. For example we obtain the following estimates from above.

Proposition 3. *Suppose that $p(x, \xi) \in S_\Gamma^{m'}$ and $u \in \mathscr{E}'(\mathbf{R}^d)$. Then there exist* *positive constants* C_1, C_2, ε *and negligible sequences* $\{a_n\}$, $\{b_n\}$ *such that, if the size of* Λ_n^Γ *is smaller than* ε, *we have for all* n

$$\|p(x, D)\Lambda_n^\Gamma u\| \leq C_1 n^{m'} \|\Lambda_n^\Gamma u\| + a_n\,, \tag{2-3}$$

$$\|\Lambda_n^\Gamma p(x, D)u\| \leq C_2 n^{m'} \|\tilde{\Lambda}_n^\Gamma u\| + b_n\,, \tag{2-4}$$

where $\Lambda_n^\Gamma \subset\subset \tilde{\Lambda}_n^\Gamma$. *The constants* C_1, C_2 *and* ε *are independent of* u.

Proof. By assumption we can take the microlocalized symbol $p_{n,loc}^\Gamma(x, \xi)$ so that $\{p_{n,loc}^\Gamma(x, \xi)\}$ is a bounded subset of $S_{\rho,\delta}^{m'}$. In other words $\{n^{-m'}p_{n,loc}^\Gamma(x, \xi)\}$ is a bounded subset of $S_{\rho,\delta}^0$, because (2-1) holds. So by the theorem of L^2-boundedness of Calderón-Vaillancourt [1] there exists a constant C such that

$$n^{-m'} \|p_{n,loc}^\Gamma(x, D)v\| \leq C\|v\| \qquad \text{for} \quad v \in \mathscr{S}(\mathbf{R}^d)\,. \tag{2-5}$$

On the other hand, Proposition 2 says that

$$\|p(x, D)\Lambda_n^\Gamma u\| \leq \|p_{n,loc}^\Gamma(x, D)\Lambda_n^\Gamma u\| + a_n \tag{2-6}$$

with a negligible sequence $\{a_n\}$. By (2-5) and (2-6) we obtain (2-3).

By the way, Proposition 1 implies that with a constant C independent of n we have

$$\begin{aligned} \|\Lambda_n^\Gamma p(x, D)u\| &\leq \|\Lambda_n^\Gamma p(x, D)\tilde{\Lambda}_n^\Gamma u\| + \tilde{a}_n \\ &\leq C\|p(x, D)\tilde{\Lambda}_n^\Gamma u\| + \tilde{a}_n\,, \end{aligned}$$

where $\{\tilde{a}_n\}$ is a negligible sequence. Hence (2-4) follows from (2-3). Q.E.D.

§3. Finely microlocal smoothness

Let us consider the smoothness of a distribution on a Γ-parabolic neighborhood. First, we deal with the smoothness on a conic neighborhood W_{Γ^0}. As stated in Mizohata [12], the wave front set is closely connected with the microlocal energy $\|\Lambda_n^{\Gamma^0} u\|$. (See also [10], [11]). That is, we can prove the following proposition.

Proposition 4. *Let* u *be a distribution in* $\mathbf{R}^d$. *In order that* $(x_0, \xi^0) \notin WF(u)$, *it is necessary and sufficient that there exist a conic neighborhood* W_{Γ^0} of (x_0, ξ^0) *and a microlocalizer* $\Lambda_n^{\Gamma^0}$ *attached to* W_{Γ^0} *such that* $\{\|\Lambda_n^{\Gamma^0} u\|\}$ *is rapidly decreasing as* $n \to \infty$.

Proof. Let us notice that the cut-off function $\beta_n^{\Gamma^0}(x)$ doesn't depend on n. So we denote $\alpha_n^{\Gamma^0}(\xi)$ and $\beta_n^{\Gamma^0}(x)$ by $\alpha_n(\xi)$ and $\beta(x)$ for short. Now if $(x_0, \xi^0) \notin WF(u)$, then there exist $\beta(x) \in C_0^\infty(\mathbf{R}^d)$ which is equal to 1 in a neighborhood of x_0 and a conic neighborhood V of ξ^0 in $\mathbf{R}^d$ such that

$$|(\beta u)\hat{}\,(\xi)| \leq C_M \langle \xi \rangle^{-M}\,, \qquad M = 1, 2, \cdots, \qquad \xi \in V\,.$$

Hence, if the support of $\alpha_n(\xi)$ is small enough to be contained in V for all n, we have

$$\|\alpha_n(D)\beta u\|^2 = (2\pi)^{-d} \int \alpha_n(\xi)^2 |(\beta u)\hat{}(\xi)|^2 \, d\xi$$
$$\leq C_M \int_{n/2 \leq |\xi| \leq 2n} \langle\xi\rangle^{-2M} \, d\xi$$
$$\leq C'_M n^{-2M+d}, \qquad M = 1, 2, \cdots.$$

Conversely let us assume that $\|\Lambda_n^{\Gamma^0} u\| = \|\alpha_n(D)\beta u\|$ is rapidly decreasing. Take $\alpha_n^j(\xi)$ and $\beta^j(x)$ $(j = 1, 2)$ in such a way that $\alpha_n^1 \subset\subset \alpha_n^2 \subset\subset \alpha_n$ and $\beta^1 \subset\subset \beta^2 \subset\subset \beta$. We consider $\alpha_n^1(\xi)(\beta^1 u)\hat{}(\xi)$. Observe that

$$\sup_\xi |\alpha_n^1(\xi)(\beta^1 u)\hat{}(\xi)| \leq \|\alpha_n^1(D)\beta^1 u\|_{L^1(\mathbf{R}_x^d)}$$
$$\leq C_0 \|(1 + |x|^2)^l \alpha_n^1(D)\beta^1 u\|$$

where $l = [d/2] + 1$ and C_0 is a constant independent of n. Furthermore, $(1 + |x|^2)^l \cdot \alpha_n^1(D)\beta^1 u$ is the image of the inverse Fourier transformation of

$$(1 - \Delta_\xi)^l \alpha_n^1(\xi)(\beta^1 u)\hat{}(\xi)$$
$$= \sum_{|\nu| \leq 2l} \frac{(-1)^{|\nu|}}{\nu!} \alpha_n^{1(\nu)}(\xi) \mathscr{F}_{x\to\xi}\{((-i\partial_x)^\nu (1 + |x|^2)^l)\beta^1 u\}.$$

($\mathscr{F}_{x\to\xi}$ stands for the Fourier transformation). Therefore

$$(1 + |x|^2)^l \alpha_n^1(D)\beta^1 u$$
$$= \sum_{|\nu| \leq 2l} \frac{(-1)^{|\nu|}}{\nu!} \alpha_n^{1(\nu)}(D)\{(1 + |x|^2)^l_{(\nu)}\beta^1 u\}$$
$$= \sum_{|\nu| \leq 2l} \alpha_n^{1(\nu)}(D)\beta_\nu(x)u.$$

Here we set

$$\beta_\nu(x) = \nu!^{-1}(-1)^{|\nu|}(1 + |x|^2)^l_{(\nu)}\beta^1(x).$$

So we have

$$\sup_\xi |\alpha_n^1(\xi)(\beta^1 u)\hat{}(\xi)| \leq C_0 \sum_{|\nu| \leq 2l} \|\alpha_n^{1(\nu)}(D)\beta_\nu(x)u\|.$$

Now

$$\text{(3-1)} \qquad \alpha_n^{1(\nu)}(D)\beta_\nu(x)u = \alpha_n^{1(\nu)}(D)\alpha_n^2(D)\beta^2(x)\beta_\nu(x)u$$
$$= \alpha_n^{1(\nu)}(D)(\alpha_n^2(D)\beta^2(x))\beta_\nu(x)(\alpha_n(D)\beta(x)u)$$
$$+ \alpha_n^{1(\nu)}(D)(\alpha_n^2(D)\beta^2(x))\beta_\nu(x)(1 - \alpha_n(D)\beta(x))u.$$

Because $\{\alpha_n^{1(\nu)}(\xi)\}$ and $\{\beta_\nu(x)\}$ are bounded subsets of $S_{1,0}^{-|\nu|}$ and $S_{1,0}^0$ respectively, the right-hand side of (3-1) is negligible by assumption and Proposition 1. Therefore for every positive M there exists a constant C_M such that

$$\sup_{\xi} |\alpha_n^1(\xi)(\beta^1 u)\hat{}(\xi)| \leq C_M n^{-M}$$

holds for all n. In other words there exists a conic neighborhood V of ξ^0 in $\mathbf{R}^d$ such that

$$|(\beta^1 u)\hat{}(\xi)| \leq C'_M \langle\xi\rangle^{-M}, \qquad M = 1, 2, \cdots, \qquad \xi \in V.$$

This shows that $(x_0, \xi^0) \notin WF(u)$. Q.E.D.

Taking this proposition into account, it is plausible to define the "smoothness of a distribution on a Γ-parabolic neighborhood" as follows.

Definition 8. A distribution u in $\mathbf{R}^d$ is said to be smooth on W_Γ if there exists a microlocalizer Λ_n^Γ attached to W_Γ such that $\{\|\Lambda_n^\Gamma u\|\}$ is rapidly decreasing as $n \to \infty$. In this case we write $u \in C_\Gamma^\infty$ (or $u \in C_{(x_0, \xi^0; \Gamma)}^\infty$).

Once this notion is established, we can also express Proposition 4 in the following way.

> $(x_0, \xi^0) \notin WF(u)$ if and only if there exists a conic neighborhood W_{Γ^0} such that $u \in C^\infty_{(x_0, \xi^0; \Gamma^0)}$.

We should also remark that, if $\{\|\Lambda_n^\Gamma u\|\}$ is rapidly decreasing and $\tilde{\Lambda}_n^\Gamma \subset\subset \Lambda_n^\Gamma$, $\{\|\tilde{\Lambda}_n^\Gamma u\|\}$ is rapidly decreasing, too. In fact

$$\tilde{\Lambda}_n^\Gamma u = \tilde{\Lambda}_n^\Gamma \Lambda_n^\Gamma u + \tilde{\Lambda}_n^\Gamma (1 - \Lambda_n^\Gamma) u .$$

Applying Proposition 1, the assumption implies that the right-hand side is negligible. In the same way we can prove

Proposition 5. *$C_\Gamma^\infty \subset C_{\Gamma'}^\infty$ provided that $\Gamma' \subset \Gamma$ and* $\max(\delta_\Gamma, \delta_{\Gamma'}) < \min(\rho_\Gamma, \rho_{\Gamma'})$.

Proposition 6. *Suppose that $p(x, \xi) \in S_{1,0}^m$ and $u \in \mathscr{E}'(\mathbf{R}^d)$. Then $u \in C_\Gamma^\infty$ implies $p(x, D)u \in C_\Gamma^\infty$. The same conclusion holds when $p(x, \xi) \in S_{\rho_1, \delta_1}^m$ and* $\max(\delta_\Gamma, \delta_1) < \min(\rho_\Gamma, \rho_1)$.

