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On bounds for Castelnuovo's index of regularity

By

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1. Introduction

For coherent sheaves on a projective space Castelnuovo's index of regularity was first defined by D. Mumford [11], who attributes the idea to G. Castelnuovo. In fact, he used it to show that certain twists of a coherent sheaf are generated by its global sections. In a more algebraic setting Castelnuovo's regularity was defined by D. Eisenbud and S. Goto [2] and A. Ooishi [12], see (2.2). It comes out that Castelnuovo's index of regularity gives an upper bound for the maximal degree of the syzygies in a minimal free resolution, see (2.8) for the precise statement. This fact is very important for the complexity of a program for a numerical computation of syzygies. As shown by examples of D. Bayer and M. Stillman, see [1], Castelnuovo's regularity can be of exponential growth with respect to the Krull dimension. While these examples have rather "wild" singularities, one might hope to get better bounds in the case of "tame" singularities. This point of view is pursued further in the papers [2], [10], [12], [16], [17], [18].

One of our main results, see (3.5), supplies an upper bound for $\operatorname{reg} R$, Castelnuovo's regularity, of a graded k-algebra R that is Cohen-Macaulay or locally Cohen-Macaulay and unmixed. In a geometric context, we get a satisfactory bound in the case of a Cohen-Macaulay projective variety. In particular, with (3.5) a) we solve a problem posed in [10] in the affirmative. The same result was shown independently and by a different argument by J. Stückrad and W. Vogel in [18].

In the case of a Cohen-Macaulay ring R it follows, see [2], that

$$\operatorname{reg} R \leqslant e(R) - c\,,$$

where $e(R)$ denotes the multiplicity and c $(:= \dim_k R_1 - \dim R)$ the codimension of R. For R a domain over k, an algebraically closed field, D. Eisenbud and S. Goto conjectured that

$$\operatorname{reg} R \leqslant e(R) - c\,,$$

see [2]. If R is the coordinate ring of a reduced, irreducible curve in projective space, this is true as shown by L. Gruson, R. Lazarsfeld, and C. Peskine in [7]. In our situation of a locally Cohen-Macaulay domain R, i.e., $\operatorname{Proj}(R)$ is a

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Cohen-Macaulay integral scheme, over an algebraically closed field of characteristic zero there are several results in this direction shown by J. Stückrad and W. Vogel, see [16], [17], and [18]. In particular they have proved that

$$\operatorname{reg} R < (e(R) - 1)/c + 1 + \Lambda(R),$$

where $\Lambda(R)$ is defined in terms of the annihilators of local cohomology. See Section 3 for the definition of $\Lambda(R)$. In Section 4 we will give a straightforward proof of the above bound. In the more particular case of an arithmetically Buchsbaum variety, i.e., R is a Buchsbaum domain, J. Stückrad and W. Vogel obtained

$$\operatorname{reg} R < (e(R) - 1)/c + 1,$$

see [17]. In (4.3) we enclose a simplified proof.

In Section 2 we give several characterizations of Castelnuovo's regularity in terms of Koszul cohomology and certain quotient modules. It turns out that for an unmixed locally Cohen-Macaulay ring R, the reduction exponent of a standard system of parameters $\underline{x} \subseteq R_1$ does not depend on $\underline{x}$ and equals $\operatorname{reg} R$. In Section 3 we investigate bounds for Castelnuovo's index of regularity in the case when R is unmixed and locally Cohen-Macaulay. In Section 4 we demand in addition R to be a domain over an algebraically closed field of characteristic zero.

2. The Castelnuovo index of regularity

First of all let us fix some notation. By $R = \bigoplus_{n \geqslant 0} R_n$ we denote a graded k-algebra such that $R_0 = k$ is a field, R is generated by R_1, and $\dim_k R_1 < \infty$. Then there is a natural isomorphism $R \cong S/I$, where $S = k[X_1, \ldots, X_e]$ is the polynomial ring in $e = \dim_k R_1$ indeterminates over k and $I \subseteq S$ is a homogeneous ideal, which contains no linear forms. We put $P = (X_1, \ldots, X_e)S$, the irrelevant ideal of S. For a graded R-module $M = \bigoplus_{n \in \mathbf{Z}} M_n$, we introduce the following notations.

(2.1) Definition. (1) $s(M) = \sup\{n \in \mathbf{Z} : M_n \neq 0\}$.
(2) $i(M) = \inf\{n \in \mathbf{Z} : M_n \neq 0\}$.
(3) $\lambda(M) = \inf\{n \in \mathbf{Z} : P^n M = 0\}$.

