J. Math. Kyoto Univ. (JMKYAZ)
30-3 (1990) 451–458

# On the family of holomorphic mappings into projective space with lacunary hypersurfaces

By

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## Introduction

After Borel's discovery of a generalization of Picard's theorem to the case of holomorphic mappings of the complex plane $\mathbf{C}$ into projective space $\mathbf{P}^n$ with lacunary hyperplanes, Bloch [1] and Cartan [4] studied families of holomorphic mappings of a disk into $\mathbf{P}^n$ with lacunary hyperplanes. After a half century, the works of Bloch and Cartan were taken up again by Kiernan-Kobayashi [6] from the view point of hyperbolic geometry and reformulated as follows: if $H$ is the union of $n+2$ hyperplanes in general position in $\mathbf{P}^n$, then $\mathbf{P}^n \backslash H$ is *tautly imbedded modulo* $\Delta$ (= diagonal hyperplanes) in $\mathbf{P}^n$, namely, for any sequence of holomorphic mappings $\{f_m\}_{m=1,2,\dots}$ of a polydisk $D^k$ to $\mathbf{P}^n \backslash H$, either it has a subsequence which converges in $\mathrm{Hol}(D^k, \mathbf{P}^n)$, or the sequence of the image $f_m(K)$ converges to $\Delta$ for any compact set $K$ of $D^k$, where $\mathrm{Hol}(D^k, \mathbf{P}^n)$ is the space of holomorphic mappings of $D^k$ to $\mathbf{P}^n$ with compact open topology.

On the other hand, Nishino [9] generalized the theorem of Picard-Borel to the case of holomorphic mappings of $\mathbf{C}^k$ to $\mathbf{P}^n \backslash A$, where $A$ is a hypersurface with $n+2$ distinct irreducible components.

In the present paper, we shall first introduce the notion of cluster sets for sequences of holomorphic mappings and apply it to the study of the behavior of sequences of holomorphic mappings of $D^k$ to $\mathbf{P}^n \backslash A$, $A$ being the same as Nishino's case above. In particular, we shall examine in detail the case $n=2$. Our results consist of the following three parts. ("Hypersurfaces" or "curves" below are all algebraic ones over $\mathbf{C}$.)

1°. **Theorem 1.** *Let $A$ be a hypersurface of $\mathbf{P}^n$ with $\ell$ ($\ell \geqq n+2$) distinct irreducible components such that the rank of $(\mathbf{P}^n, A)$ is $n$ (see Definition 2 below). Then $\mathbf{P}^n \backslash A$ is tautly imbedded modulo some algebraic subset $B$ in $\mathbf{P}^n$.*

2°. Let $A$ be a hypersurface (curve) of $\mathbf{P}^2$. An irreducible curve $C \not\subset A$ will be called a *nonhyperbolic curve* with respect to $A$ if the normalization of $C \backslash A$ is isomorphic to $\mathbf{C}$ or $\mathbf{C}^* = \mathbf{C} \backslash \{0\}$. If $C$ is an irreducible component of $A$, we shall say that $C$ is nonhyperbolic with respect to $A$ if the normalization of $C \backslash A'$ is

Communicated by Prof. Y.Kusunoki, October 1, 1988

isomorphic to $\mathbf{C}$, $\mathbf{C}^*$, $\mathbf{P}$ or an elliptic curve, where $A'$ is the union of the components of $A$ except $C$.