Proof. Assume that $\{\|\Lambda_n^\Gamma u\|\}$ is rapidly decreasing. Take $\tilde{\Lambda}_n^\Gamma$ so that $\tilde{\Lambda}_n^\Gamma \subset\subset \Lambda_n^\Gamma$. Then we have

$$\tilde{\Lambda}_n^\Gamma p(x, D)u = \tilde{\Lambda}_n^\Gamma p(x, D)\Lambda_n^\Gamma u + \tilde{\Lambda}_n^\Gamma p(x, D)(1 - \Lambda_n^\Gamma)u .$$

The second term is negligible by Proposition 1. As for the first term, there exists a constant C_0 such that for all n

$$\|\tilde{\Lambda}_n^\Gamma p(x, D)\Lambda_n^\Gamma u\| \leq C_0 n^m \|\Lambda_n^\Gamma u\|$$

by Proposition 3, so it is negligible, too. Q.E.D.

The following theorem is fundamental.

Theorem 2. *Let u be a distribution in $\mathbf{R}^d$.*

1) *If $(x_0, \xi^0) \notin WF(u)$, there exists a Γ^0 such that $u \in C^\infty_{(x_0,\xi^0;\Gamma)}$ for any $\Gamma \subset \Gamma^0$.*

2) *Assume that $u \in C^\infty_{(x_0,\xi^0;\Gamma_j)}$ $(1 \leqslant j \leqslant J)$ for some $\{\Gamma_j\}_{1\leqslant j\leqslant J}$ satisfying $\max \delta_j < \min \rho_j$. If $\bigcup_{j=1}^{J} W_{\Gamma_j}$ contains a conic neighborhood of (x_0, ξ^0), then $(x_0, \xi^0) \notin WF(u)$.*

Proof. 1) follows from Propositions 4 and 5. Let us prove 2). We assume $x_0 = 0$, $\xi^0 = (0, \cdots, 0, 1)$ and

$$\bigcup_{j=1}^{J} W_{\Gamma_j} \supset \{(x, \xi) \in \mathbf{R}^d \times \mathbf{R}^d \,|\, \xi_d > 0,\ |\xi'| \leqslant \varepsilon_0 \xi_d,\ |x| \leqslant \varepsilon_0\}\,.$$

Put

$$\Gamma_j = \begin{pmatrix} a_j & b_j & g_j & h_j & \\ A_j & B_j & G_j & H_j & \end{pmatrix}\!\!\!\!\begin{matrix}\\ \varepsilon_j\end{matrix} \qquad \text{for} \quad 1 \leqslant j \leqslant J\,,$$

$$\varepsilon = \min_{1\leqslant j\leqslant J} \varepsilon_j\,.$$

First we arrange $A_j\xi_d^{a_j}$ and $B_j\xi_d^{b_j}$ $(1 \leqslant j \leqslant J)$ together in the increasing order of magnitude and number them as follows:

$$S_1 < S_2 < S_3 < \cdots < S_P\,.$$

Note that this order is independent of ξ_d when ξ_d is large. Doing the same procedure as to $G_j\xi_d^{-g_j}$ and $H_j\xi_d^{-h_j}$, we get $T_1, T_2, \cdots, T_Q$. Then define

$$W_q^p = \{(x, \xi) \in \mathbf{R}^d \times \mathbf{R}^d \,|\, S_{p-1} \leqslant [\xi']_K \leqslant S_p,$$
$$T_{q-1} \leqslant [x]_L \leqslant T_q,\ \xi_d > 0,\ |\xi'| \leqslant \varepsilon\xi_d,\ |x| \leqslant \varepsilon\}$$

for $1 \leqslant p \leqslant P$, $1 \leqslant q \leqslant Q$. Here we regard $S_0 = T_0 = 0$. Each W_q^p is nothing but a Γ-parabolic neighborhood. It is easy to see that

1°) $\{W_q^p\}$ is a refinement of $\{W_{\Gamma_j}\}$, that is for any (p, q) there exists a j such that $W_q^p \subset W_{\Gamma_j}$,
2°) $\max \delta_q^p = \max \delta_j$, $\min \rho_q^p = \min \rho_j$,
3°) $\bigcup_{p,q} W_q^p \supset \{(x, \xi) \,|\, \xi_d > 0,\ |\xi'| \leqslant \varepsilon_0\xi_d,\ |x| \leqslant \varepsilon_0\}$.

The properties 1°) and 2°) imply that u is smooth on every W_q^p due to the assumption and Proposition 5. Let $\Lambda_n^{p,q} = \alpha_n^p(D)\beta_n^q(x)$ be a microlocalizer attached to W_q^p such that $\{\|\alpha_n^p(D)\beta_n^q(x)u\|\}$ is rapidly decreasing. Now we will show that the smoothness of u on W_q^p and W_{q+1}^p implies the smoothness of u on

$$\tilde{W} = \{S_{p-1} \leqslant [\xi']_K \leqslant S_p,\ T_{q-1} \leqslant [x]_L \leqslant T_{q+1}\}\,.$$

Take $\tilde{\alpha}_n^p$, $\tilde{\beta}_n^q$, $\tilde{\beta}_n^{q+1}$ so that $\tilde{\alpha}_n^p \subset\subset \alpha_n^p$, $\tilde{\beta}_n^q \subset\subset \beta_n^q$, $\tilde{\beta}_n^{q+1} \subset\subset \beta_n^{q+1}$, and put

$$\tilde{\beta}_n(x) = \min\left(\tilde{\beta}_n^q(x) + \tilde{\beta}_n^{q+1}(x)\ ,\ 1\right).$$

Then $\tilde{\Lambda}_n = \tilde{\alpha}_n^p(D)\tilde{\beta}_n(x)$ is a microlocalizer attached to $\tilde{W}$. Now $\tilde{\Lambda}_n(x, D)u$ is decomposed into three terms:

$$\tilde{\Lambda}_n(x, D)u = \tilde{\alpha}_n^p(D)\tilde{\beta}_n^q u + \tilde{\alpha}_n^p(D)\tilde{\beta}_n^{q+1} u + \tilde{\alpha}_n^p(D) f_n u \tag{3-2}$$

where

$$f_n(x) = \tilde{\beta}_n(x) - (\tilde{\beta}_n^q(x) + \tilde{\beta}_n^{q+1}(x)) .$$

The first two terms on the right-hand side of (3-2) are negligible. Taking ζ_n^q in such a way that $\tilde{\beta}_n^q \subset\subset \zeta_n^q \subset\subset \beta_n^q$, we obtain

$$\begin{aligned}\tilde{\alpha}_n^p(D) f_n u &= \tilde{\alpha}_n^p(D)\zeta_n^q f_n u \\ &= (\tilde{\alpha}_n^p(D)\zeta_n^q) f_n (\alpha_n^p(D)\beta_n^q) u + (\tilde{\alpha}_n^p(D)\zeta_n^q) f_n (1 - \alpha_n^p(D)\beta_n^q) u ,\end{aligned}$$

since $\operatorname{supp} f_n \subset \operatorname{supp} \tilde{\beta}_n^q$. So the third term on the right-hand side of (3-2) is also negligible. Hence u is smooth on $\tilde{W}$.

Repeating this procedure, we see that u is smooth on $W^p = \{S_{p-1} \leqslant [\xi']_K \leqslant S_p, |x| \leqslant \varepsilon_0\}$ from the property 3°). In the similar way we can prove that the smoothness of u on W^p and W^{p+1} implies the smoothness of u on $\{S_{p-1} \leqslant [\xi']_K \leqslant S_{p+1}, |x| \leqslant \varepsilon_0\}$. Therefore u is smooth on $\{\xi_d > 0, |\xi'| \leqslant \varepsilon_0 \xi_d, |x| \leqslant \varepsilon_0\}$. Then the theorem follows from Proposition 4. Q.E.D.

Here let us consider the smoothness on W_Γ in the sense of Sobolev space.

Definition 9. *We say that* $u \in \mathscr{D}'(\mathbf{R}^d)$ *belongs to* H_Γ^s (or $H^s_{(x_0, \xi^0; \Gamma)}$) if there exists a microlocalizer Λ_n^Γ attached to W_Γ such that $\|\Lambda_n^\Gamma u\| = O(n^{-s})$ as $n \to \infty$.

We note that $a_n = O(n^{-M})$ means that there exists a constant C such that $|a_n| \leqslant Cn^{-M}$ for large n. For H_Γ^s we have the analogous results to C^∞ case studied so far. For example,

1) $H_\Gamma^s \subset H_{\Gamma'}^s$ provided that $\Gamma' \subset \Gamma$ and $\max(\delta_\Gamma, \delta_{\Gamma'}) < \min(\rho_\Gamma, \rho_{\Gamma'})$.
2) Suppose that $p(x, \xi) \in S_\Gamma^{m'}$ and $u \in \mathscr{E}'(\mathbf{R}^d)$. Then $u \in H_\Gamma^s$ implies $p(x, D)u \in H_\Gamma^{s-m'}$.

However, to state the analogy of Proposition 4, we need a slight modification as shown in Mizohata [12]. Let us recall here the definition of the wave front set in the sense of H^s.

Definition 10. For $u \in \mathscr{D}'(\mathbf{R}^d)$ and $(x_0, \xi^0) \in \mathbf{R}^d \times (\mathbf{R}^d \backslash 0)$ we say that $(x_0, \xi^0) \notin WF_{(s)}(u)$ (or $u \in H^s_{(x_0, \xi^0)}$), if there exists a function $\zeta(x) \in C_0^\infty(\mathbf{R}^d)$ which is equal to 1 in a neighborhood of x_0 and a conic neighborhood V of ξ^0 such that

$$\int_V \langle \xi \rangle^{2s} |(\zeta u)\hat{}(\xi)|^2 \, d\xi < +\infty .$$

(See Hörmander [6], p. 11).

Proposition 7. *The following two conditions for a distribution u in $\mathbf{R}^d$ are equivalent.*

(a) $(x_0, \xi^0) \notin WF_{(s)}(u)$.

(b) *There exist a conic neighborhood* W_{Γ^0} *of* (x_0, ξ^0) *and a microlocalizer* $\Lambda_n^{\Gamma^0}$ *attached to* W_{Γ^0} *such that*

$$\sum_n \|\Lambda_n^{\Gamma^0} u\|^2 n^{2s-1} < +\infty . \tag{3-3}$$

Corollary. *Let u be a distribution in* $\mathbf{R}^d$.

1) $(x_0, \xi^0) \notin WF_{(s)}(u)$ *implies* $u \in H^s_{(x_0, \xi^0; \Gamma^0)}$ *for some conic neighborhood* W_{Γ^0}.

2) $(x_0, \xi^0) \notin WF_{(s)}(u)$ *if* $u \in H^{s+\varepsilon}_{(x_0, \xi^0; \Gamma^0)}$ *for some* Γ^0 *and an* $\varepsilon > 0$.