Note that $s(M) < \infty$ (resp. $i(M) > -\infty$, $\lambda(M) < \infty$) if M is an artinian R-module (resp. a finitely generated R-module, an R-module of finite length). See [16] for the basic facts about these integers. In the following we will use the local cohomology modules $H_P^i(M)$, $i \in \mathbf{Z}$, and the Koszul cohomology modules $H^i(Q; M)$, $i \in \mathbf{Z}$, with respect to a fixed (minimal) generating set of a homogeneous ideal Q. Note that changing a basis of Q yields isomorphic Koszul cohomology modules. For further basic results about local cohomology and Koszul cohomology we refer to A. Grothendieck [6].

(2.2) Definition. Set $\operatorname{reg} M = \max\{s(H_P^i(M)) + i : i \in \mathbf{Z}\}$, the Castelnuovo index of regularity of a finitely generated graded R-module M.

The Castelnuovo index of regularity was studied by D. Mumford [11]. For further investigations see also [12], [2], [1], and [16]. Because $H_P^i(M)$, $i \in \mathbf{Z}$, are artinian R-modules, see [6], (2.1) yields that (2.2) is well-defined. In the sequel we shall need the notion of a filter-regular sequence for M.

(2.3) Definition. Let $\{x_1, \ldots, x_r\}$ be a system of homogeneous elements of R that is a part of a system of parameters for M. It is called a filter-regular sequence for M if

$$\lambda((x_1, \ldots, x_{i-1})M : x_i/(x_1, \ldots, x_{i-1})M) < \infty$$

or equivalently,

$$s((x_1, \ldots, x_{i-1})M : x_i/(x_1, \ldots, x_{i-1})M) < \infty$$

for $i = 1, \ldots, r$.

In the case of a local ring the notion of a filter-regular sequence was introduced in [15]. See [15] for basic facts. If $\underline{x} = \{x_1, \ldots, x_d\}$ denotes a homogeneous system of parameters consisting of a filter-regular sequence for M, we put

$$r_{i-1}(\underline{x}; M) = s((x_1, \ldots, x_{i-1})M : x_i/(x_1, \ldots, x_{i-1})M)$$

for $i = 1, \ldots, d$. In addition we set

$$r_d(\underline{x}; M) = s(M/\underline{x}M),$$

which we call the reduction exponent of $\underline{x}$ with respect to M. For $i \in \mathbf{Z}\setminus\{0, \ldots, d\}$, we define $r_i(\underline{x}; M)$ to be $-\infty$.

(2.4) Proposition. *Let M be a finitely generated graded R-module and $\{x_1, \ldots, x_r\}$ a system of forms of R with $d_j = \deg x_j$, $j = 1, \ldots, r$. For all i and j, there is the following short exact sequence of homomorphisms of degree* 0:

$$0 \to (H^{i-1}(x_1, \ldots, x_{j-1}; M)/x_j H^{i-1}(x_1, \ldots, x_{j-1}; M))(d_j)$$
$$\to H^i(x_1, \ldots, x_j; M) \to 0_{H^i(x_1, \ldots, x_{j-1}; M)} : x_j \to 0\,.$$

The proof is well-known.

Now we are ready to prove several characterizations of the Castelnuovo index of regularity.

(2.5) Theorem. *For a finitely generated graded R-module M, the following integers are equal*:

a) $\operatorname{reg} M$.

b) $\max\{s(H^i(Q; M)) + i : i \in \mathbf{Z}\}$ *for any P-primary ideal Q generated by forms of degree* 1.

c) $\max\{s(H^i(\underline{x}; M)) + i : i \in \mathbf{Z}\}$ *for any system of parameters $\underline{x} = \{x_1, \ldots, x_d\}$ for M consisting of forms of degree* 1.

d) $\max\{r_i(\underline{x}; M) : i \in \mathbf{Z}\}$, *where* $\underline{x} = \{x_1, \ldots, x_d\}$, $d = \dim M$, *is any filter-regular sequence for M consisting of forms of degree* 1.

In particular, the integers defined in b), c), d) *are independent of the choice of Q and* $\underline{x}$ *respectively.*

Proof. Denote by A, B, C, D the integers defined in a), b), c), d) respectively.