**Theorem 2.** *Let $A$ be a curve with $\ell$ ($\ell \geqq 4$) distinct irreducible components in $\mathbf{P}^2$. Suppose that the number of nonhyperbolic curves in $\mathbf{P}^2$ with respect to $A$ is finite. Then, there is a curve $S$ in $\mathbf{P}^2$ such that $\mathbf{P}^2 \backslash A$ is tautly imbedded modulo $S$ in $\mathbf{P}^2$. Here we may take $S = \phi$ if there is no nonhyperbolic curve in $\mathbf{P}^2$ with respect to $A$.*

3°. Let $M$ be a complex analytic manifold of arbitrary dimension. For $p, q \in M$, we denote the Kobayashi pseudodistance from $p$ to $q$ by $d_M(p, q)$ (see [8] for its definition and basic properties). We shall say that $p \in M$ is a *hyperbolic* (resp. *nonhyperbolic*) *point* of $M$ when $d_M(p, q) > 0$ for every $q \neq p$ (resp. $d_M(p, q) = 0$ for some $q \neq p$). We can deduce from Theorem 2 the following

**Theorem3.** *Let $A$ be a curve with $\ell$ ($\ell \geqq 4$) distinct irreducible components in $\mathbf{P}^2$.*

(1) *If the number of the nonhyperbolic curves in $\mathbf{P}^2$ with respect to $A$ is finite, then the set of the nonhyperbolic points of $\mathbf{P}^2 \backslash A$ is contained in some curve.*

(2) *If the number of the nonhyperbolic curves in $\mathbf{P}^2$ with respect to $A$ is infinite, then there exists a regular rational function $f$ on $\mathbf{P}^2 \backslash A$ such that all the irreducible components of the level curves $f^{-1}(a)$ ($a \in \mathbf{P}$) are isomorphic to either $\mathbf{C}$ or $\mathbf{C}^*$, consequently $\mathbf{P}^2 \backslash A$ has no hyperbolic point.*

We would like to mention here that the curves $A$ in $\mathbf{P}^2$ such that $\mathbf{P}^2 \backslash A$ have a regular rational function $f$ of $\mathbf{C}$ or $\mathbf{C}^*$-type such as the case (2) of Theorem 3 above are all determined by Kashiwara [5] and Kizuka [7]. From their results and our Theorem 3, we have the following

**Corollary.** *Let $A$ be a curve with $\ell$ ($\ell \geqq 4$) distinct irreducible components in $\mathbf{P}^2$.*

(1) *If at least one irreducible component of $A$ is of genus $\geqq 1$, the set of nonhyperbolic points of $\mathbf{P}^2 \backslash A$ is contained in some curve.*

(2) *If at least two irreducible components of $A$ are hyperbolic with respect to $A$, the set of the nonhyperbolic points of $\mathbf{P}^2 \backslash A$ is contained in some curve.*

## 1. Cluster sets

Let $D$ be a domain of $\mathbf{C}^k$ and $M$ be a complex analytic manifold of arbitrary dimension. Let $F = \{f_m\}_{m=1,2,\ldots}$ be a sequence of holomorphic mappings of $D$ to $M$.

**Definition 1.** We define the *cluster set* $F(a; M)$ of $F$ at a point a of $D$ by

$$F(a; M) = \bigcap_{\varepsilon>0} \bigcap_{N=1}^{\infty} \overline{\bigcap_{m \geqq N} f_m(U_\varepsilon(a))}$$

where $U_\varepsilon(a)=\{z\in D;\ \|z-a\|<\varepsilon\}$.

Clearly, we have

**Proposition 1.** *If $F(a;M)$ consists of a single point of $M$, there is a neighborhood $U$ of $a$ such that $\{f_m\}$ has a convergent subsequence in* $\mathrm{Hol}(U, M)$.

Let $S(M)$ be the set of the nonhyperbolic points of $M$. When $M$ is compact and $S(M)\neq\phi$, it is proved by Brody [3] that there exists a nonconstant holomorphic mapping $h$ of $\mathbf{C}$ to $M$ such that $h(\mathbf{C})\subset S(M)$. We can get a generalization of Brody's theorem as follows.

**Proposition 2.** *Suppose that $M$ is compact and $F$ is not normal at a point $a\in D$, then there exists a nonconstant holomorphic mapping $h$ of $\mathbf{C}$ to $M$ such that $h(\mathbf{C})\subset F(a;M)$.*

(We say "$F$ is *normal* at $a\in D$" in case $M$ is compact if there exists a neighborhood $U$ of $a$ such that every subsequence of $F$ has a convergent subsequence in $\mathrm{Hol}(U, M)$.)

