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# On the product space $D([0, 1]: \mathbf{R}^d) \times D([0, 1]: \mathbf{R}^d)$

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## § 1. Introduction

Let $D^d = D([0, 1]: \mathbf{R}^d)$ be the space of $\mathbf{R}^d$-valued functions on [0, 1] that are *càdlàg* (right-continuous with left-limits). We endow this space with Skorohod's $J_1$-topology: i.e., let $\Lambda$ denote the class of strictly increasing, continuous mapping of [0, 1] onto itself. Then elements $x_n$ of $D^d$ converge to a limit $x$ in $D^d$ if and only if there exist functions $\lambda_n$ in $\Lambda$ such that

$$\lim_{n \to \infty} \sup_t \{|x_n \circ \lambda_n(t) - x(t)| + |\lambda_n(t) - t|\} = 0.$$

Now let us consider the product topological space $D^d \times D^d = (D^d, J_1) \times (D^d, J_1)$. In the set theoretical sense $D^d \times D^d$ may clearly be identified with $D^{2d}$ in the obvious manner, whereas these two spaces are endowed with destinct topologies. Indeed the convergence of $z_n = (x_n^{(1)}, x_n^{(2)}) \in D^d \times D^d$ to $z = (x^{(1)}, x^{(2)}) \in D^d \times D^d$ in the product topology is, by definition, equivalent to that of each component; i.e., for each $i = 1, 2$, there exist functions $\lambda_n^{(i)}$ in $\Lambda$ such that

$$\lim_{n \to \infty} \sup_t \{|x_n^{(i)} \circ \lambda_n^{(i)}(t) - x^{(i)}(t)| + |\lambda_n^{(i)}(t) - t|\} = 0.$$

Here it is important that $\lambda_n^{(i)}$ may depend on $i$. On the other hand, $J_1$-convergence in $D^{2d}$ requires that

$$\lim_{n \to \infty} \sup_t \{|x_n^{(i)} \circ \lambda_n(t) - x^{(i)}(t)| + |\lambda_n(t) - t|\} = 0, \ i = 1, 2,$$

for a *common* sequence $\{\lambda_n\}_n \subset \Lambda$. Thus the product topology is weaker than $J_1$. We refer to [4] and [5] for details (see also p. 293 of [3]) but for our later use we remark that $(x, y) \in D^2 \mapsto x + y \in D^1$ is continuous in $J_1$ but not in the product topology of $D^1 \times D^1$.

The above fact often causes difficulties in proving the tightness of stochastic processes. (We say that stochastic processes are tight if and only if their laws are tight.) Tightness of a sequence of $\mathbf{R}^d$-valued stochastic processes with continuous paths is clearly equivalent to that of each component. In the case of càdlàg stochastic processes, however, $\{(X_n, Y_n)\}$ is not necessarily tight in $D^2$ even if both

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$\{X_n\}$ and $\{Y_n\}$ are tight in $D^1$. Similarly, the tightness of $\{X_n + Y_n\}$ follows from that of $\{(X_n, Y_n)\}$ in $D^2$ but not from that of $\{X_n\}$ and $\{Y_n\}$. Therefore, it might be significant to find supplementary conditions under which the convergence (or relative compactness) in $D^d \times D^d$ implies that in $(D^{2d}, J_1)$. On this subject we know the following result.

**Theorem A** (Holley-Stroock). *Let $x_n$, $y_n$, $x$ and $y$ be elements of $D^d$. Then*

$$(J_1) \qquad (x_n, y_n) \longrightarrow (x, y) \text{ in } (D^{2d}, J_1)$$

*holds if and only if*

$$(1.1) \qquad ax_n + by_n \longrightarrow ax + by \text{ in } (D^d, J_1), \quad \text{for all } a, b \in \mathbf{R}.$$

This theorem is an immediate consequence of Lemma (A. 28) on page 391 of Holley-Stroock [2], and if we read the proof carefully we also have

**Theorem A′**. *Condition $(J_1)$ holds if and only if*

$$x_n \longrightarrow x, \quad y_n \longrightarrow y \text{ and } x_n + y_n \longrightarrow x + y \quad \text{in } D^d.$$