$A \geqslant B$: Consider the following spectral sequence

$$H^p(Q; H_P^q(M)) \Rightarrow H^{p+q}(Q; M),$$

see e.g. [6]. By the definition of A, we have

$$H_P^i(M)_j = 0 \qquad \text{for all} \quad i + j > A .$$

Therefore

$$[H^p(Q; H_P^q(M))]_n = 0 \qquad \text{whenever} \quad p + q + n > A$$

because it is a subquotient of

$$[\oplus H_P^q(M)(p)]_n = 0 .$$

From the spectral sequence, it follows that $[H^i(Q; M)]_n = 0$ for all i, n with $i + n > A$. This yields the inequality.

$B \geqslant C$ holds trivially.

$C \geqslant D$: We have $H^i(\underline{x}; M)_n = 0$ whenever $i + n > C$ by the definition of C. It follows easily from (2.4) that $[H^i(x_1, \ldots, x_j; M)]_n = 0$ for all i, n with $i+n> C$ and all $j = 1, \ldots, d$. Again by (2.4), we get epimorphisms

$$H^{j-1}(x_1, \ldots, x_j; M) \to ((x_1, \ldots, x_{j-1})M : x_j/(x_1, \ldots, x_{j-1})M)(j-1) \to 0$$

for $j = 1, \ldots, d$ and $H^d(\underline{x}; M) \cong (M/\underline{x}M)(d)$. These conclusions imply the inequality.

$D \geqslant A$: For the proof we use induction on $\dim M$. The case of $\dim M = 0$ is obvious. Let $\dim M > 0$. By the induction hypothesis, it follows from the definition of D that

$$[H_P^i(M/x_1 M)]_n = 0 \qquad \text{as soon as} \quad i + n > D .$$

Furthermore we have $[0_M : x_1]_n = 0$ for all $n > D$. Because x_1 is a filter-regular element, we get an exact sequence

$$[H_P^{i-1}(M/x_1 M)]_n \to [H_P^i(M)]_{n-1} \xrightarrow{f_i} [H_P^i(M)]_n$$

for all $i > 0$, see [14], (2.5). By the induction hypothesis, f_i is injective whenever $i - 1 + n > D$. Because $s(H_P^i(M)) < \infty$, it follows

$$[H_P^i(M)]_n = 0 \qquad \text{for all } i, n \text{ with } i + n > D \text{ and } i > 0 .$$

Now let us consider the case $i = 0$. By [14], (2.5), we have an injection

$$0 \to [H_P^0(M)]_{n-1}/[0_M : x_1]_{n-1} \to [H_P^0(M)]_n .$$

Because $[0_M : x_1]_{n-1} = 0$ for all $n - 1 > D$, we get $[H_P^0(M)]_n = 0$ for all $n > D$ as required.

In the case of a Cohen-Macaulay module, (2.5) yields the well-known fact that

$$r_d(\underline{x}; M) = \operatorname{reg} M .$$

That is, the reduction exponent of $\underline{x}$ with respect to M is independent of $\underline{x}$, an arbitrary system of parameters consisting of forms of degree 1.

(2.6) Corollary. *Let M be as above, and let $x \in R_1$ be a filter-regular element. Then*

$$\operatorname{reg} M = \max \{\operatorname{reg}(M/xM), s(0_M : x)\} .$$

This is clear by view of (2.5). We will apply it to the case where $\underline{x} = \{x_1, \ldots, x_d\}$ is a filter-regular d-sequence, see [9] for the definition. It follows easily that there is an injection

$$0 \to 0_M : x_1 \to x_2 M : x_1 / x_2 M ,$$

see [9]. That is, $s(0_M : x_1) \leqslant s(x_2 M : x_1 / x_2 M)$.

(2.7) Corollary. *Let $\underline{x} = \{x_1, \ldots, x_d\}$, $d = \dim M$, be a filter-regular d-sequence of forms of degree* 1. *Then*

$$\operatorname{reg} M = r_d(\underline{x}; M) .$$

In particular, in the case of a Buchsbaum module M, the reduction exponent $r_d(\underline{x}; M)$ of a system of parameters $\underline{x}$ of forms of degree 1 *is independent of $\underline{x}$ and equal to* $\operatorname{reg} M$.

Proof. By the previous remark, we have

$$\operatorname{reg} M = \operatorname{reg}(M/x_1 M)$$

and induction proves the first part. Then the second claim follows because any system of parameters for a Buchsbaum module is a d-sequence.