*Proof.* We may assume $a=o$ (origin). Let $V$ be any open neighborhood of $F(o;M)$. For each $\rho>0$, we set $U_\rho=\{z\in\mathbf{C}^k;\ \|z\|<\rho\}$. We can take $\rho>0$ and an integer $N$ such that $f_m(U_{2\rho})\subset V$ for every $m\geqq N$. Since $\{f_m\}$ is not equicontinuous in $U_\rho$, there exist a sequence of points $p_\nu (\nu=1,2,\dots)$ of $U_\rho$, a sequence of tangent vectors $v_\nu\in T_{p_\nu}(U_\rho)$ at $p_\nu$ respectively and a subsequence $\{f_{m_\nu}\}\subset F$ such that $\dfrac{\|f_{m_\nu *}v_\nu\|}{\|v_\nu\|}\nearrow\infty\ (\nu\to\infty)$, where $\|f_{m_\nu *}v_\nu\|$ is the norm of tangent vector $f_{m_\nu *}v_\nu$ on $M$ with respect to a Hermitian metric on $M$ and $\|v_\nu\|$ is the norm of $v_\nu$ with respect to Euclidean metric on $\mathbf{C}^k$. Considering the restriction of $f_{m_\nu}$ to the complex line which passes through $p_\nu$ and includes $v_\nu$, we get a sequence $\{g_\nu\}$ of $\mathrm{Hol}(\Delta, M)$ such that $g_\nu(\Delta)\subset f_{m_\nu}(U_{2\rho})$ and $|g'_\nu(0)|\nearrow\infty\ (\nu\to\infty)$, where $\Delta=\{x\in\mathbf{C};\ |x|<\rho\}$ and $|g'_\nu(0)|=\left\|g_{\nu *}\left(\dfrac{\partial}{\partial x}\right)_{x=0}\right\|$. By Brody's method we can find $h_\nu\in\mathrm{Hol}(\Delta_{R_\nu}, V)$ such that $h_\nu(\Delta_{R_\nu})\subset g_\nu(\Delta)\subset V$, $\lim_{\nu\to\infty}R_\nu=\infty$, $|h'_\nu(0)|=1$ and

$$\sup_{x\in\Delta_{R_\nu}}|h'_\nu(x)|\times\left(\frac{R_\nu^2-|x|^2}{R_\nu^2}\right)\leqq 1,$$

where $\Delta_{R_\nu}=\{x\in\mathbf{C};\ |x|<R_\nu\}$.

Now, let us consider a sequence of open sets $\{V_k\}_{k=1,2,\dots}$ such that

$$(*)\qquad V_1\supset\supset V_2\supset\supset\cdots\supset\supset V_k\supset\supset\cdots\longrightarrow F(o,M).$$

Then for each integer $k>0$, there exists, according to the above discussion, $h_k\in\mathrm{Hol}(\Delta_{R_k}, V_k)$ such that $R_k>k$, $|h'_k(0)|=1$ and

$$\sup_{x\in\Delta_{R_k}}|h'_k(x)|\times\left(\frac{R_k^2-|x|^2}{R_k^2}\right)\leqq 1.$$

Since $\{h_k\}_{k=1,2,\dots}$ is equicontinuous on $|x| < R$ for every $R > 0$, it has, taking account of (*), a subsequence which converges to $h \in \mathrm{Hol}(\mathbf{C}, M)$ with $|h'(0)| = 1$ and $h(\mathbf{C}) \subset F(o; M)$. Q.E.D.