Proof of Proposition 7 *and Corollary.* As in the proof of Proposition 4, we denote the cut-off functions $\alpha_n^{\Gamma^0}(\xi)$ and $\beta_n^{\Gamma^0}(x)$ attached to a conic neighborhood by $\alpha_n(\xi)$ and $\beta(x)$ respectively. Also in this proof we denote by C constants independent of n in general.

First we assume $(x_0, \xi^0) \notin WF_{(s)}(u)$. By definition, there exist $\beta(x) \in C_0^\infty(\mathbf{R}^d)$ which is equal to 1 near x_0 and a conic neighborhood V of ξ^0 in $\mathbf{R}^d$ such that

$$\int_V |(\beta u)^\wedge(\xi)|^2 \langle \xi \rangle^{2s} \, d\xi < +\infty . \tag{3-4}$$

Take a cut-off function $\alpha_n(\xi)$ so that $\operatorname{supp} \alpha_n(\xi) \subset V$. Then we have by (2-1)

$$\begin{aligned} \|\alpha_n(D)\beta u\|^2 n^{2s} &= (2\pi)^{-d} \int_V \alpha_n(\xi)^2 |(\beta u)^\wedge(\xi)|^2 \, d\xi \cdot n^{2s} \\ &\leqslant C \int_V \alpha_n(\xi)^2 |(\beta u)^\wedge(\xi)|^2 \langle \xi \rangle^{2s} \, d\xi . \end{aligned} \tag{3-5}$$

Corollary 1) is an immediate consequence of (3-4) and (3-5). On the other hand, (2-1) also tells us that for any ξ

$$\sum_n \alpha_n(\xi)^2/n \leqslant \sum_{\langle \xi \rangle/C_2 \leqslant n \leqslant \langle \xi \rangle/C_1} 1/n \leqslant C_2/C_1 .$$

Hence

$$\begin{aligned} \sum_n \|\alpha_n(D)\beta u\|^2 n^{2s-1} &\leqslant C \sum_n \int_V (\alpha_n(\xi)^2/n) |(\beta u)^\wedge(\xi)|^2 \langle \xi \rangle^{2s} \, d\xi \\ &\leqslant C \int_V |(\beta u)^\wedge(\xi)|^2 \langle \xi \rangle^{2s} \, d\xi \\ &< +\infty . \end{aligned}$$

Therefore (a) implies (b).

Conversely let us assume (b), that is (3-3) holds for some microlocalizer $\Lambda_n^{\Gamma^0} = \alpha_n(D)\beta(x)$. We can choose positive constants C_1', C_2' and a conic neighborhood V of ξ^0 such that

$$\alpha_n(\xi) = 1 \qquad \text{on} \quad \{\xi \in V \mid C_1' n \leqslant \langle \xi \rangle \leqslant C_2' n\} .$$

Then, when $|\xi|$ is large, we have

$$\sum_n \alpha_n(\xi)^2/n \geqslant \sum_{\langle\xi\rangle/C_2' \leqslant n \leqslant \langle\xi\rangle/C_1'} 1/n \geqslant C_0 > 0$$

where C_0 is independent of ξ. Therefore

$$\begin{aligned}\int_{V\cap\{|\xi|\geqslant R\}} |(\beta u)^{\wedge}(\xi)|^2\langle\xi\rangle^{2s}\,d\xi &\leqslant C\sum_n \int (\alpha_n(\xi)^2/n)|(\beta u)^{\wedge}(\xi)|^2\langle\xi\rangle^{2s}\,d\xi \\ &\leqslant C\sum_n \int \alpha_n(\xi)^2|(\beta u)^{\wedge}(\xi)|^2\,d\xi\cdot n^{2s-1} \\ &= C\sum_n \|\Lambda_n^{\Gamma^0}u\|^2 n^{2s-1} < +\infty,\end{aligned}$$

when R is large enough. Therefore (a) holds.

Finally Corollary 2) follows easily from Proposition 7, because $u \in H_{\Gamma^0}^{s+\varepsilon}$ implies

$$\begin{aligned}\sum_n \|\Lambda_n^{\Gamma^0}u\|^2 n^{2s-1} &\leqslant \sum_n (n^{-s-\varepsilon})^2 n^{2s-1} \\ &= \sum_n n^{-1-2\varepsilon} < +\infty\,.\end{aligned} \qquad \text{Q.E.D.}$$

The next theorem is the analogy of Theorem 2 for H_Γ^s. The proof is the same as Theorem 2, so we omit it.

Theorem 2′. *Let u be a distribution in $\mathbf{R}^d$.*

1) *If $(x_0, \xi^0) \notin WF_{(s)}(u)$, there exists a Γ^0 such that $u \in H^s_{(x_0,\xi^0;\Gamma)}$ for any $\Gamma \subset \Gamma^0$.*

2) *Assume that $u \in H^s_{(x_0,\xi^0;\Gamma_j)}$ $(1 \leqslant j \leqslant J)$ for some $\{\Gamma_j\}_{1\leqslant j\leqslant J}$ satisfying $\max \delta_j < \min \rho_j$. If $\bigcup_{j=1}^{J} W_{\Gamma_j}$ contains a conic neighborhood $W_{(x_0,\xi^0;\Gamma^0)}$, then $u \in H^s_{(x_0,\xi^0;\Gamma^0)}$.*

At the end of this section we give some examples.

Example 1. Let us consider a function $u_k(x) = \exp(ix^{-k})$ on $\mathbf{R}^1$, where k is a positive integer. u_k has a singularity at $x = 0$. Observing this singularity from the viewpoint of Γ-parabolic neighborhoods, we see that u_k is smooth on the following W_Γ. Here we are considering at $(x_0, \xi^0) = (0, 1)$. (The same conclusion holds at $(x_0, \xi^0) = (0, -1)$).

(a) $\Gamma_1 = \begin{pmatrix} \phi & \phi & \begin{matrix}\delta_1 \\ 1\end{matrix} & \begin{matrix}0 \\ 1\end{matrix} & 1\end{pmatrix}$, $\quad \delta_1 < 1/(k+1)$.

(b) $\Gamma_2 = \begin{pmatrix} \phi & \phi & +\infty & \begin{matrix}\delta_2 \\ 1\end{matrix} & 1\end{pmatrix}$, $\quad 1/(k+1) < \delta_2 < 1$.

In fact, if we put

$$v_n^j(\xi) = \int e^{-ix\xi}\beta_n^{\Gamma_j}(x)u_k(x)\,dx \qquad \text{for} \quad j = 1, 2\,,$$

we get the following estimates for $v_n^1(\xi)$:

$$|(i\xi)^M v_n^1(\xi)| \leqslant C_M n^{\delta_1(k+1)M}\,, \qquad M = 1, 2, \cdots,$$

where C_M denotes a constant independent of n. Hence we have with another constant C'_M

$$|\alpha_n(\xi)v_n^1(\xi)| \leqslant C'_M n^{\{\delta_1(k+1)-1\}M}$$

for all M, showing the smoothness of u_k on W_{Γ_1}.

Let us now observe u_k on W_{Γ_2}. u_k is a solution of the differential equation:

$$x^{k+1}(\partial u/\partial x) + kiu = 0\,.$$

So we have for all M

$$v_n^2(\xi) = (-i/k)^M \int u_K(x)p(x, D)^M(e^{-i\xi x}\beta_n^{\Gamma_2}(x))\,dx\,.$$

Here

$$p(x, D)u = (\partial/\partial x)(x^{k+1}u) = (k+1)x^k u + x^{k+1}(\partial u/\partial x)\,.$$

By this formula we get

$$|\alpha_n(\xi)v_n^2(\xi)| \leqslant C_M n^{\{1-\delta_2(k+1)\}M}$$

for all M with a constant C_M independent of n. Therefore u_k is smooth also on W_{Γ_2}.

Example 2. Next we take up a function $u(x) = f(x_2/x_1^2)$ on $\mathbf{R}^2(f(t) \in \mathscr{S}(\mathbf{R}^1))$. It is easily seen that u is infinitely differentiable except at $x = 0$. Furthermore the wave front set of u consists of two points in $\mathbf{R}^2 \times S^1$, that is $(0, (0, 1))$ and $(0, (0, -1))$. Considering $(0, (0, 1))$, getwe get the following results.

(a) $u \in C_\Gamma^\infty$ where $\Gamma = \begin{pmatrix} \rho_1 & 1 & \phi & \phi & 1 \\ 1 & 1 & & & \end{pmatrix}$, if $\rho_1 > 1/2$.

(b) $u \in H_\Gamma^{(\rho_2/2)-1}$ where $\Gamma = \begin{pmatrix} -\infty & \rho_2 & \phi & \phi & 1 \\ & 1 & & & \end{pmatrix}$, if $\rho_2 \leqslant 1/2$.

In order to show these results, we have to consider the Fourier transform of $\beta(x)u(x)$ where $\beta(x)$ is a function of $C_0^\infty(\mathbf{R}^d)$ and equal to 1 near $x = 0$. However, it suffices to look at $\hat{u}(\xi)$, because the Fourier transform of $(1 - \beta(x))u(x)$ is rapidly decreasing with respect to ξ. Now $\hat{u}(\xi)$ is easily calculated:

$$\hat{u}(\xi) = \begin{cases} \xi_2^{-3/2}\hat{g}_1(\xi_1/\sqrt{\xi_2}) & (\xi_2 > 0) \\ -\displaystyle\int_{-\infty}^{+\infty} f(t)\,dt \cdot \delta''(\xi_1) & (\xi_2 = 0) \\ (-\xi_2)^{-3/2}\hat{g}_2(\xi_1/\sqrt{-\xi_2}) & (\xi_2 < 0) \end{cases}$$

where δ'' denotes the second derivative of the δ-function on $\mathbf{R}^1$ and g_1 and g_2 are the functions of single variable defined by

$$g_1(t) = \hat{f}(t^2)t^2, \qquad g_2(t) = \hat{f}(-t^2)t^2.$$

The above results follow from this explicit form of $\hat{u}(\xi)$. Moreover, if $\hat{g}_1(\tau)$ satisfies

$$\hat{g}_1(0) = (\partial_\tau \hat{g}_1)(0) = \cdots = (\partial_\tau^{l-1} \hat{g}_1)(0) = 0,$$

then we get a better result than (b):

(c) $u \in H_\Gamma^{-((1/2)-\rho_2)(l+(1/2))-3/4}$

$$\text{where} \quad \Gamma = \begin{pmatrix} -\infty & \begin{smallmatrix}\rho_2 \\ 1\end{smallmatrix} & \phi & \phi & 1 \end{pmatrix} \quad \text{and} \quad \rho_2 \leqslant 1/2.$$

We should remark that, if we estimate the order of singularity of u in a conic neighborhood of (0, (0, 1)), we can assert merely that $u \in H_{\Gamma^0}^{-3/4}$. This value $-3/4$ is equal to the one we get by choosing $\rho_2 = 1/2$ in (b) or (c).

We can prove similar results for a function $f(x_2/x_1^k)$ on $\mathbf{R}^2$ where k is a positive integer and $f(t) \in \mathscr{S}(\mathbf{R}^1)$.