(2.8) Corollary. *We have*

$$\operatorname{reg} M = \max \{s(\operatorname{Tor}_i^S(k, M)) - i : i \in \mathbf{Z}\} .$$

Proof. First $\operatorname{Tor}_i^S(k, M) \cong H_i(P; M)$ for all i. Secondly, note that

$$[H_i(P; M)]_n \cong [H^{e-i}(P; M)]_{n-e} \qquad \text{for all } i, n$$

as follows from properties of the Koszul (co-)complex. Then

$$s(H^i(P; M)) + i = s(H_{e-i}(P; M)) - (e - i)$$

and the desired formula follows by (2.5).

The previous corollary shows that the Castelnuovo index of regularity of M is determined as a degree bound of the syzygies occuring in a minimal free resolution of M as an S-module. Thus reasonable estimates of reg M lead to estimates for the degree of syzygies. This is an important note related to computational problems as done by D. Bayer and M. Stillman [1], where one can find also the equivalence of a) and d) of (2.5) shown by a different argument.

(2.9) Definition. A finitely generated graded R-module M has a t-linear resolution if $t := i(M) = \operatorname{reg} M$.

If $M \neq 0$ and $p = \operatorname{proj.dim} M$ ($\leqslant \dim S = e$ by Hilbert's syzygy theorem), one has immediately the inequalities

$$\operatorname{reg} M \geqslant s(H^n(P; M)) + n \geqslant i(H^n(P; M)) + n \geqslant i(M)$$

for all $n = p, \ldots, e$ and hence in the case of t-linear resolution

$$\operatorname{Tor}_i^S(k, M) \cong k^{b_i}(-t-i) \qquad \text{for all } i = 0, \ldots, p\,.$$

That is, M has a minimal free resolution as an S-module of the type

$$0 \to S^{b_p}(-t-p) \to \cdots \to S^{b_1}(-t-1) \to S^{b_0}(-t) \to M \to 0\,.$$

Finally note that

$$\operatorname{reg}(S/I) + 1 = \operatorname{reg}(I)$$

as easily seen.

3. Rings with finitely generated local cohomology

As before let $R = \bigoplus_{n \geqslant 0} R_n$ be a k-algebra of finite type such that $R_0 = k$ and R is generated by R_1 over k. As above we consider the local cohomology modules $H_P^i(R)$, $i \in \mathbf{Z}$. With L we denote the length of an R-module, $e(\underline{x}; R)$ denotes the multiplicity of a system of parameters $\underline{x}$ of R.

(3.1) Definition and Theorem. *The k-algebra R is called a ring with finitely generated local cohomology if R satisfies one of the following equivalent conditions:*

(i) *$H_P^i(R)$ is a finitely generated R-module for all $i <$ dim R.*

(ii) *There is an invariant $I(R)$ and an integer m such that*

$$L(R/\underline{x}R) - e(\underline{x}; R) \leqslant I(R)$$

for all systems of parameters $\underline{x} = \{x_1, \ldots, x_d\}$ with equality for all $\underline{x} \subseteq P^m$.

(iii) *R is equidimensional and $R_{\mathfrak{p}}$ is a Cohen-Macaulay ring for all relevant homogeneous prime ideals $\mathfrak{p}$ of R.*

The proof of (3.1) is given in [15], where the study of these generalized Cohen-Macaulay rings was started. Recall that

$$I(R) = \sum_{i=0}^{d-1} \binom{d-1}{i} L(H_P^i(R))\,, \qquad d = \dim R\,.$$

In accordance with [10] and [16] we put

$$\Lambda(R) = \sum_{i=0}^{d-1} \binom{d-1}{i} \lambda(H_P^i(R))$$

for rings R with finitely generated local cohomology. In [14] a system of parameters $\underline{x}$ for which the equality in (3.1) (ii) holds is called a standard system of parameters. On the other side, every system of parameters of a ring with finitely generated local cohomology is a filter-regular sequence. This follows easily from (iii) of (3.1). In particular, a Buchsbaum ring has finitely generated local cohomology.

Next we show several results concerning the behaviour of reg R by passing to R/xR.

(3.2) Lemma. *Let R denote a graded k-algebra and $x \in R_1$ a filter-regular element. Then it follows*

a) $\operatorname{reg} R \leqslant \operatorname{reg} R/xR + \lambda(H_P^0(R/0:x))$,

b) $\operatorname{reg} R \leqslant \operatorname{reg} R/xR + \lambda(H_P^0(R))$, *and*

c) $\operatorname{reg} R \leqslant \operatorname{reg} R/xR + L(H_P^0(R)) - L(0:x)$.