**Corollary.** *Let $D$ be a domain of $\mathbf{C}^k$, $A$ a curve in $\mathbf{P}^2$ and $F = \{f_m\}_{m=1,2,\dots}$ a sequence of holomorphic mappings of $D$ to $\mathbf{P}^2 \backslash A$. Suppose that, for a point $a$ of $D$, there exists a curve $C$ in $\mathbf{P}^2$ with no nonhyperbolic irreducible component with respect to $A$ such that $F(a; \mathbf{P}^2) \subset C$. Then, $F$ is normal at $a$ as a sequence of holomorphic mappings of $D$ to $\mathbf{P}^2$.*

*Proof.* If we assume that $F$ is not normal at a, there exists, by the proof of Proposition 2, a sequence of holomorphic mappings $\{h_k\}_{k=1,2,\dots}$ of $\Delta_k = \{x \in \mathbf{C};\ |x| < k\}$ to $\mathbf{P}^2 \backslash A$ which converges to a nonconstant holomorphic mapping $h$ of $\mathbf{C}$ to $\mathbf{P}^2$ such that $h(\mathbf{C}) \subset F(a; \mathbf{P}^2) \subset C$. Clearly, $h(\mathbf{C})$ is contained in an irreducible component $C_0$ of $C$. Further $h(\mathbf{C}) \cap A = \phi$ or $h(\mathbf{C}) \subset A$. In the former case, the normalization of $C_0 \backslash A$ is isomorphic to $\mathbf{C}$ or $\mathbf{C}^*$. In the latter case, letting $A'$ be the union of the irreducible components of $A$ except $C_0$, we have $h(\mathbf{C}) \cap A' = \phi$ and the normalization of $C_0 \backslash A'$ is isomorphic to $\mathbf{C}$, $\mathbf{C}^*$, $\mathbf{P}$ or an elliptic curve. This is a contradiction. Q.E.D.

## 2. Rank of the complementary domains of hypersurface of $p^n$ and the proof of Theorem 1

Let $A_1, \dots, A_\ell$ be $\ell\,(\ell \geqq n+2)$ distinct irreducible hypersurfaces of $\mathbf{P}^n$ and set $A = A_1 \cup \cdots \cup A_\ell$. Let $P_i(x_0, \dots, x_n)$ be homogeneous polynomials which take zeros only on $A_i$ respectively, where $(x_0, \dots, x_n)$ are the homogeneous coordinates for $\mathbf{P}^n$. We may assume that $P_i\,(i = 1, 2, \dots, \ell)$ are of the same degree $d$. Let $F$ be the rational mapping of $\mathbf{P}^n$ to $\mathbf{P}^{\ell-1}$ defined by $y_1 = P_1, \dots, y_\ell = P_\ell$, where $(y_1, \dots, y_\ell)$ are the homogeneous coordinates for $\mathbf{P}^{\ell-1}$. Since the rank of $F$ is $\leqq n$, the image of $F$ is contained in a hypersurface $S$ of $\mathbf{P}^{\ell-1}$. Let us write the defining equation of $S$ as follows

$$\sum_\lambda c_\lambda \times y_1^{\lambda_1} \times \cdots \times y_\ell^{\lambda_\ell} = 0,$$

where $c_\lambda \neq 0$, $\lambda = (\lambda_1, \dots, \lambda_\ell)$ and $\lambda_i$'s are nonnegative integers satisfying $\lambda_1 + \cdots + \lambda_\ell = N$ (a positive integer). Set $G_\lambda = c_\lambda \times P_1^{\lambda_1} \times \cdots \times P_\ell^{\lambda_\ell}$, then $\{G_\lambda\}$ are homogeneous polynomials of $x_0, \dots, x_n$ of degree $d \times N$ and satisfy $\sum_\lambda G_\lambda \equiv 0$. Let $\{G_0, \dots, G_p\}$ be a subset of $\{G_\lambda\}$ which satisfies $G_0 + \cdots + G_p \equiv 0$ and every subtotal of $G_0, \dots, G_p$ is not identically zero. We shall consider the rational mapping $G$ of $\mathbf{P}^n$ to $\mathbf{P}^p$ defined by $(G_0, \dots, G_p)$.