Therefore, under condition

$$(J_{pr}) \qquad (x_n, y_n) \longrightarrow (x, y) \quad \text{in } (D^d, J_1) \times (D^d, J_1),$$

a necessary and sufficient condition for $(J_1)$ in Theorem A is

$$(J_{sm}) \qquad x_n + y_n \longrightarrow x + y \text{ in } (D^d, J_1).$$

In some cases where the choice of the coordinate plays no essential role, condition $(J_{sm})$ (or (1.1)) is usually easier to check than $(J_1)$. However, in other cases $(J_{sm})$ is not always easier than $(J_1)$ itself and in fact we often try to prove $(J_1)$ in order to verify $(J_{sm})$ (see [4] for example). Therefore, it is of interest to study when $(J_{sm})$ (or $(J_1)$) holds under condition $(J_{pr})$. Throughout we shall denote $z(t) - z(t-0)$ by $\Delta z(t)$.

**Theorem B** (*W. Whitt* [5]). *Under condition $(J_{pr})$, a sufficient condition for $(J_{sm})$ (and hence for $(J_1)$) is*

$$(W) \qquad |\Delta x(t)| \cdot |\Delta y(t)| = 0, \quad \text{for all } t \in (0, 1).$$

Although this theorem is useful in various cases, condition $(W)$ is by no means necessary for $(J_{sm})$ and is sometimes too restrictive; for example, Theorem B is not applicable even in the simplest case where the limit process is an isotropic stable Lévy process. Therefore it might be natural to ask what we should assume for those $t$'s such that $|\Delta x(t)| \cdot |\Delta y(t)| > 0$. The aim of the present paper is to relax Whitt's condition $(W)$ so that it becomes not only sufficient but also necessary for $(J_1)$ given $(J_{pr})$. (Another elementary criterion is given on page 302 of [3].) We shall also give a few sufficient conditions which are useful in some problems where Whitt's theorem is not applicable.

## §2. Main results

Let $a \wedge b$ denote $\min\{a, b\}$.

**Theorem 1.** *Under condition* $(J_{pr})$, $(J_1)$ *holds if and only if the following condition is satisfied.*
(A) *For every* $t \in (0, 1)$ *and real numbers* $t_n, s_n \in (0, 1)$ *both converging to* $t$,

$$\lim_{n\to\infty} |\Delta x_n(s_n)| \wedge |\Delta y_n(t_n)| = 0 \tag{2.1}$$

*provided that* $t_n \neq s_n$ *for all* $n \geq 1$.

**Remark.** (2.1) holds automatically if $t$ is a continuity point of $x$ or $y$. Indeed, under $(J_{pr})$, we easily have

$$\limsup_{n\to\infty} |\Delta x_n(s_n)| \leq |\Delta x(t)|.$$

Since a similar inequality holds for $y_n$, we obtain

$$\limsup_{n\to\infty} |\Delta x_n(s_n)| \wedge |\Delta y_n(t_n)| \leq |\Delta x(t)| \wedge |\Delta y(t)|.$$

Whitt's condition $(W)$ is therefore a sufficient condition for (A).

We shall postpone the proof until the next section and give a sufficient condition for (A).

**Theorem 2.** *Under condition* $(J_{pr})$, *a sufficient condition for* $(J_1)$ *is*
(B) *There exist nonnegative numbers* $a_n$ $(\to 0$ *as* $n \to \infty)$ *and a nondecreasing function* $h\colon [0, \infty) \to [0, \infty)$ $(h(+0) = 0)$ *such that*

$$|\Delta x_n(t)| \wedge 1 \leq h(|\Delta y_n(t)|) + a_n, \quad \textit{for all} \quad t \in (0, 1).$$

*Proof.* Suppose $s_n \neq t_n$ $(n \geq 1)$ and $s_n, t_n \to t$. As we shall see in the next section (Lemma 1 (i)), $y_n \to y$ implies

$$\limsup_{n\to\infty} |\Delta y_n(s_n)| \wedge |\Delta y_n(t_n)| = 0. \tag{2.2}$$

Now if (B) holds, we obtain

$$\limsup_{n\to\infty} |\Delta x_n(s_n)| \wedge |\Delta y_n(t_n)| \wedge 1 \leq \limsup_{n\to\infty} h(|\Delta y_n(s_n)|) \wedge |\Delta y_n(t_n)|.$$

The right-hand side vanishes by (2.2) and the assumption $h(+0) = 0$ ($h(0) = 0$ by monotonicity). Thus, under condition $(J_{pr})$, we see that (B) implies (A) (and hence $(J_1)$ by Theorem 1).