§4. Proof of Theorem 1

First we remark that by Propositions 4 and 6 we have $WF(p(x, D)u) \subset WF(u)$ for any $p(x, \xi) \in S_{1,0}^m$ and $u \in \mathscr{E}'(\mathbf{R}^d)$. Therefore, in order to prove Theorem 1, it suffices to show that $(x_0, \xi^0) \notin WF(p(x, D)u)$ implies $(x_0, \xi^0) \notin WF(u)$ for $u \in \mathscr{E}'(\mathbf{R}^d)$. This follows from Theorem 2 and

Theorem 3. *Let (x_0, ξ^0) be a point of $\mathbf{R}^d \times S^{d-1}$ and W_Γ be a Γ-parabolic neighborhood of (x_0, ξ^0). If $p(x, \xi) \in S_{1,0}^m$ is Γ-elliptic, then $p(x, D)u \in C_\Gamma^\infty$ implies $u \in C_\Gamma^\infty$ for $u \in \mathscr{E}'(\mathbf{R}^d)$.*

Admitting this theorem for the moment, let us prove Theorem 1.

Proof of Theorem 1. Let $p(x, \xi)$ be a symbol satisfying the conditions in the theorem. Assume that $(x_0, \xi^0) \notin WF(p(x, D)u)$. Then there exists a conic neighborhood W_{Γ^0} such that $p(x, D)u \in C_\Gamma^\infty$ for any $\Gamma \subset \Gamma^0$. We put $W_j' = W_{\Gamma_j} \cap W_{\Gamma^0}$ for $j = 1, \cdots, J$. Then $p(x, D)u$ is smooth on W_j' for every j and $\{W_j'\}_{1 \leqslant j \leqslant J}$ is a family of Γ-parabolic neighborhoods of (x_0, ξ^0) satisfying the conditions (a), (b) and (c) in the assumption of Theorem 1. Hence u is smooth on W_j' for every j by Theorem 3, and so we have $(x_0, \xi^0) \notin WF(u)$ by Theorem 2. Thus we have proved Theorem 1. Q.E.D.

The rest of this section is devoted to the proof of Theorem 3. We need an estimate from below.

Proposition 8. *Let $(x_0, \xi^0) \in \mathbf{R}^d \times S^{d-1}$ and W_Γ be a Γ-parabolic neighborhood* *of* (x_0, ξ^0). *Suppose that* $p(x, \xi) \in S_\Gamma^{m'}$ *is* Γ*-elliptic.* (*See* (1-2) *and* (1-3)). *Then there exist positive constants* ε, N *and a negligible sequence* $\{a_n\}$ *such that, if the size of* Λ_n^Γ *is smaller than* ε *and* n *is larger than* N, *we have*

$$\|p(x, D)\Lambda_n^\Gamma u\| \geqslant (C_0/2)((1-\varepsilon)n)^{m'-\sigma}\|\Lambda_n^\Gamma u\| + a_n$$

for any $u \in \mathscr{E}'(\mathbf{R}^d)$.

Proof. First we assume (1-2). If we choose ε so small and the size of $\Lambda_n^\Gamma = \alpha_n^\Gamma(D)\beta_n^\Gamma(x)$ is smaller than ε, then we can take cut-off functions $\alpha_n^j(\xi)$ and $\beta_n^j(x)$ ($j = 1, 2$) in such a way that $\alpha_n^\Gamma(\xi) \subset\subset \alpha_n^2(\xi) \subset\subset \alpha_n^1(\xi)$, $\beta_n^\Gamma(x) \subset\subset \beta_n^2(x) \subset\subset \beta_n^1(x)$ and $\operatorname{supp}(\beta_n^1(x)\alpha_n^1(\xi)) \subset \tilde{W}_\Gamma$. In this case we have

$$\operatorname{supp}\alpha_n^\Gamma(\xi) \subset \{(1-\varepsilon)n \leqslant |\xi| \leqslant (1+\varepsilon)n\}\,. \tag{4-1}$$

Now let us define

$$q_n(x, \xi) = zp(x, \xi)\beta_n^1(x)\alpha_n^1(\xi) + C_0\langle\xi\rangle^{m'-\sigma}(1 - \beta_n^1(x)\alpha_n^1(\xi))\,.$$

By assumption $\{q_n\}$ is a bounded subset of $S_{\rho,\delta}^{m'}$. Hence we have by Proposition 1

$$\begin{aligned}&(zp(x, D) - q_n(x, D))\Lambda_n^\Gamma(x, D)\\ &\quad\equiv \alpha_n^2(D)\beta_n^2(x)(zp(x, D) - q_n(x, D))\Lambda_n^\Gamma(x, D) \pmod{S_{neg}}\,.\end{aligned}$$

The right-hand side is equal to zero, because

$$\begin{aligned}&\beta_n^2(x)(zp(x, \xi) - q_n(x, \xi))\alpha_n^\Gamma(\xi)\\ &\quad= (zp(x, \xi) - C_0\langle\xi\rangle^{m'-\sigma})(1 - \beta_n^1(x)\alpha_n^1(\xi))\beta_n^2(x)\alpha_n^\Gamma(\xi)\\ &\quad= 0\,.\end{aligned}$$

So we have only to consider $\|q_n(x, D)\Lambda_n^\Gamma u\|$ instead of $\|p(x, D)\Lambda_n^\Gamma u\|$. As mentioned above, $q_n \in S_{\rho,\delta}^{m'}$. Furthermore by (1-2)

$$\begin{aligned}\operatorname{Re} q_n(x, \xi) &= \operatorname{Re}(zp(x, \xi))\beta_n^1(x)\alpha_n^1(\xi) + C_0\langle\xi\rangle^{m'-\sigma}(1 - \beta_n^1(x)\alpha_n^1(\xi))\\ &\geqslant C_0\langle\xi\rangle^{m'-\sigma}\beta_n^1(x)\alpha_n^1(\xi) + C_0\langle\xi\rangle^{m'-\sigma}(1 - \beta_n^1(x)\alpha_n^1(\xi))\\ &= C_0\langle\xi\rangle^{m'-\sigma} \qquad \text{on} \quad \{|\xi| \geqslant R\}\,.\end{aligned}$$

Therefore, by the sharp form of Gårding's inequality (See Kumano-go [8], Chap. 3, §4), there exists a constant C such that

$$\operatorname{Re}(q_n(x, D)\Lambda_n^\Gamma u, \Lambda_n^\Gamma u) \geqslant C_0\|\Lambda_n^\Gamma u\|^2_{(m'-\sigma)/2} - C\|\Lambda_n^\Gamma u\|^2_{(m'-(\rho-\delta))/2}\,, \tag{4-2}$$

where $\|\ \|_s$ stands for the norm in the Sobolev space of order s. Here we can easily see that by (4-1)

$$(1 + (1-\varepsilon)^2n^2)^{s/2}\|\Lambda_n^\Gamma u\| \leqslant \|\Lambda_n^\Gamma u\|_s \leqslant (1 + (1+\varepsilon)^2n^2)^{s/2}\|\Lambda_n^\Gamma u\|\,.$$

Hence (4-2) implies, for sufficiently large n,

$$\operatorname{Re}(q_n(x, D)\Lambda_n^\Gamma u, \Lambda_n^\Gamma u) \geqslant (C_0/2)((1-\varepsilon)n)^{m'-\sigma}\|\Lambda_n^\Gamma u\|^2$$

due to the assumption $0 \leqslant \sigma < \rho - \delta$.

Next we assume (1-3). Observe that

$$\|p(x, D)\Lambda_n^\Gamma u\|^2 = (p(x, D)\Lambda_n^\Gamma u, p(x, D)\Lambda_n^\Gamma u) = (p^*(x, D)p(x, D)\Lambda_n^\Gamma u, \Lambda_n^\Gamma u) \tag{4-3}$$

where $p^*(x, D)$ is the formal adjoint of $p(x, D)$. The calculus of pseudo-differential operators tells us that $p^*(x, D)p(x, D)$ has the following decomposition:

$$p^*(x, D)p(x, D) = q_1(x, D) + q_2(x, D), \tag{4-4}$$

$$\begin{cases} q_1(x, \xi) = |p(x, \xi)|^2 \in S_\Gamma^{2m'}, \\ q_2(x, \xi) \in S_\Gamma^{2m'-(\rho-\delta)}. \end{cases}$$

Since $\operatorname{Re} q_1(x, \xi) \geqslant C_0^2\langle\xi\rangle^{2(m'-\sigma)}$ holds by assumption, we can apply the above argument to $q_1(x, \xi)$, where (1-2) is satisfied in this case, and obtain that

$$\operatorname{Re}(q_1(x, D)\Lambda_n^\Gamma u, \Lambda_n^\Gamma u) \geqslant (C_0^2/2)((1-\varepsilon)n)^{2(m'-\sigma)}\|\Lambda_n^\Gamma u\|^2 + \text{negligible terms}. \tag{4-5}$$

On the other hand, Proposition 3 says that there exists a constant C such that

$$|(q_2(x, D)\Lambda_n^\Gamma u, \Lambda_n^\Gamma u)| \leqslant Cn^{2m'-(\rho-\delta)}\|\Lambda_n^\Gamma u\|^2 + \text{negligible terms}. \tag{4-6}$$

In view of (4-3) $\sim$ (4-6) and the assumption $0 \leqslant \sigma < (\rho-\delta)/2$, we have

$$\|p(x, D)\Lambda_n^\Gamma u\|^2 \geqslant (C_0^2/4)((1-\varepsilon)n)^{2(m'-\sigma)}\|\Lambda_n^\Gamma u\|^2 + \text{negligible terms}$$

when n is large, which completes the proof. Q.E.D.

Proof of Theorem 3. Let us assume that $p(x, \xi) \in S_\Gamma^{m'}$ and $\{\|\Lambda_n^\Gamma p(x, D)u\|\}$ is rapidly decreasing. We take another microlocalizer $\tilde{\Lambda}_n^\Gamma = \tilde{\alpha}_n^\Gamma(D)\tilde{\beta}_n^\Gamma(x)$ and a neighborhood $\tilde{W}_\Gamma$ of W_Γ satisfying

1°) $\operatorname{supp}(\tilde{\beta}_n^\Gamma(x)\tilde{\alpha}_n^\Gamma(\xi)) \subset\subset \tilde{W}_\Gamma$,
2°) there exists a microlocalizer which is attached to $\tilde{W}_\Gamma$ and subordinate to Λ_n^Γ.

We shall show that $\|\tilde{\Lambda}_n^\Gamma u\|$ is rapidly decreasing.

