Spectral and scattering theory for the Schrödinger operators with penetrable wall potentials

Dedicated to Professor Tosio Kato on his 70th birthday

By

Teruo Ikebe and Shin-ichi Shimada

§ 0. Introduction

In this paper we shall consider the Schrödinger operator with a penetrable wall potential in $\mathbb{R}^3$ formally of the form

$$H_{\text{formal}} = -\Delta + q(x)\delta(|x| - a),$$

where $q(x)$ is real and smooth on $S_a = \{x; |x| = a\}$ ($a > 0$) and $\delta$ denotes the one-dimensional delta function. This operator is said to provide a simple model for the $\alpha$-decay (Petzold [15]). Other applications may be found in the references cited in Antoine-Gesztesy-Shabani [3]. Dolph-McLeod-Thoe [5] treated this operator ($q(x) \equiv \text{const.}$) with concern for the analytic continuation of the scattering matrix, yet at the formal level.

The first problem one meets is to define properly $H_{\text{formal}}$ as a self-adjoint operator in $L^2(\mathbb{R}^3)$. For this purpose, let us consider the quadratic form $h$ (which is associated with $H_{\text{formal}}$)

$$h[u, v] = (H_{\text{formal}}u, v) = (Vu, Vv) + (q\gamma u, \gamma v)_a,$$

$$\text{Dom}[h] = H^1(\mathbb{R}^3).$$

Here $\gamma$ is the trace operator from $H^1(\mathbb{R}^3)$ to $L^2(S_a)$, $\text{Dom}[h]$ denotes the form domain of $h$, $(\cdot, \cdot)$ means the $L^2(\mathbb{R}^3)$ inner product, $(\cdot, \cdot)_a$ the $L^2(S_a)$ inner product, and $H^m(G)$ the Sobolev space of order $m$ over $G$. $h$ is shown to be a lower semibounded closed form, and thus determines a lower semibounded selfadjoint operator $H$. More precisely, $H$ is seen to be the negative Laplacian with the boundary condition

$$q(x)(\gamma u)(x) - \left\{ \frac{\partial u}{\partial n_+}(x) + \frac{\partial u}{\partial n_-}(x) \right\}_{|S_a} = 0,$$

where $n_+$ ($n_-$) denotes the outward (inward) normal to $S_a$. We should note here that while $h$ is a "small" perturbation of $h_0$, which is defined by

$$h_0[u, v] = (Vu, Vv), \quad \text{Dom}[h_0] = H^1(\mathbb{R}^3),$$

Received June 28, 1989
via an infinitesimally $h_0$-bounded form, $H - H_0$ is not $H_0$-bounded, where $H_0 = -\Delta$, $\text{Dom}(H_0) = H^2(\mathbb{R}^3)$, is the selfadjoint operator associated with $h_0$. We shall adopt this operator $H$ as the rigorous selfadjoint realization of the formal expression $H_{\text{formal}}$. Antoine et al. [3] defined the Hamiltonians corresponding to $H_{\text{formal}}$ as the selfadjoint extensions of $(-\Delta|_{\mathcal{C}_c^\infty(\mathbb{R}^3)\otimes\mathbb{R}^3})$ making use of the decomposition of $L_2(\mathbb{R}^3)$ with respect to angular momenta. Here $\mathcal{C}^\infty_p(G)$ denotes the set of all infinitely continuously differentiable functions with compact support in $G$ and $\sim$ means the closure.

After having determined the proper selfadjoint operator $H$ corresponding to $H_{\text{formal}}$, we take interest in the spectral structure of $H$. It can be seen that the negative part of the spectrum of $H$ consists of a finite number of eigenvalues of finite multiplicity (Theorem 6.5). Further, we can show the difference of the resolvents of $H$ and $H_0$ is a compact operator, which implies that the essential spectrum of $H$ coincides with the interval $[0, \infty)$. A most interesting problem in the spectral theory for $H$ is that of absolute continuity. Namely, let $E(\cdot)$ be the spectral measure associated with $H$. Then the problem is: Is $H$ restricted to $E((0, \infty))L_2(\mathbb{R}^3)$ an absolutely continuous operator? This problem is affirmatively answered by making use of the so-called limiting absorption principle. Our limiting absorption principle for $H$ states that the resolvent $(H - z)^{-1}$ can be extended to a $\mathcal{B}(L_2^2(\mathbb{R}^3), L_2^{-2}(\mathbb{R}^3))$-valued continuous function of $z$ on $\Pi\setminus(\sigma_p(H)\cup\{0\})$ when $s > 1/2$. Here $\Pi$ is the complex plane with the upper and lower edges of $(0, \infty)$ distinguished such that the upper (lower) edge is the boundary points from above (below) (see Kuroda [11, Appendix to Chap. IV]), and $\sigma_p(H)$ denotes the point spectrum of $H$, $\mathcal{B}(X, Y)$ the Banach space of bounded linear operators on $X$ to $Y$, and $L_2^2(\mathbb{R}^3)$ the weighted $L_2$ space defined by