**Corollay.** *If* $x_n + y_n$ *is continuous for all* $n \geq 1$, *then* $(J_{pr})$ *implies* $(J_1)$ *(and hence* $(J_{sm})$*).*

*Proof.* Since $|\Delta x_n(t)| = |\Delta y_n(t)|$, condition (B) is satisfied with $a_n = 0$ and $h(u) = u$.

**Theorem 3.** *Suppose $x_n^{(k)} \to x^{(k)}$ in $(D^d, J_1)$ for $k = 1, \ldots, p$. If there exist nonnegative numbers $a_n$ ($\to 0$ as $n \to \infty$), a nondecreasing function $h: [0, \infty) \to [0, \infty)$ ($h(+0) = 0$), an integer $q \geq 1$ and a convergent sequence $\{y_n\}_n$ in $(D^q, J_1)$ such that*

$$\max_k |\Delta x_n^{(k)}(t)| \wedge 1 \leq h(|\Delta y_n(t)|) + a_n, \quad \textit{for all} \quad t \in (0, 1),$$

*then $(x_n^{(1)}, \cdots, x_n^{(p)}) \to (x^{(1)}, \cdots, x^{(p)})$ in $(D^{pd}, J_1)$.*

*Proof.* By mathematical induction on $r(\geq 1)$ it follows from Theorem 2 that

$$(x_n^{(1)}, \cdots, x_n^{(r)}, y_n) \longrightarrow (x^{(1)}, \cdots, x^{(r)}, y) \text{ in } (D^{rd+q}, J_1),$$

for every $r$ $(1 \leq r \leq p)$, which proves the assertion.

## §3. Proof of Theorem 1

We first prepare

**Lemma 1.** *Suppose $z_n \to z$ in $D^d$. Let $c \in [0, 1]$ and let $0 \leq a_n < b_n \leq 1$ be real numbers both converging to $c$. Then;*

(i) $$\limsup_{n \to \infty} |\Delta z_n(a_n)| \wedge |\Delta z_n(b_n)| = 0;$$

(ii) *There exist numbers $u_n \in (a_n, b_n]$ such that*

$$\lim_{n \to \infty} |(z_n(b_n) - z_n(a_n)) - \Delta z_n(u_n)| = 0.$$

*Proof.* Take $\lambda_n \in \Lambda$ so that $\sup_t \{|z_n \circ \lambda_n(t) - z(t)| + |\lambda_n(t) - t|\} \to 0$, and let $a_n' = \lambda_n^{-1}(a_n)$, $b_n' = \lambda_n^{-1}(b_n)$. Then it is easy to see that the left-hand side of (i) is equal to

$$\delta := \limsup_{n \to \infty} |\Delta z(a_n')| \wedge |\Delta z(b_n')|.$$

If $\delta > 0$ then $c$ is an accumulating point of $D_\delta := \{u: |\Delta z(u)| > \delta/2\}$, which is a contradiction because $D_\delta$ is a finite set. Thus we conclude that $\delta = 0$, proving (i). For the proof of (ii) we define $\{u_n'\}$ as follows; for $n$ such that $c \in (a_n', b_n']$ we put $u_n' = c$ and for other $n$'s let $u_n'$ be any continuity point in $(a_n', b_n']$ of $z(\cdot)$. Now let $u_n = \lambda_n(u_n')$. Since $a_n' < u_n' \leq b_n'$, it holds $a_n < u_n \leq b_n$. Keeping $\sup_t |z_n \circ \lambda_n(t) - z(t)| \to 0$ in mind, we have

$$\limsup_{n\to\infty} |(z_n(b_n) - z_n(a_n)) - \Delta z_n(u_n)|$$

$$= \limsup_{n\to\infty} |(z_n \circ \lambda_n(b'_n) - z_n \circ \lambda_n(a'_n)) - \Delta z_n \circ \lambda_n(u'_n)|$$

$$= \limsup_{n\to\infty} |(z(b'_n) - z(a'_n)) - \Delta z(u'_n)|.$$

Now it is easy to see that this vanishes by the definition of $u'_n$.