Proof. By the definition of λ, it is enough to show the first inequality. Because $x \in R_1$ is filter-regular, there are exact sequences

$$H_P^{i-1}(R/xR)_j \to H_P^i(R)_{j-1} \xrightarrow{x} H_P^i(R)_j \qquad \text{for } i \geqslant 1 \quad \text{and}$$

$$0 \to H_P^0(R)_{j-1}/[0:x]_{j-1} \xrightarrow{x} H_P^0(R)_j \to H_P^0(R/xR)_j\,,$$

see [14], (2.5). Let $r = \operatorname{reg} R/xR$. Then the first sequence shows

$$H_P^i(R)_j = 0 \qquad \text{for all } i, j \text{ with } i+j > r \text{ and } i \geqslant 1\,.$$

Now suppose $\operatorname{reg} R > r + l$, where $l = \lambda(H_P^0(R/0:x)) \geqslant 0$. Then $\operatorname{reg} R = s(H_P^0(R)) =: s$ taking into account the conclusion from the first sequence. Thus, there is an element $0 \neq m \in H_P^0(R)_s$. Because $s > r + l$, the second sequence provides an element $m_1 \in H_P^0(R)_{s-1}$ such that

$$m = x(m_1 + [0:x]_{s-1}) = xm_1\,.$$

Iterating this argument $l + 1$ times, we obtain an element $m_{l+1} \in H_P^0(R)_{s-l-1}$ such that

$$0 \neq m = x^{l+1} m_{l+1} \in xP^l([\mathrm{H}_P^0(R)]_{s-l-1}/[0:x]_{s-l-1}) = 0\,.$$

But this is a contradiction. Therefore, the desired inequality holds.

Another inequality concerning reg R is given in the following lemma. To this end we put $\bar{R} = R/H_P^0(R)$. Note that

$$\operatorname{reg} R = \max\{\operatorname{reg} \bar{R}, s(H_P^0(R))\}\,.$$

(3.3) Lemma. *There is the following bound*

$$\operatorname{reg} R \leqslant \operatorname{reg} \bar{R} + \lambda(H_P^0(R)).$$

Proof. Assume the contrary. Then

$$s > r + \lambda(H_P^0(R)),$$

where $s = s(H_P^0(R))$ and $r = \operatorname{reg} \bar{R}$. That means $P^{s-r-1} H_P^0(R) = 0$ by the definition of λ. Now we present $R = S/I$ and $\bar{R} = S/J$ for a homogeneous ideal J, i.e.,

$$H_P^0(R) \cong J/I.$$

Because $r = \operatorname{reg} \bar{R}$ we have $\operatorname{reg} J = r + 1$. By (2.8) this means that J is minimally generated by forms of degree $\leqslant r + 1$. Thus $P_{s-r-1} J_{r+1} = J_s$. On the other side $P_{s-r-1} J_{r+1} \subseteq I_s$ as follows from $s - r - 1 \geqslant \lambda(H_P^0(R))$. Combining both results, we have

$$J_s = P_{s-r-1} J_{r+1} \subseteq I_s,$$

i.e., there is no non-zero form of degree s in $H_P^0(R)$, a contradiction to $s = s(H_P^0(R))$.

(3.4) Theorem. *Let R be a graded k-algebra with finitely generated local cohomology and $d = \dim R$. For a system of parameters $\underline{x} \subseteq R_1$ of R it yields*

$$r_d(\underline{x}; R) \leqslant \operatorname{reg} R \leqslant r_d(\underline{x}; R) + I(R) - L(R/\underline{x}R) + e(\underline{x}; R).$$

In particular, for a standard system of parameters $\underline{x} \subseteq R_1$ we have $r_d(\underline{x}; R) = \operatorname{reg} R$.

Proof. First note that $r_d(\underline{x}; R) \leqslant \operatorname{reg} R$ follows by (2.5). We show the second claim by induction on d. In the case $d = 1$, (3.2) c) shows

$$\operatorname{reg} R \leqslant r_1(x; R) + I(R) - L(R/xR) + e(x; R)$$

because $L(0:x) = L(R/xR) - e(x; R)$, where $x = x_1$. Now let $d > 1$. By the induction hypothesis we get

$$\operatorname{reg} R/x_1 R \leqslant r_d(\underline{x}; R) + I(R/x_1 R) - L(R/\underline{x}R) + e(\underline{x}; R).$$

By (3.2) c), it follows

$$\operatorname{reg} R \leqslant \operatorname{reg} R/x_1 R + L(H_P^0(R/0:x_1)).$$

Now the inductive step is complete because

$$I(R/x_1 R) + L(H_P^0(R/0:x_1)) \leqslant I(R)$$

as follows from the long local cohomology sequence of

$$0 \to (R/0:x_1)(-1) \xrightarrow{x_1} R \to R/x_1 R \to 0$$

and the fact that x_1 is a filter-regular element.

