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A note on the coefficients of Hilbert polynomial

By

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Let (Q, m) be a local ring of dimension d and q an m-primary ideal. It is well known that for sufficiently large values of n (notation: $n \gg 0$) $l_Q(Q/q^{n+1})$, the length of the Q-module Q/q^{n+1} is a uniquely determined polynomial of degree d, called the Hilbert polynomial and is given by

$$l_Q(Q/q^{n+1}) = e_0\binom{n+d}{d} - e_1\binom{n+d-1}{d-1} + \cdots + (-1)^d e_d$$

where the integers $e_0, e_1, \ldots, e_d$ depend on q and are known as normalized Hilbert coefficients. e_i is sometimes written as $e_i(q)$ to emphasize its dependence on q.

It is easily seen that e_0 is positive. In case Q is a Cohen-Macaulay ring, Northcott [2] showed that e_1 is non-negative and Narita [1] showed that in this case e_2 is non-negative as well. Narita gave an example of a Cohen-Macaulay ring and an m-primary ideal q such that $e_3(q)$ is negative.

The purpose of this note is to show that for any integer $d \geqslant 3$, it is possible to construct an example of a Cohen-Macaulay ring (Q, m) of dimension d and an m-primary ideal q such that $e_d(q)$ is negative.

We use the arguements given in [1] to obtain explicit values of the normalized Hilbert coefficients and use them subsequently to test our examples for the claim made above. To give a general treatment we must introduce some auxilliary notations which are explained at the appropriate place. Throughout this note (Q, m) denotes a Cohen-Macaulay ring with infinite residue field.

§1:

To start with we quote the following result

1.1 (Northcott [2]). *Let* $\dim Q = d$ *and* w *a superfical element of* q. *Suppose that* w *is not a zero divisor. Let* $\bar{Q} = Q/Qw$ *and* $\bar{q} = q/Qw$. *Then we have*
(i) $l_{\bar{Q}}(\bar{Q}/\bar{q}^{n+1}) = l_Q(q^{n+1} : w/q^{n+1})$ *for all* $n > 0$
(ii) $l_{\bar{Q}}(\bar{Q}/\bar{q}^{n+1}) = l_Q(Q/q^{n+1}) - 1_Q(Q/q^n)$ *for all* $n \gg 0$
(iii) $e_i(q) = e_i(\bar{q}),\quad 0 \le i \le d-1.$

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Let $\dim Q = d$ and q an m-primary ideal. Let $w_1, w_2, \ldots, w_d$ be a system of parameters in q. We denote by $Q_{(i)}$ the quotient ring of Q modulo $\sum_{j=1}^{i-1} Qw_j$. $q_{(i)}$ denotes the image of q in $Q_{(i)}$ and $w_{(i)}$ denotes the image of w_i in $Q_{(i)}$. For the sake of uniformity $Q_{(1)}$, $q_{(1)}$ and $w_{(1)}$ will denote Q, q and w_1 respectively.

1.2. Theorem. *Let (Q, m) be a Cohen-Macaulay ring of dimension d ($\geqslant 1$). Suppose q is an m-primary ideal and $w_1, w_2, \ldots, w_d$ a system of parameters in q such that $w_{(i)}$ is a superficial element of $q_{(i)}$. Then*

$$l_Q(Q/q^{n+1}) = l_Q(Q_{(d+1)})\binom{n+d}{d} - \sum_{i_{d-1}=0}^{n} \sum_{i_{d-2}=0}^{i_{d-1}} \cdots \sum_{i_1=0}^{i_2} \sum_{i=0}^{i_1} (A_i + B_{d,i})$$

$$- \sum_{i_{d-2}=0}^{n} \sum_{i_{d-3}=0}^{i_{d-2}} \cdots \sum_{i_1=0}^{i_2} \sum_{i=0}^{i_1} B_{d-1,i} - \cdots - \sum_{i=0}^{n} B_{1,i}$$

where $A_i = l_Q(Q/\sum_{k=1}^{d} Qw_k) - l_Q(Q/q^{i+1} + \sum_{k=1}^{d} Qw_k), \quad i \geqslant 0$

and $B_{j,i} = l_{Q_{(j)}}(q_{(j)}^{i+1} : w_{(j)}/q_{(j)}^{i}), \quad j = 1, 2, \ldots, d; \; i \geqslant 0$

Proof. The proof is by induction on d. We shall denote the terms A_i and $B_{j,i}$ for $Q_{(2)}$ by bars over the corresponding letters.