$$L_2^2(\mathbb{R}^3) = \{u(x); (1 + |x|^2)^{1/2}u(x) \in L_2(\mathbb{R}^3)\}$$

with the norm $\|u\|_{0,2} = \|(1 + |\cdot|^2)^{1/2}u\|$ ($\|u\| = \|u\|_{0,0}$ is the usual $L_2$-norm).

Let us recall some notions from scattering theory. In the situation described above the wave operators $W_\pm$ interwining the pair $(H, H_0)$, defined as

$$W_\pm = \text{strong limit } e^{itH}e^{-itH_0},$$

are shown to exist and to be complete. Thus, let us define the generalized Fourier transform $\mathcal{F}_\pm$ by

$$\mathcal{F}_\pm = \mathcal{F}W_\pm^*,$$

where $\mathcal{F}$ is the ordinary Fourier Transform defined by

$$(\mathcal{F}u)(\xi) = (2\pi)^{-3/2}\int_{\mathbb{R}^3} e^{-ix\cdot\xi}u(x)dx,$$

and $*$ means adjoint. Then, with the aid of the limiting absorption principle for $H$ we can construct the distorted plane waves $\varphi_\pm(x, \xi)$ which are the integral kernels of $\mathcal{F}_\pm$ and satisfy the following Lippmann-Schwinger equation
\[ \varphi_{\pm}(x, \xi) = e^{ix \cdot \xi} - \frac{1}{4\pi} \int_{S_a} \frac{e^{iy|\xi|}}{|x - y|} \varphi_{\pm}(y, \xi) dS_y. \]

On the other hand, let \( \lambda_1, \lambda_2, \ldots \) be the nonpositive eigenvalues of \( H \) (counting multiplicity) and \( \varphi_1(x), \varphi_2(x), \ldots \) the corresponding normalized eigenfunctions of \( H \). Then we have the following eigenfunction expansion formula

\[ u(x) = \sum (u, \varphi_n) \varphi_n(x) + \text{l.i.m.}(2\pi)^{-3/2} \int_{\mathbb{R}^3} d\xi (\mathcal{F}_\pm u)(\xi) \varphi_{\pm}(x, \xi), \]

where l.i.m. means limit in the mean.

We shall outline here the contents of the present paper. In §1 we shall define the proper selfadjoint operator \( H \) corresponding to \( H_{\text{formal}} \) and characterize the domain of \( H \). §2 will be devoted to studying some integral operators connected with the resolvent of \( H \). The second resolvent equation for \( H \) and \( H_0 \) will be discussed in §3. The existense and completeness of the wave operators will be shown in §4. In §5 we shall investigate the spectrum of \( H \). An upper bound on the total number of the bound states of \( H \) will be given in §6. In §7 we shall show the limiting absorption principle for \( H \), and in §8 the eigenfunction expansion theorem concerning \( H \).

Part of the results obtained here has been announced in LNM 1285, 211–214 (ed. I. W. Knowles and Y. Saitō). Also, a detailed discussion of the scattering matrices will be given elsewhere by one of the authors (S.S.).

§1. The Schrödinger operator \( H \)

Throughout the paper we shall make the following assumption.

**Assumption 1.1.** \( q(x) \) is a real-valued, smooth function on \( S_a \).

For a rigorous definition of the Schrödinger operator \( H_{\text{formal}} \), we need some lemmas concerning the trace operators.

**Lemma 1.2.** Let \( \gamma_+ \) and \( \gamma_- \) be the trace operators from \( H^1(\{x;|x|<a\}) \) and \( H^1(\{x;|x|>a\}) \), respectively, to \( L_2(S_a) \). Let \( u \in H^1(\mathbb{R}^3) \). Then \( \gamma_+ u = \gamma_- u \).