Since $A_1, \dots, A_\ell$ are all irreducible and distinct, $\lambda \neq \lambda'$ implies $G_\lambda / G_{\lambda'} \not\equiv$ constant. Therefore, we have $p \geqq 2$ and

**Proposition 3.** *The rank of $G$ is always $\geqq 1$.*

**Definition 2.** Fixing $A_1,\dots,A_\ell$, we call the maximum of rank $G$ for various choice of $P_1,\dots,P_\ell$ and $G_0,\dots,G_p$ *the rank of* $(\mathbf{P}^n, A)$.

**Remark.** It is easy to see that, in case $A$ consists of hyperplanes, we have rank $(\mathbf{P}^n, A) = n$ if $A$ consists of hyperplanes in general position in $\mathbf{P}^n$.

Before proving Theorem 1, let us prepare a lemma.

**Lemma 1.** *Let $A_1,\dots,A_\ell$ be $\ell$ $(\ell \geqq 1)$ distinct irreducible hypersurfaces of $\mathbf{P}^n$ and set $M = \mathbf{P}^n \backslash (A_1 \cup \cdots \cup A_\ell)$. Let $\{f_m\}_{m=1,2,\dots}$ be a sequence of holomorphic mappings of a domain $D$ of $\mathbf{C}^k$ to $M$ which converges to $f$ in* $\mathrm{Hol}(D\backslash E, \mathbf{P}^n)$, *$E$ being a proper analytic subset of $D$. Then, either $\{f_m\}$ converges in* $\mathrm{Hol}(D, \mathbf{P}^n)$ *or* $f(D\backslash E) \subset \bigcap_{i=1}^{\ell} A_i$.

*Proof.* Since $f_m(D) \cap A_i = \phi$ for all $m$, we have, by Hurwitz's theorem, either $f(D\backslash E) \cap A_i = \phi$ or $f(D\backslash E) \subset A_i$ for each $i = 1,\dots,\ell$. Therefore, if $f(D\backslash E) \not\subset \bigcap_{i=1}^{\ell} A_i$, then $f(D\backslash E) \cap A_i = \phi$ for some $i$. Then $\{f_m\}$ converges in $\mathrm{Hol}(D, \mathbf{P}^n \backslash A)$, since $\mathbf{P}^n \backslash A$ is a Stein manifold. Q.E.D.

*Proof of Theorem* 1. Let us consider $\{f_m\}_{m=1,2,\dots}$ in $\mathrm{Hol}(D^k, \mathbf{P}^n \backslash A)$ and assume that rank $G = n$. Set $G \circ f_m = g_m = (g_0^m,\dots,g_p^m)$, then $g_i^m = G_i \circ f_m \neq 0$ and $g_0^m + \cdots + g_p^m \equiv 0$. Therefore $g_m \in \mathrm{Hol}(D^k, M)$, where $M$ is defined as follows. Letting $Y$ be the hyperplanes in $\mathbf{P}^p$ defined by $y_0 + \cdots + y_p = 0$, we set

$$H_j = \{(y_0,\dots,y_p) \in Y;\ y_j = 0\}\ (j = 0,\dots,p),$$

and

$$M = Y \backslash (H_0 \cup \cdots \cup H_p).$$

Let $\mathscr{I}$ be the set of subsets of $(0,\dots,p)$ which consists of at least two elements and not more than $p$-1 elements and set

$$\Delta_I = \{(y_0,\dots,y_p) \in Y;\ y_{j_1} + \cdots + y_{j_s} = 0,\ I = (j_1,\dots,j_s) \in \mathscr{I}\}$$

and

$$\Delta = \bigcup_{I \in \mathscr{I}} \Delta_I.$$

Then, from Theorem 6 in Kiernan-Kobayashi [6], either $\{g_m\}$ has a convergent subsequence in $\mathrm{Hol}(D^k, Y)$ or the sequence of the image $g_m(K)$ converges to $\Delta$ (diagonal hyperplanes) for any compact set $K$ of $D^k$.