Now we are going to prove one of our main results in this section. Part a) of (3.5) gives an affirmative answer to a problem posed by W. Vogel during his talk at the 5th National School in Algebra, Varna 1986, see also [10].

(3.5) Theorem. *Let k be an infinite field, let $R = S/I$ denote a graded k-algebra with finitely generated local cohomology, and let $c =$ height I $(= \dim_k R_1 - \dim R)$ and $t + 1 = i(I)$. Then it holds:*

a) $t \leqslant \operatorname{reg} R \leqslant e(R) - \binom{c+t}{c} + t + I(R)\,.$

b) *If R is in addition a Gorenstein ring with $c > 1$, then*

$$2t + 1 \leqslant \operatorname{reg} R\,, \qquad \textit{and}$$

$$\operatorname{reg} R \leqslant e(R) - 2\binom{c+t}{c} + 2(t+1)$$

provided $\operatorname{reg} R > 2t + 1$. *In the case* $2t + 1 = \operatorname{reg} R$, *it yields*

$$e(R) = \binom{c+t}{c} + \binom{c+t-1}{c}.$$

Proof. First let us treat the case a). Because

$$t + 1 = i(I) \leqslant \operatorname{reg} I = \operatorname{reg} R + 1$$

we get the lower bound. In order to prove the upper bound choose in R_1 a system of parameters $\underline{x} = \{x_1, \ldots, x_d\}$ such that $e(R) = e(\underline{x}; R)$. Put $r = r_d(\underline{x}; R)$. Then $R/\underline{x}R \cong k[Y_1, \ldots, Y_c]/J$ for an ideal J in the polynomial ring $k[Y_1, \ldots, Y_c]$ in c indeterminates. With $t' := i(J) - 1$, there is the following bound for the length

$$L(R/\underline{x}R) \geqslant \binom{c+t'}{c} + r - t'\,.$$

Note that $r = s(R/\underline{x}R)$. Now it is easy to see that $t' \geqslant t$ and

$$\binom{c+t'}{c} - t' \geqslant \binom{c+t}{c} - t\,.$$

Thus $L(R/\underline{x}R) \geqslant \binom{c+t}{c} + r - t$. Therefore, the claim follows by (3.4).

In order to prove b), choose a regular sequence $\underline{x} = \{x_1, \ldots, x_d\}$ of degree 1 such that $e(R) = e(\underline{x}; R) = L(R/\underline{x}R)$. By (3.4) we get $r := \operatorname{reg} R = r_d(\underline{x}; R)$. Since $\underline{x}$ is a regular sequenc, it follows easily $t = t'$, where t' is defined as in the proof of a). Because R is a Gorenstein ring the same holds for $B := R/\underline{x}R$. Then there is a perfect pairing

$$B_i \times B_{r-i} \to B_r \cong k\,, \qquad i = 0, \ldots, r\,.$$

Therefore it follows $\dim B_i = \dim B_{r-i}$ for $i = 0, \ldots, r$. Because of the self-duality of the finite free resolution of R over S we get the lower bound. Now let

$r > 2t + 1$. Then

$$L(R/\underline{x}R) \geqslant 2\binom{c+t}{c} + r - 2(t+1)\,,$$

which proves the upper bound for r. The case $r = 2t + 1$ is done by the same argument.

In [13] a Cohen-Macaulay (resp. Gorenstein) ring R is called extremal if $\operatorname{reg} R = t$ (resp. $\operatorname{reg} R = 2t + 1$). In this case R has a "simple" resolution over S.

(3.6) Corollary. *Let R be a Cohen-Macaulay ring. Then the following conditions are fulfilled*:

a) $t \leqslant \operatorname{reg} R \leqslant e(R) - \binom{c+t}{c} + t\,.$

b) $e(R) \geqslant \binom{c+t}{c}$ *and equality holds if and only if I has a $(t+1)$-linear resolution.*

Proof. a) is clear by (3.5) a) because $I(R) = 0$ for a Cohen-Macaulay ring R. The bound in b) is obvious by a). If equality holds, it follows from a) that $\operatorname{reg} I = t + 1 = i(I)$ and I has a $(t + 1)$-linear resolution. The converse comes from the fact that

$$\operatorname{reg} R/\underline{x}R = r_d(\underline{x};\, R) = t \qquad \text{and}$$

$$i(I) = i((I, \underline{x}R)/\underline{x}R)\,.$$

One may ask for the sharpening of (3.5) a) with $\Lambda(R)$ instead of $I(R)$. But this does not hold as follows from an example given by D. Lazard, see [16], Section 6, ex. 1.