First notice that there exists a real number s such that $\|\Lambda_n^\Gamma u\| = O(n^{-s})$, since $u \in \mathscr{E}'(\mathbf{R}^d)$. Given a positive integer k arbitrarily, we take microlocalizers $\Lambda_n^1, \cdots, \Lambda_n^k$ attached to $\tilde{W}_\Gamma$ in such a way that

$$\tilde{\Lambda}_n^\Gamma \subset\subset \Lambda_n^k \subset\subset \Lambda_n^{k-1} \subset\subset \cdots \subset\subset \Lambda_n^1 \subset\subset \Lambda_n^\Gamma.$$

Let us observe that for $0 \leqslant l \leqslant k$

$$\begin{aligned} \Lambda_n^{l+1}p(x, D)u &= p(x, D)\Lambda_n^{l+1}u + [\Lambda_n^{l+1}, p(x, D)]u \\ &= p(x, D)\Lambda_n^{l+1}u + [\Lambda_n^{l+1}, p(x, D)]\Lambda_n^l u + \text{negligible terms}. \end{aligned} \tag{4-7}$$

Here we set $\Lambda_n^0 = \Lambda_n^\Gamma$ and $\Lambda_n^{k+1} = \tilde{\Lambda}_n^\Gamma$. Since $[\Lambda_n^{l+1}, p(x, D)] \in S_\Gamma^{m'-(\rho-\delta)}$, (4-7) implies that there exist positive constants C and C' such that

$$\|\Lambda_n^{l+1} p(x, D)u\| \geqslant C n^{m'-\sigma} \|\Lambda_n^{l+1} u\| - C' n^{m'-(\rho-\delta)} \|\Lambda_n^l u\| + \text{negligible terms}$$

due to Proposition 3 and Proposition 8. Using this formula for $0 \leqslant l \leqslant k$, we easily see that

(4-8) $\|\tilde{\Lambda}_n^\Gamma u\| \leqslant C_0 n^{-(m'-\sigma)} \|\Lambda_n^\Gamma p(x, D)u\| + C_1 n^{(k+1)(\sigma-(\rho-\delta))} \|\Lambda_n^\Gamma u\| + \text{negligible terms}$

holds with constants C_0 and C_1, since $\sigma - (\rho - \delta) < 0$. Now $\{\|\Lambda_n^\Gamma p(x, D)u\|\}$ is rapidly decreasing by assumption, so we have

$$\|\tilde{\Lambda}_n^\Gamma u\| = O(n^{-s+(k+1)(\sigma-(\rho-\delta))}).$$

This means that $\{\|\tilde{\Lambda}_n^\Gamma u\|\}$ is rapidly decreasing, because $\sigma - (\rho - \delta) < 0$ and k is arbitrary. Thus we have proved Theorem 3. Q.E.D.

§5. Applications to known examples

In this section we apply Theorem 1 to some operators which are well-known to be hypoelliptic.

Application 1. First let us consider a differential operator

$$P = p(x, D) = |x|^{2l}(-\Delta)^m + (-\Delta)^{m'}$$

with a symbol

$$p(x, \xi) = |x|^{2l}|\xi|^{2m} + |\xi|^{2m'}.$$

Here Δ denotes Laplacian $\partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_d^2$ and l, m and m' are non-negative integers. This operator was studied by Grushin [2]. By applying Theorem 1, we shall show the (micro)hypoellipticity of P when the condition $l > m - m' > 0$ is satisfied.

It is clear that P is microhypoelliptic at $(x, \xi) \in (\mathbf{R}^d \backslash 0) \times S^{d-1}$, because P is elliptic at such a point. (cf. Theorem 3). So we have only to consider at $(0, \xi)$. Without loss of generality we may assume $\xi = (0, \cdots, 0, 1)$. We define

$$K = (0, \cdots, 0), \qquad L = (1, \cdots, 1),$$
$$[x]_L = [x]_{(1,\cdots,1)} = |x|.$$

The Γ-parabolic neighborhoods we treat here are as follows:

$$\Gamma_{gh} = \begin{pmatrix} \phi & \phi & \begin{matrix} g \\ 1 \end{matrix} & \begin{matrix} h \\ 1 \end{matrix} & 1 \end{pmatrix},$$

$$W_{gh} = \{(x, \xi) \mid \xi_d^{-g} \leqslant |x| \leqslant \xi_d^{-h},\ |x| \leqslant 1,\ |\xi'| \leqslant \xi_d,\ \xi_d > 0\}.$$

1°) P is $\Gamma_{+\infty,h}$-elliptic if $m - lh \leqslant m'$, that is $(m - m')/l \leqslant h$. In fact, such a h $(0 \leqslant h < 1)$ exists due to the assumption of $l > m - m'$.

2°) When we consider P on W_{gh} where $0 \leqslant h < g < 1$ and $m - lh \geqslant m'$, we have

$$P \in S_\Gamma^{2(m-lh)}, \qquad \operatorname{Re} p(x, \xi) \geqslant |\xi|^{2(m-lg)}.$$

Hence P is $\Gamma_{g,h}$-elliptic if

$$2lg - 2lh < 1 - g,$$

that is

$$l'(1-h) < 1 - g \qquad (l' = (2l)/(2l+1)).$$

Putting the above results together, we easily see that we can take up a finite family $\{W_{g_j,h_j}\}$ of Γ-parabolic neighborhoods satisfying

(a) P is Γ_{g_j,h_j}-elliptic for every j,
(b) $\bigcup W_{g_j,h_j}$ contains a conic neighborhood of $(0, (0,\cdots, 0, 1))$.

So we have proved the microhypoellipticity of P at $(0, (0,\cdots, 0,1))$ because of Theorem 1.

Application 2. Let $\mathfrak{m} = (m_1,\cdots, m_d)$ be a vector whose components are positive integers and we set $|\alpha:\mathfrak{m}| = (\alpha_1/m_1) + \cdots + (\alpha_d/m_d)$ for a multi-index α. Let us consider a differential operator

$$P = p(x, D) = \Sigma_{|\alpha:\mathfrak{m}|\leqslant 1}\, a_\alpha(x) D_x^\alpha.$$

Here $a_\alpha(x) \in C^\infty(\mathbf{R}^d)$, $D_x = (-i\partial_1,\cdots, -i\partial_d)$ $(\partial_j = \partial/\partial x_j)$ and we define

$$p(x, \xi) = \sum_{|\alpha:\mathfrak{m}|\leqslant 1} a_\alpha(x)\xi^\alpha,$$

$$p_0(x, \xi) = \sum_{|\alpha:\mathfrak{m}|=1} a_\alpha(x)\zeta^\alpha.$$

If the condition $p_0(x_0, \xi) \neq 0$ holds for all $\xi \neq 0$, we say that P is semi-elliptic at x_0. For example, the heat operator $\partial^2/\partial x_1^2 - \partial/\partial x_2$ is semi-elliptic in this sense with $\mathfrak{m} = (2, 1)$. We are going to show that P is microhypoelliptic at (x_0, ξ^0) for all $\xi^0 \in S^{d-1}$, when P is semi-elliptic at x_0.

For simplicity we assume that $m_1 \geqslant m_2 \geqslant \cdots \geqslant m_d$. First we remark that, if we define

$$R(\xi) = \left(\sum_{j=1}^{d} \xi_j^{2m_j}\right)^{1/2},$$

the condition $p_0(x_0, \xi) \neq 0$ $(\xi \neq 0)$ implies that there exist positive constants C_1 and C_2 such that

(5-1) $$C_1^{-1} R(\xi) \leqslant |p_0(x, \xi)| \leqslant C_1 R(\xi),$$

$$C_2^{-1} R(\xi) \leqslant |p(x, \xi)| \leqslant C_2 R(\xi)$$

in a neighborhood of x_0 when $|\xi|$ is large. Also for all multi-indices α and β there exists a constant $C_{\alpha,\beta}$ such that

(5-2) $$|p_{(\beta)}^{(\alpha)}(x, \xi)| \leqslant C_{\alpha,\beta} R(\xi)^{1-|\alpha:\mathfrak{m}|}.$$

Now we put

$$m = \max\{m_j \mid 1 \leqslant j \leqslant d\} \; (= m_1),$$

$$m_0 = \max\{m_j \mid j \text{ such that } \xi_j^0 \neq 0\}.$$

In case $m = m_0$, P is elliptic at (x_0, ξ^0) by (5-1), so P is microhypoelliptic there. Hence we suppose $m > m_0$ from now on. Let $\{m_1, \cdots, m_l\}$ be the set of $\{m_j \mid m_j > m_0\}$. Note that $e_1 = (1, 0, \cdots, 0)$, $e_2 = (0, 1, 0, \cdots, 0)$, $\cdots$, $e_l = (0, \cdots, 1, \cdots, 0)$ (only the l-th component is equal to 1) and ξ^0 form an orthonormal system, since $\xi_j^0 = 0$ for $1 \leqslant j \leqslant l$. We take an orthonormal basis of $\mathbf{R}_\xi^d$ including this system and consider the following Γ-parabolic neighborhoods of (x_0, ξ^0).

$$W_{a,b} = \{(x, \xi) \mid r^a \leqslant R_0(\xi) \leqslant r^b\} \cap W_{\Gamma^0}.$$

Here r denotes a ξ^0-component of ξ, that is $r = \xi^0 \cdot \xi$, and

$$R_0(\xi) = \left(\sum_{j=1}^{l} \xi_j^{2m_j}\right)^{1/(2m)}.$$

We choose a conic neighborhood W_{Γ^0} of (x_0, ξ^0) so small that (5-1) and (5-2) hold in a neighborhood of W_{Γ^0}. We study the differential operator P on these Γ-parabolic neighborhoods.

1°) P is Γ-elliptic on $W_{-\infty,b}$ if $bm \leqslant m_0$. In fact (5-1) and (5-2) tell us that, if $bm \leqslant m_0$,

$$P \in S_\Gamma^{m_0}, \qquad |p(x, \xi)| \geqslant C|\xi|^{m_0}$$

where C is a constant independent of (x, ξ).

2°) When we consider P on $W_{a,b}$ where $0 < a < b \leqslant 1$ and $bm \geqslant m_0$, we have with a constant C'

$$P \in S_\Gamma^{bm}, \qquad |p(x, \xi)| \geqslant C'|\xi|^{am},$$

by (5-1) and (5-2). Hence P is Γ-elliptic on $W_{a,b}$ if

$$bm - am < a/2,$$

that is

$$m'b < a \qquad (m' = (2m)/(2m+1)).$$

Therefore P is microhypoelliptic at (x_0, ξ^0) as in the case of Application 1.

§6. Hypoellipticity of $\partial_x^{2m} + x\partial_y$

In this section we deal with a differential operator

$$P_m = (\partial/\partial x)^{2m} + x(\partial/\partial y)$$

where m is a positive integer. Because there is a point where the symbol of P_m vanishes, Theorem 1 is no more applicable to this case. However the Γ-

parabolic neighborhoods are very useful to this operator. We shall show that P_m is (micro) hypoelliptic for all $m \geqslant 1$. As mentioned in the Introduction, the hypoellipticity of P_1 is derived from the well-known theorem of Hörmander [4].