We have now the commutative diagram as follows

$$\begin{tikzcd} \mathbf{P}^n \arrow[r, "G"] & V \subset Y \subset \mathbf{P}^p \\ D \arrow[u, "f_m"] \arrow[ur, "g_m"'] \arrow[u, phantom, shift left=4ex, "\bigcirc"] & \end{tikzcd}$$

where $V = G(\mathbf{P}^n)$ is an algebraic subvariety of $\mathbf{P}^p$. Set $B_{\Delta_I} = \{(x_0, \dots, x_n) \in \mathbf{P}^n;\ G_{j_1}((x)) + \cdots + G_{j_s}((x)) = 0\}$ for $I = (j_1, \dots, j_s) \in \mathscr{I}$. Then $B_\Delta = \bigcup_{I \in \mathscr{I}} B_{\Delta_I}$ is a hypersurface of $\mathbf{P}^n$. Further, since the rank of $G$ is equal to $n$, there exist a hypersurface $C$ of $\mathbf{P}^n$ and $C'$ of $V$ such that

$$(*) \qquad G|_{\mathbf{P}^n \setminus C} : \mathbf{P}^n \setminus C \longrightarrow V \setminus C'$$

is an unramified covering ($\dim V = n$). We set $B = B_\Delta \cup C$.

Assume now that, for some compact set $K$ of $D^k$, the sequence of the image $f_m(K)$ does not converge to $B$. Then the sequence of the image $g_m(K)$ not converging to $\Delta$, $g_m$ has a subsequence $\{g_{m_\nu}\}$ which converges to $g \in \mathrm{Hol}(D^k, V)$ with $g(K) \not\subset B'$ where $B' = \Delta \cup C'$. $E = g^{-1}(C')$ is then a proper analytic subset of $D^k$ or empty. Since $(*)$ is a finitely sheated unramified covering, $\{f_{m_\nu}\}$ has a subsequence which converges to $f \in \mathrm{Hol}(D^k \setminus E, \mathbf{P}^n \setminus C)$. Since $\bigcap_{i=1}^{\ell} A_i \subset C$, we have $f(D^k \setminus E) \not\subset \bigcap_{i=1}^{\ell} A_i$. Therefore, by Lemma 1, $\{f_{m_\nu}\}$ has a convergent subsequence in $\mathrm{Hol}(D^k, \mathbf{P}^n)$. Q.E.D.

## 3. Proof of Theorem 2

We use the same notations as in the proof of Theorem 1. We have the commutative diagram as follows.

$$\begin{tikzcd} \mathbf{P}^2 \arrow[r, "G"] & V \subset Y \subset \mathbf{P}^p \\ D \arrow[u, "f_m"] \arrow[ur, "g_m"'] \arrow[u, phantom, shift left=4ex, "\bigcirc"] & \end{tikzcd}$$