Suppose $d = 1$. Write w_1 as w. Since w is a superficial element which is not a zero divisor, we have

$$\begin{aligned} l_Q(Q/Qw + q^{i+1}) &= l_Q(Q/q^{i+1}) - l_Q(Qw + q^{i+1}/q^{i+1}) \\ &= l_Q(Q/q^{i+1}) - l_Q(Q/q^{i+1} : w) \\ &= l_Q(Q/q^{i+1}) - l_Q(Q/q^{i}) + l_Q(q^{i+1} : w/q^{i}) \end{aligned}$$

Thus

$$l_Q(Q/q^{i+1}) - l_Q(Q/q^{i}) = l_Q(Q/Qw + q^{i+1}) - B_{1,i}$$

Putting $i = 0, 1, \ldots, n$ and summing up the respective sides, we get

$$\begin{aligned} l_Q(Q/q^{n+1}) &= \sum_{i=0}^{n} l_Q(Q_{(2)}/q_{(2)}^{i+1}) - \sum_{i=0}^{n} B_{1,i} \qquad \cdots (1) \\ &= l_Q(Q_{(2)})\binom{n+1}{1} - \sum_{i=0}^{n} A_i - \sum_{i=0}^{n} B_{1,i} \end{aligned}$$

Assume the result holds when $\dim Q = d - 1$. Since $\dim(Q_{(2)}) = d - 1$, the given hypothesis ensures that

$$l_{Q_{(2)}}(Q_{(2)}/q_{(2)}^{n+1}) = l_{Q_{(2)}}((Q_{(2)})_{(d)})\binom{n+d-1}{d-1}$$

$$- \sum_{i_{d-2}=0}^{n} \sum_{i_{d-3}=0}^{i_{d-2}} \cdots \sum_{i_1=0}^{i_2} \sum_{i=0}^{i_1} (\bar{A}_i + \bar{B}_{d-1,i})$$

$$- \sum_{i_{d-3}=0}^{n} \sum_{i_{d-4}=0}^{i_{d-3}} \cdots \sum_{i_1=0}^{i_2} \sum_{i=0}^{i_1} \bar{B}_{d-2,i} - \ldots - \sum_{i=0}^{n} \bar{B}_{1,i}$$

Using (1) we get

$$l_Q(Q/q^{n+1}) = l_{Q_{(2)}}((Q_{(2)})_{(d)}) \left(\sum_{i=0}^{n} \binom{i+d-1}{d-1} \right) - \sum_{k=0}^{n} \sum_{i_{d-2}=0}^{k} \cdots \sum_{i_1=0}^{i_2} \sum_{i=0}^{i_1} (\bar{A}_i + \bar{B}_{d-1,i})$$

$$- \sum_{k=0}^{n} \sum_{i_{d-3}=0}^{k} \cdots \sum_{i_1=0}^{i_2} \sum_{i=0}^{i_1} \bar{B}_{d-2,i} - \cdots - \sum_{k=0}^{n} \sum_{i=0}^{k} \bar{B}_{1,i} - \sum_{i=0}^{n} B_{1,i}$$

Now observe that $\bar{B}_{j-1,i} = B_{j,i}$ etc. Consequently, the last expression readily produces the desired result.

1.3. Corollary. *With the same assumptions as in Theorem 1.2, we have the following*:

$$e_0 = l_Q(Q_{(d+1)})$$

$$e_1 = \sum_{i=0}^{\infty} (A_i + B_{d,i})$$

$$e_2 = \sum_{i_1=0}^{\infty} \sum_{i=i_1+1}^{\infty} (A_i + B_{d,i}) - \sum_{i=0}^{\infty} B_{d-1,i}$$

$$e_3 = \sum_{i_2=0}^{\infty} \sum_{i_1=i_2+1}^{\infty} \sum_{i=i_1+1}^{\infty} (A_i + B_{d,i}) - \sum_{i_1=0}^{\infty} \sum_{i=i_1+1}^{\infty} B_{d-1,i} + \sum_{i=0}^{\infty} B_{d-2,i}$$

$$\cdots$$

$$\cdots$$

$$e_d = \sum_{i_{d-1}=0}^{\infty} \sum_{i_{d-2}=i_{d-1}+1}^{\infty} \cdots \sum_{i_1=i_2+1}^{\infty} \sum_{i=i_1+1}^{\infty} (A_i + B_{d,i}) - \sum_{i_{d-2}=0}^{\infty} \cdots \sum_{i_1=i_2+1}^{\infty} \sum_{i=i_1+1}^{\infty} B_{d-1,i}$$

$$+ \cdots + (-1)^{d-1} \sum_{i=0}^{\infty} B_{1,i}$$

Proof. Assume that $d = 1$. Clearly, all but finitely many A_i and $B_{1,i}$ vanish. Taking n sufficiently large, we get

$$e_0 = l_Q(Q_{(2)}) \text{ and } e_1 = \sum_{i=0}^{\infty} (A_i + B_{1,i})$$

The corollary now follows on using the inductive process and the following identity:

If atmost finitely many real numbers a_i are nonzero then

$$\sum_{j=0}^{k} \sum_{i=0}^{j} a_i = \sum_{i=0}^{\infty} a_i(k+1) - \sum_{j=0}^{\infty} \sum_{i=j+1}^{\infty} a_i$$

1.4. Remarks. 1. Theorem 1.2 includes as a special case Proposition 6 of Narita [1].

2. When $d = 3$, $e_0 = l_Q(Q/\sum_{i=1}^{3} Qw_i)$ is also mentioned in Narita [1].

§2:

2.1. Example. Let $x_1, x_2, \ldots, x_d, x_{d+1}$ be $d+1$ indeterminates over an infinite field F. Let $Q = F[[x_1, x_2, \ldots, x_{d+1}]]/(x_{d+1}^d)$. Then Q is a local Cohen-Macaulay ring of dimension d. Assume $d \geqslant 3$. Let $A = \{x_1, x_2, \ldots, x_{d-2}, x_{d-1}^{d-1}, x_d^{d-1}\}$ and $B = \{x_{d-1}x_{d+1}, x_d x_{d+1}\}$. Put $C = A \cup B$. Further, suppose that for two sets U and V of monomials in the above indeterminates UV denotes the set of all products of monomials in U with all monomials in V. A simple calculation shows that

$$\underbrace{C.C. \cdots C}_{d\ times} = \underbrace{C.C. \cdots C}_{d-1\ times}.\ A \cup T$$

where T is a set of monomials in which x_{d+1} occurs in degree d.

Let $\xi_1, \xi_2, \ldots, \xi_d, \xi_{d+1}$ denote respectively the images of $x_1, x_2, \ldots, x_d, x_{d+1}$ modulo x_{d+1}^d. Let

$$q = (\xi_1, \xi_2, \ldots, \xi_{d-2}, \xi_{d-1}^{d-1}, \xi_d^{d-1}, \xi_{d-1}\,\xi_{d+1}, \xi_d\,\xi_{d+1}).$$

As argued in the previous paragraph, it is immediate that

$$q^d = q^{d-1}(\xi_1, \xi_2, \ldots, \xi_{d-2}, \xi_{d-1}^{d-1}, \xi_d^{d-1}) \qquad \cdots (*)$$

We claim that e_d is negative. It is clear that

$$(q_{(j)}^{t+1} : \xi_{(j)}) = q_{(j)}^t \quad \text{for all } t \geqslant 0 \text{ and } j = 1, 2, \ldots, d-2.$$

Again it is easily seen that

$$(q_{(d-1)}^{d-1} : \xi_{(d-1)}^{d-1}) = q_{(d-1)}^{d-2} + Q_{(d-1)}\,\bar{\xi}_{d+1}^{d-1} \quad \text{and} \quad (q_{(d)}^{d-1} : \xi_{(d)}^{d-1}) = q_{(d)}^{d-2} + Q_{(d)}\,\bar{\bar{\xi}}_{d+1}^{d-1}$$

where $\bar{\xi}_{d+1}^{d-1}$ denotes the image of ξ_{d+1}^{d-1} in $Q_{(d-1)}$ and $\bar{\bar{\xi}}_{d+1}^{d-1}$ denotes the image of ξ_{d+1}^{d-1} in $Q_{(d)}$. Further, using (*) it is seen that

$$(q_{(d-1)}^{t+1} : \xi_{(d-1)}^{d-1}) = q_{(d-1)}^t \quad \text{and} \quad (q_{(d)}^{t+1} : \xi_{(d)}^{d-1}) = q_{(d)}^t \quad \text{for} \quad t \geqslant d-1.$$

Thus we find that

$$B_{1,i} = B_{2,i} = \cdots = B_{d-2,i} = 0 \text{ for all } i \geqslant 0 \text{ and } B_{d-1,d-2} = B_{d,d-2} \neq 0.$$

Now substituting these values in the expression for e_d as deduced in Corollary 1.3, we find that

$$e_d(q) = -\,B_{d-1,d-2} < 0.$$

2.2. Remark. Using the same construction as above it is possible to give another class of examples by replacing $\xi_1, \xi_2, \ldots, \xi_{d-2}$ by powers of the respective elements in q.

2.3. Remark. With reference to the above example, computations show that for $d \geqslant 3$

(i) $l_Q(Q_{(d+1)}) = d^3 - 2d^2 + 2d$

(ii) $A_0 = d^3 - 3d^2 + 3d$

(iii) $A_i = (d - i - 1)[(d - 1)(d - i - 3) + (i + 2)d - (i + 2)(i + 3)/2]$
for $1 \leq i \leq d - 2$

(iv) $A_i = 0$ for $i \geqslant d - 1$

(v) $B_{j,i} = 0$ for $j \leqslant d - 2$ and $i \geqslant 0$

(vi) $B_{d,0} = B_{d-1,0} = 0$ and

(vii) $B_{d,i} = B_{d-1,i} = d - 2$ for $1 \leq i \leq d - 2$

These values can be used to calculate the various Hilbert coefficients in respect of Example 2.1.

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References

[1] M. Narita, A note on the coefficients of Hilbert characteristic functions in semi-regular local rings, Proc. Camb. Phil. Soc., **59** (1963), 269–275.

[2] D. G. Northcott, The coefficients of the abstract Hilbert functions, J. London Math. Soc., **35** (1960), 209–214.