According to the notations used in §1 ~ §4, we write x_1 and x_2 instead of x and y below. So the operator P_m and the symbol $p_m(x, \xi)$ of P_m are

$$P_m = (\partial/\partial x_1)^{2m} + x_1(\partial/\partial x_2)$$

$$p_m(x, \xi) = (-1)^m \xi_1^{2m} + i x_1 \xi_2 .$$

Theorem 4. 1) *The operator P_m is (micro)hypoelliptic in $\mathbf{R}^2$.*

2) *Let τ be any positive number satisfying $\tau < (2m)/(2m+1)$. Then any solution of the equation $P_m u = f$ gains τ derivative at every point (x_0, ξ^0) in $\mathbf{R}^2 \times S^1$. More precisely, $u \in H^s_{(x_0, \xi^0)}$ and $P_m u \in H^s_{(x_0, \xi^0)}$ imply $u \in H^{s+\tau}_{(x_0, \xi^0)}$ for $u \in \mathscr{E}'(\mathbf{R}^2)$.*

Proof of Theorem 4. We shall prove only 2). 1) can be shown in the similar way. We denote $(2m)/(2m+1)$ simply by λ in this proof. Note that P_m is elliptic at (x_0, ξ^0) provided that $\xi^0 \neq (0, \pm 1)$. Hence we have only to show 2) at $(x_0, (0, \pm 1))$. As the situation is the same, we are going to treat only $(x_0, (0, 1))$. Suppose that $u \in H^s_{(x_0, (0,1))}$ and $f = P_m u \in H^s_{(x_0, (0,1))}$. Let us consider P_m on the following Γ-parabolic neighborhoods of $(x_0, (0, 1))$.

$$\Gamma_{a,b} = \begin{pmatrix} a & b & \phi & \phi & \varepsilon \\ 1 & 1 & & & \end{pmatrix},$$

$$W_{a,b} = \{(x, \xi) \mid \xi_2^a \leqslant |\xi_1| \leqslant \xi_2^b,\ |\xi_1| \leqslant \varepsilon \xi_2,\ \xi_2 > 0,\ |x - x_0| \leqslant \varepsilon\} .$$

Here ε is chosen so small that u and f belong to $H^s_{\Gamma^0}$ if W_{Γ^0} is defined by

$$W_{\Gamma^0} = \{(x, \xi) \mid \xi_2 > 0,\ |\xi_1| \leqslant 2\varepsilon \xi_2,\ |x - x_0| \leqslant 2\varepsilon\} .$$

Case I. Suppose that the first component of x_0 is not zero, that is $x_0 \neq (0, x_{0,2})$. In this case P_m is semi-elliptic. (cf. §5. Application 2.)

1°) Assume that $0 < a \leqslant b \leqslant 1$ and $b \geqslant 1/(2m)$. Then we have

$$P_m \in S_\Gamma^{2mb}, \qquad \mathrm{Re}\,((-1)^m p(x, \xi)) \geqslant C|\xi|^{2ma}$$

with a constant C. Therefore P_m is $\Gamma_{a,b}$-elliptic if

$$2mb - 2ma < a, \qquad \text{that is} \quad \lambda b < a .$$

Moreover, in view of (4-8) in the proof of Theorem 3,

$$u \in H^{s+2ma}_{\Gamma_{a,b}} \quad \text{if} \quad 1/(2m) \leqslant b \quad \text{and} \quad \lambda b < a .$$

Note that $u \in H_\Gamma^{s+1}$ if $a \geqslant 1/(2m)$.

2°) P_m is Γ-elliptic on $W_{-\infty, b}$ if $b \leqslant 1/(2m)$. This time we have with a constant C'

$$P_m \in S_\Gamma^1, \qquad \mathrm{Im}\, p(x, \xi) \geqslant C'|\xi| .$$

Hence, in view of (4-8), $u \in H_{\Gamma}^{s+1}$ on $W_{-\infty, 1/(2m)}$.

From the assertions 1°) and 2°), we have $u \in H_{\Gamma^0}^{s+1}$ for some conic neighborhood W_{Γ^0}. This implies $u \in H_{(x_0, \xi^0)}^{s+\sigma}$ for any $\sigma < 1$ by Proposition 7 and its Corollary.

Case II. Suppose that $x_0 = (0, x_{0,2})$. Without loss of generality we may assume $x_0 = 0$.

1°) Assume that $0 < a \leqslant b \leqslant 1$ and $b \geqslant 1/(2m)$. In this case, the same conclusion holds as in Case I, 1°). Hence

$$u \in H_{\Gamma_{a,b}}^{s+\lambda} \quad \text{if} \quad 1/(2m) \leqslant b \quad \text{and} \quad \lambda b < a .$$

2°) Now let us consider P_m on $W_{-\infty,b}$. In the following argument we use ρ instead of b and denote $\Gamma_{-\infty,b}$, $W_{-\infty,b}$, Λ_n^{Γ}, α_n^{Γ}, β_n^{Γ} simply by Γ_ρ, W_ρ, $\Lambda_{n,\rho}$, $\alpha_{n,\rho}$, β. That is,

$$W_\rho = \{(x, \xi) \,|\, |\xi_1| \leqslant \xi_2^\rho,\ |\xi_1| \leq \varepsilon\xi_2,\ \xi_2 > 0,\ |x| \leqslant \varepsilon\} .$$

Also we denote several constants independent of n by C_1, C_2, $\cdots$ and sometimes omit negligible terms in estimates.

Taking the result of 1°) into account, we may assume $\rho > 1/(2m+1)$. Let us estimate $\|\Lambda_{n,\rho}u\| = \|\alpha_{n,\rho}(D)\beta(x)u\|$. Because

$$\operatorname{supp} \alpha_{n,\rho}(\xi) \subset \{\xi \,|\, |\xi_1| \leqslant 2\xi_2^\rho,\ n/2 \leqslant \xi_2 \leqslant 2n\} ,$$

we have, by Poincaré's inequality,

$$\begin{aligned} \|\Lambda_{n,\rho}u\| &= \|\alpha_{n,\rho}(\xi)(\beta u)\hat{}(\xi)\| \\ &\leqslant C_1 n^\rho \|(\partial/\partial\xi_1)\{\alpha_{n,\rho}(\xi)(\beta u)\hat{}(\xi)\}\| . \end{aligned}$$

Hence

$$\begin{aligned} \text{(6-1)} \qquad \|\Lambda_{n,\rho}u\| &\leqslant C_2 n^{\rho-1} \|\xi_2(\partial/\partial\xi_1)\{\alpha_{n,\rho}(\xi)(\beta u)\hat{}(\xi)\}\| \\ &= C_2 n^{\rho-1} \|x_1(\partial/\partial x_2)(\Lambda_{n,\rho}u)\| . \end{aligned}$$

Observe that

$$\begin{aligned} \text{(6-2)} \qquad [x_1(\partial/\partial x_2), \Lambda_{n,\rho}]u &= i(\partial\alpha_{n,\rho}/\partial\xi_1)(D)(\partial\beta/\partial x_2)(x)u \\ &\quad + i(\partial\alpha_{n,\rho}/\partial\xi_1)(D)\beta(x)(\partial u/\partial x_2) \\ &\quad + \alpha_{n,\rho}(D)(\partial\beta/\partial x_2)(x)x_1 u . \end{aligned}$$

Here

$$\operatorname{supp} \partial\alpha_{n,\rho}/\partial\xi_1 \subset \{\xi \,|\, \xi_2^\rho \leqslant |\xi_1| \leqslant 2\xi_2^\rho,\ n/2 \leqslant \xi_2 \leqslant 2n\}$$

from the definition of $\alpha_{n,\rho}(\xi)$. So, owing to the result of 1°), the norm of the first two terms on the right-hand side of (6-2) is smaller than a constant times of

$n^{-s-\lambda+1-\rho}$. Hence we have

(6-3) $$\|[x_1(\partial/\partial x_2), \Lambda_{n,\rho}]u\| \leqslant C_3\|\tilde{\Lambda}_{n,\rho}u\| + O(n^{-s-\lambda+1-\rho}),$$

where $\Lambda_{n,\rho} \subset\subset \tilde{\Lambda}_{n,\rho}$. Putting (6-1) and (6-3) together, we obtain

(6-4) $$\|\Lambda_{n,\rho}u\| \leq C_4 n^{\rho-1}\{\|\Lambda_{n,\rho}(x_1\partial u/\partial x_2)\| + \|\tilde{\Lambda}_{n,\rho}u\|\} + O(n^{-s-\lambda}).$$

On the other hand,

(6-5) $$\begin{aligned}\|\Lambda_{n,\rho}(x_1\partial u/\partial x_2)\| &= \|\Lambda_{n,\rho}(P_m - (\partial/\partial x_1)^{2m})u\| \\ &\leqslant \|\Lambda_{n,\rho}f\| + \|\Lambda_{n,\rho}(\partial/\partial x_1)^{2m}u\|.\end{aligned}$$

Here we claim that

(6-6) $$\begin{aligned}\|\Lambda_{n,\rho}(\partial/\partial x_1)^{2m}u\| \leqslant\ & C_5 n^{m\rho}\{n^{\varepsilon-m\rho}\|\tilde{\Lambda}_{n,\rho}f\| + n^{m\rho-\varepsilon}\|\tilde{\Lambda}_{n,\rho}u\|\} \\ & + O(n^{-s-\lambda+1-\rho+\varepsilon}),\end{aligned}$$

where ε is any positive number satisfying $0<\varepsilon \leqslant 1/(4m+2)$. We admit (6-6) for the moment. Then, in view of (6-4), (6-5) and (6-6), we obtain

(6-7) $$\begin{aligned}\|\Lambda_{n,\rho}u\| \leqslant\ & C_6\{n^{\rho-1+\varepsilon}\|\tilde{\Lambda}_{n,\rho}f\| + n^{(2m+1)\rho-1-\varepsilon}\|\tilde{\Lambda}_{n,\rho}u\|\} \\ & + O(n^{-s-\lambda+\varepsilon}).\end{aligned}$$

From this estimate and the result of 1°), we can prove the assertion 2) of the Theorem as follows. First we set $\varepsilon' = \varepsilon + 1 - (2m+1)\rho$. Choosing ρ closely enough to $1/(2m+1)$, we may assume $\varepsilon' > 0$. On the other hand, we have $\|\tilde{\Lambda}_{n,\rho}f\| = O(n^{-s})$ and $\|\tilde{\Lambda}_{n,\rho}u\| = O(n^{-s})$ by assumption. Hence $\|\Lambda_{n,\rho}u\| = O(n^{-s-\varepsilon'})$ holds due to (6-7). Putting this and the result of 1°) together, we obtain $u \in H^{s+\varepsilon'}_{\Gamma^0}$ for some conic neighborhood W_{Γ^0} by Theorem 2′. Then, replacing $\|\tilde{\Lambda}_{n,\rho}u\| = O(n^{-s})$ by $\|\tilde{\Lambda}_{n,\rho}u\| = O(n^{-s-\varepsilon'})$, we can do the same argument and see that $\|\Lambda_{n,\rho}u\| = O(n^{-s-2\varepsilon'})$. As is easily seen, we can repeat this procedure until we obtain $u \in H^{s+1-\rho-\varepsilon}_{\Gamma^0}$. (See the first term on the right-hand side of (6-7)). Because we are able to choose $\rho - 1/(2m+1)$ and ε as closely to zero as possible, this shows 2) at (0, (0, 1)). Q.E.D.