If rank $G = 2$, then by Theorem 1, there exists a curve $S$ in $\mathbf{P}^2$ such that $\mathbf{P}^2 \setminus A$ is tautly imbedded modulo $S$ in $\mathbf{P}^2$. Therefore we have only to consider the case of rank $G = 1$. In this case the normalization of $V$ is isomorphic to $\mathbf{P}$, so we identify $V$ with $\mathbf{P}$. Let $\beta = \{b_j\}_{j=1,2,\dots,t}$ be the points of $V$ such that $B_j = \overline{G^{-1}(b_j)}$ contains at least one nonhyperbolic curve with respect to $A$ and set $S = B_\Delta \cup B_1 \cup \cdots \cup B_t$. Assume that the sequence of the image $f_m(K)$ does not converge to $S$ for some compact set $K$ of $D^k$. Since $B_\Delta \subset S$, $\{g_m\}$ has a subsequence $\{g_{m_\nu}\}$ which converges compact set $K$ of $D^k$. Since $B_\Delta \subset S$, $\{g_m\}$ has a subsequence $\{g_{m_\nu}\}$ which converges to $g$ in $\mathrm{Hol}(D^k, V)$. Then, we have $g(D^k) \not\subset \beta$. So, $E = g^{-1}(\beta)$ is an analytic subset of codimension $\geqq 1$ or empty. For each point $a \in D^k \setminus E$, the cluster set $F(a; \mathbf{P}^2)$ of the sequence $F = \{f_{m_\nu}\}$ is contained in the fiber $\overline{G^{-1}(g(a))}$ which has no nonhyperbolic curve with respect to $A$. Therefore $F$ is normal at a by Corollary of Proposition 2. $F$ is thus normal in $D^k \setminus E$ and has a subsequence which converges to $f \in \mathrm{Hol}(D^k \setminus E, \mathbf{P}^2)$. Since $\bigcap_{i=1}^{\ell} A^i$ is isolated or empty, either $f \equiv$ constant or $f(D^k \setminus E) \not\subset \bigcap_{i=1}^{\ell} A_i$. In the former case $F$ has a convergent subsequence in $\mathrm{Hol}(D^k, \mathbf{P}^2)$, since $g \equiv$ constant and $E = \phi$. In the latter case $F$ has a convergent subsequence in $\mathrm{Hol}(D^k, \mathbf{P}^2)$ by Lemma 1.

The first assertion of Theorem 2 being established, the second assertion follows easily from Corollary of Proposition 2. Q.E.D.

## 4. Proof of Theorem 3

We continue to use the same notations as in the proof of Theorem 1. If the number of nonhyperbolic curves in $\mathbf{P}^2$ with respect to $A$ is finite, there is a curve $S$ in $\mathbf{P}^2$ such that $\mathbf{P}^2 \setminus A$ is hyperbolically imbedded modulo $S$ in $\mathbf{P}^2$ by Theorem 2. (cf. Theorem 1 in [6]). Therefore, the set of the nonhyperbolic points of $\mathbf{P}^2 \setminus A$ is contained in $S$.

If the number of nonhyperbolic curves in $\mathbf{P}^2$ with respect to $A$ is infinite, then $\mathrm{rank}\, G = 1$, by Theorem 1. In this case the normalization of $V = G(\mathbf{P}^2)$ is isomorphic to $\mathbf{P}$. Set $V_0 = G(\mathbf{P}^2 \setminus A)$. Since $\dim V_0 = 1$ and $V_0 \not\subset \Delta$, $V_0 \cap \Delta$ is finite and discrete. Let us consider $f = G|_{\mathbf{P}^2 \setminus A} : \mathbf{P}^2 \setminus A \to V_0$. Then, we can consider $f$ as a regular rational function on $\mathbf{P}^2 \setminus A$ since normalization of $V$ is isomorphic to $\mathbf{P}$. Let $C$ be a nonhyperbolic curve with respect to $A$ such that $C \not\subset A$. Then, there exists a nonconstant holomorphic mapping $h$ of $\mathbf{C}$ to $C \setminus A$. Set $g = f \circ h$. Then, $g$ is a holomorphic mapping of $\mathbf{C}$ to $V_0 \subset M$. Suppose that $g$ is not constant. Then, by Borel's theorem, we have $g(\mathbf{C}) \subset \Delta \cap V_0$, which is a contradiction since $\Delta \cap V_0$ is discrete. So, $g$ is constant and $C$ is contained in a fiber of $G$. Therefore, the normalizations of infinite irreducible components of the level curves $f^{-1}(a)$ $(a \in V_0)$ are isomorphic to $\mathbf{C}$ or $\mathbf{C}^*$, which implies that, by Lemma 6 of [7], for every $a \in V_0$, each irreducible component of $f^{-1}(a)$ is nonsingular and isomorphic to $\mathbf{C}$ or $\mathbf{C}^*$. Q.E.D.

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