*Proof of Theorem* 1. If $(J_1)$ holds, then we have from Lemma 1 (i) that, for every $t \in [0, 1]$,

$$\limsup_{\substack{s_n, t_n \to t \\ s_n \neq t_n}} |\Delta(x_n, y_n)(s_n)| \wedge |\Delta(x_n, y_n)(t_n)| = 0.$$

Therefore, $(J_1)$ implies (A). Conversely, suppose $(J_{pr})$ holds but $(J_1)$ does not. This means that $\{(x_n, y_n)\}_n$ does not have compact closure in $D^{2d}$. By a well-known result (see Theorem 14.4 of [1]) we have, denoting $(x_n, y_n)$ by $w_n$,

$$\lim_{\delta \downarrow 0} \limsup_{n\to\infty} \sup_{\substack{0<s<t<u<1 \\ u-s<\delta}} |w_n(s) - w_n(t)| \wedge |w_n(t) - w_n(u)| > 0, \tag{3.1}$$

because other supplementary conditions for relative compactness are satisfied under condition $(J_{pr})$. Since

$$(a + b) \wedge (c + d) \leq a \wedge c + b \wedge c + a \wedge d + b \wedge d \quad (a, b, c, d \geq 0),$$

(3.1) implies that at least one of the following four inequalities holds.

$$\lim_{\delta \downarrow 0} \limsup_{n\to\infty} \sup_{\substack{0<s<t<u<1 \\ u-s<\delta}} |x_n(s) - x_n(t)| \wedge |x_n(t) - x_n(u)| > 0;$$

$$\lim_{\delta \downarrow 0} \limsup_{n\to\infty} \sup_{\substack{0<s<t<u<1 \\ u-s<\delta}} |y_n(s) - y_n(t)| \wedge |y_n(t) - y_n(u)| > 0;$$

$$\lim_{\delta \downarrow 0} \limsup_{n\to\infty} \sup_{\substack{0<s<t<u<1 \\ u-s<\delta}} |x_n(s) - x_n(t)| \wedge |y_n(t) - y_n(u)| > 0; \tag{3.2}$$

$$\lim_{\delta \downarrow 0} \limsup_{n\to\infty} \sup_{\substack{0<s<t<u<1 \\ u-s<\delta}} |y_n(s) - y_n(t)| \wedge |x_n(t) - x_n(u)| > 0.$$

Since the first two cases are impossible under condition $(J_{pr})$ and the last case may be reduced to (3.2) by exchanging the role of $x_n$ and $y_n$, we shall consider the case (3.2) only. We can choose real numbers $a_n$, $b_n$, $c_n$ $(0 < a_n < b_n < c_n < 1,\ c_n - a_n \to 0)$ so that

$$\limsup_{n\to\infty} |x_n(a_n) - x_n(b_n)| \wedge |y_n(b_n) - y_n(c_n)| > 0. \tag{3.3}$$

Choosing a subsequence, if necessary, we may and do assume that $a_n$ converges to a $c \in [0, 1]$. In fact we have $0 < c < 1$ because, if $c = 0$ or 1, it follows from Theorem 14.4 of [1] that

$$\limsup_{n\to\infty} |x_n(a_n) - x_n(b_n)| = 0,$$

which contradicts (3.3). By Lemma 1, we can choose $s_n \in (a_n, b_n]$ and $t_n \in (b_n, c_n]$ so that

$$\lim_{n\to\infty} |(x_n(b_n) - x_n(a_n)) - \Delta x_n(s_n)| = 0$$

and

$$\lim_{n\to\infty} |(y_n(c_n) - y_n(b_n)) - \Delta y_n(t_n)| = 0.$$

Combining these two with (3.3) we obtain

$$\limsup_{n\to\infty} |\Delta x_n(s_n)| \wedge |\Delta y_n(t_n)| > 0.$$

Thus if $(J_1)$ fails then so does (A), which completes the proof of Theorem 1.

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