Proof of (6-6). Assume that $\Lambda_{n,\rho} \subset\subset \Lambda^1_{n,\rho} \subset\subset \Lambda^2_{n,\rho} \subset\subset \tilde{\Lambda}_{n,\rho}$. Because

$$\begin{aligned}\|\Lambda_{n,\rho}(\partial/\partial x_1)^{2m}u\| &\leqslant \|\Lambda_{n,\rho}(\partial/\partial x_1)^m\Lambda^1_{n,\rho}(\partial/\partial x_1)^m\Lambda^2_{n,\rho}u\| \\ &\leqslant C_7 n^{m\rho}\|(\partial/\partial x_1)^m\Lambda^2_{n,\rho}u\|,\end{aligned}$$

we have only to show that

(6-8) $$\begin{aligned}\|(\partial/\partial x_1)^m\Lambda^2_{n,\rho}u\| \leqslant\ & C_8\{n^{\varepsilon-m\rho}\|\tilde{\Lambda}_{n,\rho}f\| + n^{m\rho-\varepsilon}\|\tilde{\Lambda}_{n,\rho}u\|\} \\ & + O(n^{-s-\lambda+(1-\rho)-m\rho+\varepsilon}),\end{aligned}$$

for any ε satisfying $0<\varepsilon \leqslant 1/(4m+2)$. Observe that

$$\begin{aligned}\|(\partial/\partial x_1)^m \Lambda_{n,\rho}^2 u\|^2 &\leqslant ((\partial/\partial x_1)^m \Lambda_{n,\rho}^2 u, (\partial/\partial x_1)^m \Lambda_{n,\rho}^2 u)\\ &= (-1)^m \operatorname{Re}(P_m \Lambda_{n,\rho}^2 u, \Lambda_{n,\rho}^2 u)\\ &= (-1)^m \operatorname{Re}(\Lambda_{n,\rho}^2 P_m u, \Lambda_{n,\rho}^2 u)\\ &\quad + (-1)^m \operatorname{Re}([(\partial/\partial x_1)^{2m}, \Lambda_{n,\rho}^2]u, \Lambda_{n,\rho}^2 u)\\ &\quad + (-1)^m \operatorname{Re}([x_1(\partial/\partial x_2), \Lambda_{n,\rho}^2]u, \Lambda_{n,\rho}^2 u).\end{aligned}$$

Here $[(\partial/\partial x_1)^{2m}, \Lambda_{n,\rho}^2]$ is of order $(2m-1)\rho$ on W_ρ since $(\partial/\partial x_1)^{2m}$ is of order $2m\rho$. Hence, noting (6-3), we have

$$\begin{aligned}\|(\partial/\partial x_1)^m \Lambda_{n,\rho}^2 u\|^2 &\leqslant \|\tilde{\Lambda}_{n,\rho} f\| \, \|\tilde{\Lambda}_{n,\rho} u\| + C_9 n^{(2m-1)\rho} \|\tilde{\Lambda}_{n,\rho} u\|^2\\ &\quad + C_{10}\{\|\tilde{\Lambda}_{n,\rho} u\| + O(n^{-s-\lambda+1-\rho})\} \, \|\tilde{\Lambda}_{n,\rho} u\|\\ &\leqslant C_{11}\{n^{2(\varepsilon-m\rho)}\|\tilde{\Lambda}_{n,\rho} f\|^2 + n^{2(m\rho-\varepsilon)}\|\tilde{\Lambda}_{n,\rho} u\|^2\}\\ &\quad + O(n^{-2s-2\lambda+2(1-\rho)-2m\rho+2\varepsilon}).\end{aligned}$$

This implies (6-8). Q.E.D.

Remark 1. The value $(2m)/(2m+1)$ appearing in Theorem 4, 2) is best-possible. In fact, let $\varphi(t)$ be an element of $C^\infty(\mathbf{R}^1)$ satisfying

$$0 \leqslant \varphi(t) \leqslant 1, \quad \varphi(t) = 1 \text{ on } (-\infty, 1], \quad = 0 \text{ on } [2, +\infty),$$

and put

$$\lambda(\xi_1, \xi_2) = \varphi(|\xi_1|^{2m+1}/\xi_2)(1-\varphi(\xi_2))\xi_2^{-(m+1)/(2m+1)},$$

$$u(x) = \mathscr{F}^{-1}\lambda = (2\pi)^{-2}\int e^{ix\cdot\xi}\lambda(\xi)\, d\xi.$$

Then we see easily that

$$u \in H^{-s}(\mathbf{R}^2) \quad \text{for} \quad s > 0 \quad \text{and}$$

$$P_m u \in H^{-s}(\mathbf{R}^2) \quad \text{for } s > (2m)/(2m+1),$$

but

$$u \notin L^2(U) \qquad \text{in any neighborhood } U \text{ of } x = (0, 0).$$

For the present we cannot prove that any solution of $P_m u = f$ gains $(2m)/(2m+1)$ derivative by our method. However, we suppose that the derivative gain of P_m is exactly $(2m)/(2m+1)$.

When $m = 1$, this value equals to $2/3$, as Rothschild-Stein obtained in [15].

Remark 2. By making full use of Γ-parabolic neighborhoods, we can also prove the hypoellipticity of

$$(\partial/\partial x_1)^{2m} \pm i x_1^k (-i\, \partial/\partial x_2)^l$$

in the similar way. But the situation is a little more complicated. As for such operators, we shall discuss in the forthcoming paper.

§7. Proofs of Propositions 1 and 2

First we remark that by (2-1) there exist functions $\chi_n(\xi)$ and $\chi_{n,\pm}(\xi)$ which are elements of $C^\infty(\mathbf{R}^d)$, depend only on $|\xi|$ and satisfy the following conditions.

$$
\text{(7-1)}\quad
\begin{aligned}
&0 \leqslant \chi_n, \quad \chi_{n,\pm} \leqslant 1, \quad \chi_n + \chi_{n,-} + \chi_{n,+} = 1, \\
&\operatorname{supp} \chi_n \subset \{C_1 n \leqslant |\xi| \leqslant C_2 n\}, \\
&\chi_n = 1 \text{ in a neighborhood of } \operatorname{supp} \alpha_n^\Gamma(\xi), \\
&\operatorname{supp} \chi_{n,-} \subset \{|\xi| \leqslant C_1' n\}, \\
&\operatorname{supp} \chi_{n,-} \bigcap \operatorname{supp} \alpha_n^\Gamma(\xi) = \phi, \\
&\operatorname{supp} \chi_{n,+} \subset \{C_2' n \leqslant |\xi|\}, \\
&\operatorname{supp} \chi_{n,+} \bigcap \operatorname{supp} \alpha_n^\Gamma(\xi) = \phi, \\
&|\partial_\xi^\mu \chi_n|, |\partial_\xi^\mu \chi_{n,\pm}| \leqslant C_\mu n^{-|\mu|} \quad \text{for any } \mu.
\end{aligned}
$$

(In this section we denote several constants independent of n and ξ by $C_1, C_2, \cdots$.)

Lemma. *Suppose that $\{p_n(x, \xi)\}$ is a bounded subset of $S^m_{1,0}$, then*

$$\{\alpha_n^\Gamma(D) p_n(x, D) \chi_{n,\pm}(D)\} \in S_{neg},$$

$$\{\alpha_n^\Gamma(D) \beta_n^\Gamma(x) p_n(x, D) \chi_{n,\pm}(D)\} \in S_{neg}.$$

The same conclusion holds if $\{p_n(x, \xi)\}$ is a bounded subset of $S^m_{\rho_1,\delta_1}$ where $\max(\delta_\Gamma, \delta_1) < \min(\rho_\Gamma, \rho_1)$.

Proof. We shall show that $\{\alpha_n^\Gamma(D)\beta_n^\Gamma(x)p_n(x, D)\chi_{n,\pm}(D)\} \in S_{neg}$ when $\{p_n(x, \xi)\}$ is a bounded subset of $S^m_{\rho_1,\delta_1}$ and $\max(\delta_\Gamma, \delta_1) < \min(\rho_\Gamma, \rho_1)$. We can do the same in the other cases.

We denote anew $\max(\delta_\Gamma, \delta_1)$ and $\min(\rho_\Gamma, \rho_1)$ by δ and ρ respectively. We put

$$S_{n,\pm}(x, D) = \alpha_n^\Gamma(D)\beta_n^\Gamma(x)p_n(x, D)\chi_{n,\pm}(D).$$

From the calculus of pseudo-differential operators, we have

$$S_{n,\pm}(x, \xi) = O_s - \iint e^{-iy\cdot\eta}\alpha_n^\Gamma(\xi + \eta)\beta_n^\Gamma(x + y)p_n(x + y, \xi)\chi_{n,\pm}(\xi)\, dy\, d\eta/(2\pi)^d.$$

Here O_s- means the oscillatory integral. Now let l be a positive integer satisfying $l > d/2$. We take such an integer l and fix it. On the other hand, let k be an arbitrary positive integer. Using

$$e^{-iy\cdot\eta} = \langle y\rangle^{-2l}(1 - \Delta_\eta)^l \langle\eta\rangle^{-2(k+l)}(1 - \Delta_y)^{k+l} e^{-iy\cdot\eta},$$

we obtain by integration by parts

$$(7\text{-}2)\quad S_{n,\pm}{}^{(\mu)}_{(\nu)}(x,\xi)=\iint e^{-iy\cdot\eta}\langle\eta\rangle^{-2(k+l)}\partial_\xi^\mu[(1-\Delta_\eta)^l\alpha_n^\Gamma(\xi+\eta)\chi_{n,\pm}(\xi)(1-\Delta_y)^{k+l}$$
$$\times D_x^\nu\{\langle y\rangle^{-2l}\beta_n^\Gamma(x+y)p_n(x+y,\xi)\}]\,dy\,d\eta/(2\pi)^d\,.$$

Because of Leibniz' formula, (2-2) and (7-1), the integrand of (7-2) is estimated from above by the sum of a finite number of terms of the following form:

$$C\langle\eta\rangle^{-2(k+l)}n^{-\rho|\mu_1|}\langle y\rangle^{-2l}n^{\delta|\gamma_1|}\langle\xi\rangle^{m+\delta|\gamma_2|-\rho|\mu_2|}$$

where $|\mu_1+\mu_2|=|\mu|$ and $|\gamma_1+\gamma_2|\leqslant\nu+2(k+l)$. Let us notice that the integral (7-2) is absolutely convergent owing to $\langle\eta\rangle^{-2l}\langle y\rangle^{-2l}$.