4. Integral domains over an algebraically closed field of characteristic zero

In this section we want to discuss bounds for the Castelnuovo index of regularity for particular cases of rings. As an application of our ideas, we are able to simplify the proofs of some of the main results of the papers [16], [17], and [18]. First let us assume that R is an integral domain with $R_0 = k$ an algebraically closed field. These assumptions are necessary in order to apply H. Flenner's Bertini theorems, see [3], Satz 5.5. More concrete, if $\dim R \geqq 3$, then there is a generic form $x \in R_1$ such that

$$R' = R/xR : \langle P \rangle$$

is again an integral domain. (For a submodule N of an S-module M, $N : \langle P \rangle := \{m \in M \mid \exists i \in \mathbf{Z}\colon P^i m \subseteq N\}$.) Note that

$$R' = S/(I, xS) : \langle P \rangle\,,$$

where I is the defining prime ideal of R. That is, $(I, xS) : \langle P \rangle$ is again a prime ideal. As before, let

$$c = \dim_k R_1 - \dim R .$$

(4.1) Proposition. *With the above notation,*

$$\dim_k R_1 - \dim R = \dim_k R'_1 - \dim R' .$$

For the proof of (4.1) see Lemma 3 of [16] or [4], p. 174. Note that (4.1) does not remain true without additional assumptions on k and R. As in Section 3, we define $t = i(I) - 1$. Now we are going to sharpen part of (3.5) a) in the case of R a domain over an algebraically closed field k. In order to dispose the General Position Lemma, we assume k to be of characteristic zero.

(4.2) Theorem (see [16], Theorem 1). *Let $R = S/I$ denote a graded k-algebra with finitely generated local cohomology. If R is an integral domain with $c = \dim_k R_1 - \dim R \geqslant 1$ and k is an algebraically closed field of characteristic zero, then*

$$t \leqslant \operatorname{reg} R < (e(R) - 1)/c + 1 + \Lambda(R) \, (\leqslant e(R) - c + 1 + \Lambda(R)) .$$

Proof. First note that the lower bound follows from (2.8). Since we have excluded the trivial case $c = 0$ (in which R is isomorphic to a polynomial ring over k), we have $d := \dim R \geqslant 2$. We induct on d. Let $d = 2$. Then, by the General Position Lemma [5], (2.13)–(2.16), there is a generic form $x \in R_1$ such that $R' = R/xR : \langle P \rangle$ is the homogeneous coordinate ring of $e(R) \geqslant c + 1$ points in general position and $[H^1_P(R')]_n = 0$ for all integers n with $e(R) \leqslant cn + 1$. Note that R' is a one-dimensional Cohen-Macaulay ring (reduced but not a domain). Hence

$$\operatorname{reg} R' < (e(R) - 1)/c + 1 .$$

Doing the inductive step, we shall see that $\operatorname{reg} R \leqslant \operatorname{reg} R' + \Lambda(R)$. This shows the case $d = 2$.

Let $d \geqslant 3$. Then, by Flenner's Bertini theorem [3], Satz 5.5, we can choose an appropriate x, see beginning of Section 4, and apply the induction hypothesis to R'. Thus

$$\operatorname{reg} R' < (e(R') - 1)/c' + 1 + \Lambda(R') .$$

Now $e(R') = e(R)$ as follows from the choice of x. By (4.1) we have $c' = c$. By virtue of (3.3), it follows

$$\operatorname{reg} R = \operatorname{reg} R/xR \leqslant \operatorname{reg} R' + \lambda(H^0_P(R/xR)) .$$

Putting this together, it yields

$$\operatorname{reg} R < (e(R) - 1)/c + 1 + \Lambda(R') + \lambda(H^0_P(R/xR)) .$$

By the definition of Λ we have

$$\Lambda(R') + \lambda(H^0_P(R/xR)) = \Lambda(R/xR) .$$

The short exact sequence

$$0 \to R(-1) \xrightarrow{x} R \to R/xR \to 0$$

yields $\Lambda(R/xR) \leqslant \Lambda(R)$ as follows easily from the corresponding long exact sequence for the local cohomology modules. This finishes the proof.