1°) $(S_{n,-}(x,\xi))$ Since $\xi\in\operatorname{supp}\chi_{n,-}$ and $\xi+\eta\in\operatorname{supp}\alpha_n$, there exist positive constants C and C' such that $|\xi|\leqslant Cn\leqslant C'|\eta|$. Therefore, when k is large, we have

$$\langle\eta\rangle^{-2k}n^{-\rho|\mu_1|}n^{\delta|\gamma_1|}\langle\xi\rangle^{m+\delta|\gamma_2|-\rho|\mu_2|}\leqslant C_3n^{m_+-\rho|\mu_1|+\delta|\nu|+2\delta l-2k(1-\delta)}$$
$$\leqslant C_4\langle\xi\rangle^{m_+-\rho|\mu_1|+\delta|\nu|+2\delta l-2k(1-\delta)}\,.$$

$(m_+=\max(0,m))$. Since k is arbitrary, we have $S_{n,-}(x,\xi)\in S^{-\infty}$ for every n. Furthermore it is easily seen that this implies $\{S_{n,-}(x,D)\}\in S_{neg}$.

2°) $(S_{n,+}(x,\xi))$ Since $\xi\in\operatorname{supp}\chi_{n,+}$ and $\xi+\eta\in\operatorname{supp}\alpha_n$, there exist positive constants C and C' such that $n\leqslant C|\xi|\leqslant C'|\eta|$. Therefore

$$\langle\eta\rangle^{-2k}n^{-\rho|\mu_1|}n^{\delta|\gamma_1|}\langle\xi\rangle^{m+\delta|\gamma_2|-\rho|\mu_2|}\leqslant C_5\langle\xi\rangle^{m-\rho|\mu_2|+\delta|\nu|+2\delta l-2k(1-\delta)}$$
$$\leqslant C_6n^{m-\rho|\mu_2|+\delta|\nu|+2\delta l-2k(1-\delta)}\,,$$

when k is large. The rest is the same as in the case of 1°). Q.E.D.

Proof of Proposition 1.

1) We devide $\Lambda_n^\Gamma(x,D)=\alpha_n^\Gamma(D)\beta_n^\Gamma(x)$ into three terms:

$$(7\text{-}3)\quad \Lambda_n^\Gamma(x,D)=\alpha_n^\Gamma(D)\beta_n^\Gamma(x)\chi_n(D)+\alpha_n^\Gamma(D)\beta_n^\Gamma(x)\chi_{n,-}(D)+\alpha_n^\Gamma(D)\beta_n^\Gamma(x)\chi_{n,+}(D)\,.$$

Here $\{\alpha_n^\Gamma(D)\beta_n^\Gamma(x)\chi_{n,-}(D)\}$ and $\{\alpha_n^\Gamma(D)\beta_n^\Gamma(x)\chi_{n,+}(D)\}$ belong to S_{neg} because of Lemma. Concerning the first term on the right-hand side of (7-3), we see that $\{\alpha_n^\Gamma(\xi)\}$ and $\{\beta_n^\Gamma(x)\chi_n(\xi)\}$ are bounded subsets of $S^0_{\rho,\delta}$ by (2-2) and (7-1). Hence $\{\alpha_n^\Gamma(D)\beta_n^\Gamma(x)\chi_n(D)\}$ is a bounded subset of $S^0_{\rho,\delta}$, and $\{\Lambda_n^\Gamma(x,D)\}$ is so, too.

2) Let us prove that $\{\Lambda_n^\Gamma(x,D)p_n(x,D)(1-\Lambda_n^{\Gamma'}(x,D))\}\in S_{neg}$ when $\{p_n(x,\xi)\}$ is a bounded subset of $S^m_{\rho_1,\delta_1}$ and $\max(\delta_\Gamma,\delta_{\Gamma'},\delta_1)<\min(\rho_\Gamma,\rho_{\Gamma'},\rho_1)$. We can do the same in the other cases.

We denote anew $\max(\delta_\Gamma,\delta_{\Gamma'},\delta_1)$ and $\min(\rho_\Gamma,\rho_{\Gamma'},\rho_1)$ by δ and ρ respectively. Owing to Lemma and 1) proved above, we have only to show that $\{\Lambda_n^\Gamma(x,D)p_n(x,D)(1-\Lambda_n^{\Gamma'}(x,D))\chi_n(D)\}\in S_{neg}$. Let us decompose it as follows:

$$(7\text{-}4)\qquad \Lambda_n^\Gamma p_n(1-\Lambda_n^{\Gamma'})\chi_n=\alpha_n(D)\beta_n(x)p_n(x,D)(1-\alpha_n'(D))\chi_n(D)$$
$$+\alpha_n(D)\beta_n(x)p_n(x,D)\alpha_n'(D)(1-\beta_n'(x))\chi_n(D)\,.$$

Here and in what follows, we write α_n, α'_n, β_n and β'_n instead of α_n^Γ, $\alpha_n^{\Gamma'}$, β_n^Γ and $\beta_n^{\Gamma'}$ for short.

1°) We put

$$S_{n,1}(x, D) = \alpha_n(D)\beta_n(x)p_n(x, D)(1 - \alpha'_n(D))\chi_n(D).$$

Let us show that $\{S_{n,1}(x, D)\} \in S_{neg}$. Note that

$$S_{n,1}(x, \xi) = O_s - \iint e^{-iy\cdot\eta}\alpha_n(\xi + \eta)\beta_n(x + y)p_n(x + y, \xi)(1 - \alpha'_n(\xi))\chi_n(\xi)\, dy\, d\eta/(2\pi)^d.$$

Since $\xi \in \operatorname{supp}\chi_n \bigcap \operatorname{supp}(1 - \alpha'_n)$ and $\xi + \eta \in \operatorname{supp}\alpha_n$, there exist positive constants C and C' such that

$$C^{-1}n \leqslant |\xi| \leqslant Cn, \qquad |\eta| \geqslant C'n^\rho.$$

If we do the same as in the proof of Lemma, we obtain

$$\begin{aligned} |S_{n,1}{}^{(\mu)}_{(\nu)}| &\leqslant C_7 n^{m-\rho|\mu|+\delta|\nu|+2\delta l-2k(\rho-\delta)} \\ &\leqslant C_8\langle\xi\rangle^{m-\rho|\mu|+\delta|\nu|+2\delta l-2k(\rho-\delta)} \end{aligned}$$

where k is an arbitrary positive integer. Hence $\{S_{n,1}(x, D)\} \in S_{neg}$.

2°) We put

$$S_{n,2}(x, D) = \beta_n(x)p_n(x, D)\alpha'_n(D)(1 - \beta'_n(x))\chi_n(D).$$

Let us show that $\{S_{n,2}(x, D)\} \in S_{neg}$. If we can prove it, we find that the second term on the right-hand side of (7-4) is also an element of S_{neg}, because $\{\alpha_n(\xi)\}$ is a bounded subset of $S^0_{\rho,0}$. Note that

$$S_{n,2}(x, \xi) = O_s - \iint e^{-iy\cdot\eta}\beta_n(x)p_n(x, \xi + \eta)\alpha'_n(\xi + \eta)(1 - \beta'_n(x + y))\chi_n(\xi)\, dy\, d\eta/(2\pi)^d.$$

Since $x \in \operatorname{supp}\beta_n$ and $x + y \in \operatorname{supp}(1 - \beta'_n)$, there exists a positive constant C such that $|y| \geqslant Cn^{-\delta}$. On the other hand, we have with constants C' and C''

$$C'^{-1}n \leqslant |\xi| \leqslant C'n, \qquad C''^{-1}n \leqslant |\xi + \eta| \leqslant C''n,$$

since $\xi \in \operatorname{supp}\chi_n$ and $\xi + \eta \in \operatorname{supp}\alpha'_n$. Now let l be a positive integer satisfying $l > d/2$ and k be an arbitrary positive integer. Using

$$e^{-iy\cdot\eta} = \langle y\rangle^{-2l}|y|^{-2k}(1 - \Delta_\eta)^l(-\Delta_\eta)^k\langle\eta\rangle^{-2l}(1 - \Delta_y)^l e^{-iy\cdot\eta},$$

we obtain by intgration by parts

$$\begin{aligned} S_{n,2}{}^{(\mu)}_{(\nu)}(x, \xi) = \iint & e^{-iy\cdot\eta}\langle\eta\rangle^{-2l}D_x^\nu[\partial_\xi^\mu(1 - \Delta_\eta)^l(-\Delta_\eta)^k \\ & \times \{p_n(x, \xi + \eta)\alpha'_n(\xi + \eta)\chi_n(\xi)\}\beta_n(x)(1 - \Delta_y)^l \\ & \times \{\langle y\rangle^{-2l}|y|^{-2k}(1 - \beta'_n(x + y))\}]\, dy\, d\eta/(2\pi)^d. \end{aligned} \tag{7-5}$$

So the integrand of (7-5) is estimated from above by the sum of a finite number of terms of the following form:

$$C\langle\eta\rangle^{-2l}n^{m-\rho|\mu|+\delta|\nu|-2k\rho}\langle y\rangle^{-2l}|y|^{-(2k+|\gamma_1|)}n^{\delta|\gamma_2|}$$

where $|\gamma_1+\gamma_2|\leqslant 2l$. Therefore

$$|S_{n,2}{}_{(\nu)}^{(\mu)}|\leqslant C_9 n^{m-\rho|\mu|+\delta|\nu|+2\delta l-2k(\rho-\delta)}$$

$$\leqslant C_{10}\langle\xi\rangle^{m-\rho|\mu|+\delta|\nu|+2\delta l-2k(\rho-\delta)}.$$

Since k is arbitrary, this implies $\{S_{n,2}(x,D)\}\in S_{neg}$. Q.E.D.

Proof of Proposition 2. We put

$$p_{n,\mathrm{loc}}^{\Gamma}(x,\xi)=\tilde{\beta}_n^{\Gamma}(x)p(x,\xi)\tilde{\alpha}_n^{\Gamma}(\xi),$$

$$\Lambda_n^{\Gamma}(x,D)=\alpha_n^{\Gamma}(D)\beta_n^{\Gamma}(x).$$

By assumption we have $\alpha_n^{\Gamma}(\xi)\subset\subset\tilde{\alpha}_n^{\Gamma}(\xi)$ and $\beta_n^{\Gamma}(x)\subset\subset\tilde{\beta}_n^{\Gamma}(x)$. Because

$$p_{n,\mathrm{loc}}^{\Gamma}(x,D)\Lambda_n^{\Gamma}(x,D)=\tilde{\beta}_n^{\Gamma}(x)p(x,D)\Lambda_n^{\Gamma}(x,D),$$

$$\Lambda_n^{\Gamma}(x,D)p_{n,\mathrm{loc}}^{\Gamma}(x,D)=\Lambda_n^{\Gamma}(x,D)p(x,D)\tilde{\alpha}_n^{\Gamma}(x,D),$$

it is sufficient to show that

$$\{(1-\tilde{\beta}_n^{\Gamma}(x))p(x,D)\alpha_n^{\Gamma}(D)\beta_n^{\Gamma}(x)\}\in S_{neg},$$

$$\{\alpha_n^{\Gamma}(D)\beta_n^{\Gamma}(x)p(x,D)(1-\tilde{\alpha}_n^{\Gamma}(D))\}\in S_{neg}.$$

These follow from the same argument as in the proof of Proposition 1, 2). (See 1°) and 2°) in the proof of Proposition 1, 2)). Q.E.D.

DEPARTMENT OF MATHEMATICS
KYOTO UNIVERSITY

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