In the case of a Buchsbaum ring R, clearly $\Lambda(R) \leqslant 2^{d-1}$, where $d = \dim R$. So there is the bound

$$\operatorname{reg} R < (e(R) - 1)/c + 1 + 2^{d-1} .$$

However, a sharper bound is true:

(4.3) Theorem (see [17], Theorem 1). *Let R be as above an integral domain over an algebraically closed field k of characteristic zero, and assume that $c = \dim_k R_1 - \dim R \geqslant 1$. If R is a Buchsbaum ring, then*

$$\operatorname{reg} R < (e(R) - 1)/c + 1 \quad (\leqslant e(R) - c + 1) .$$

Proof. If R has the lowest possible dimension $d = 2$, as in the proof of (4.2), the General Position Lemma yields an element $x \in R_1$ such that for $R' = R/xR : \langle P \rangle$ it holds

$$\operatorname{reg} R' < (e(R) - 1)/c + 1 .$$

Clearly $\operatorname{reg} R = \operatorname{reg} R/xR = \max \{\operatorname{reg} R', s(H_P^0(R/xR))\}$, and further

$$s(H_P^0(R/xR)) = s(H_P^1(R)) + 1 \leqslant s(H_P^1(R/xR)) + 1 \leqslant \operatorname{reg} R'$$

as follows from the local cohomology sequence

$$0 \to H_P^0(R/xR) \to H_P^1(R)(-1) \xrightarrow{x} H_P^1(R) \to H_P^1(R/xR)$$

using the fact that x is the zero map for a Buchsbaum ring R. Thus $\operatorname{reg} R' = \operatorname{reg} R$ and the case $d = 2$ is shown.

We proceed by induction on d. Let $d \geqslant 3$. Applying H. Flenner's Bertini theorem, see [3], Satz 5.5, we can choose an element $x \in R_1$ such that $R' = R/xR : \langle P \rangle$ is a domain with $e(R') = e(R)$. By (4.1) we have $c = c' := \dim_k R'_1 - \dim R'$. Now R' is a Buchsbaum ring of dimension $d - 1$. Thus, by the induction hypothesis,

$$\operatorname{reg} R' < (e(R) - 1)/c + 1 .$$

As above, $\operatorname{reg} R' = \operatorname{reg} R$ and the inductive step is complete.

By virtue of (4.2) and (3.5) one should ask for

$$\operatorname{reg} R \leqslant e(R) - \binom{c+t}{c} + t + \Lambda(R) ,$$

where R is a graded integral domain over an algebraically closed field k of characteristic zero. Here $t = i(I) - 1$, where I is the homogeneous prime ideal given by $R = S/I$. The following example shows that this does not hold.

(4.4) Example. In [8] L. Gruson and C. Peskine have shown that there is a reduced, irreducible curve $C \subset \mathbf{P}^3_{\mathbf{C}} =: P$ of degree 10 and genus 6 with the following resolution

$$0 \to \mathcal{O}_P^5(-7) \to \mathcal{O}_P^{15}(-6) \to \mathcal{O}_P^{11}(-5) \to \mathcal{O}_P \to \mathcal{O}_C \to 0 .$$

For the homogeneous coordinate ring R, it follows that $\operatorname{reg} R = 4$, use (2.8). Therefore $[H_P^1(R)]_n = 0$ for all $n \geqslant 4$ and all $n \leqslant 0$, i.e., $\Lambda(R) \leqslant 3$. Because $t = 4$ the requested bound does not hold.

Moreover, the example shows also that the lower bound $\binom{c+t}{c}$ for the multiplicity of R in the Cohen-Macaulay case is no longer true for R an integral domain over an algebraically closed field k of characteristic zero and $t > 1$. Recall that the case $t = 1$, i.e., $e(R) \geqslant c + 1$, is true for R an integral domain over an algebraically closed field k, see e.g. [2] or [4], p. 173.

In their paper [2], D. Eisenbud and S. Goto conjectured that

$$\operatorname{reg} R \leqslant e(R) - c$$

for a graded domain over an algebraically closed field k. This is shown to be true for R a Cohen-Macaulay domain. The Theorems (4.2) and (4.3) of J. Stückrad and W. Vogel contain results in this direction. In particular the Buchsbaum case is a good support for this conjecture.

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