**Proof.** Since \( u \in H^1(\mathbb{R}^3) \) and \( C_0^\infty(\mathbb{R}^3) \) is dense in \( H^1(\mathbb{R}^3) \) we can choose a sequence \( \{u_n\} \subset C_0^\infty(\mathbb{R}^3) \) such that \( u_n \to u \) in \( H^1(\mathbb{R}^3) \) as \( n \to \infty \). Since \( \gamma_\pm \) are bounded operators from \( H^1(\mathbb{R}^3) \) to \( L_2(S_a) \) (see, e.g. Mizohata [13, Chap. III]), respectively, there exists a constant \( C \) such that

\[ \|\gamma_{\pm} f\|_a \leq C \|f\|_{H^1(\{x;|x| \geq a\})} \quad \text{for } f \in H^1(\{x;|x| \geq a\}), \]

where \( \|u\|_a = \sqrt{(u, u)_a} \). In view of \( (\gamma_+ u_n)(x) = (\gamma_- u_n)(x) = (u_n|S_a)(x) \) for each \( n \), we have by (1.1)
Letting \( n \) tend to \( \infty \) in (1.2), we obtain that \( \gamma_+ u = \gamma_- u \). Q.E.D.

By the above lemma, we can define the trace operator \( \gamma \) from \( H^1(\mathbb{R}^3) \) to \( L_2(S_0) \) by \( \gamma u = \gamma_+ u(= \gamma_- u) \) for \( u \in H^1(\mathbb{R}^3) \).

**Lemma 1.3.** Let \( u \) belong to \( H^1(\mathbb{R}^3) \). Then we have for any \( \varepsilon > 0 \)

\[
\| \gamma u \|_a^2 \leq \varepsilon \| \mathcal{V} u \|_a^2 + \frac{1}{\varepsilon} \| u \|_a^2.
\]

**Proof.** Since \( C_0^\infty(\mathbb{R}^3) \) is dense in \( H^1(\mathbb{R}^3) \) and \( \gamma \) is a bounded operator from \( H^1(\mathbb{R}^3) \) to \( L_2(S_0) \), it suffices to prove the lemma for \( u \in C_0^\infty(\mathbb{R}^3) \) and \( \varepsilon > 0 \). Using the inequality \( 2|p \cdot q| \leq \varepsilon |p|^2 + \varepsilon^{-1} |q|^2 \), we have for any \( \omega \in S^2 \) (the unit sphere of \( \mathbb{R}^3 \))

\[
|\gamma u(\omega)|^2 = -2 \text{Re} \int_a^\infty \frac{\partial u}{\partial r}(r\omega) \overline{u(r\omega)} \, dr
\]

\[
\leq \varepsilon \int_a^\infty \left| \frac{\partial u}{\partial r}(r\omega) \right|^2 \, dr + \varepsilon^{-1} \int_a^\infty |u(r\omega)|^2 \, dr
\]

\[
\leq \varepsilon \int_a^\infty \frac{r^2}{a^2} \left| \frac{\partial u}{\partial r}(r\omega) \right|^2 \, dr + \varepsilon^{-1} \int_a^\infty \frac{r^2}{a^2} |u(r\omega)|^2 \, dr.
\]

Multiplying both sides of (1.5) by \( a^2 \) and integrating with respect to \( \omega \) over the unit sphere \( S^2 \) yield

\[
\int_{S_0} |u(x)|^2 \, dS_x \leq \varepsilon \int_{|x| \geq a} \left| \frac{\partial u}{\partial r}(x) \right|^2 \, dx + \varepsilon^{-1} \int_{|x| \geq a} |u(x)|^2 \, dx
\]

\[
\leq \varepsilon \left\| \frac{\partial u}{\partial r} \right\|_a^2 + \varepsilon^{-1} \| u \|_a^2.
\]

(1.3) follows from (1.6) and \( \left| \frac{\partial u}{\partial r}(x) \right| \leq |\mathcal{V} u(x)| \). To prove (1.4), we have by Schwarz' inequality

\[
|u(a\omega)|^2 = \left| \int_a^\infty \frac{\partial u}{\partial r}(r\omega) \, dr \right|^2 \leq \int_a^\infty \int_a^\infty \frac{r^2}{r^2} \left| \frac{\partial u}{\partial r}(r\omega) \right|^2 \, dr \]

\[
= \frac{1}{a} \int_a^\infty r^2 \left| \frac{\partial u}{\partial r}(r\omega) \right|^2 \, dr.
\]

Thus we have
\[ \int_{S_a} |u(x)|^2 dS_x \leq a \int_{|x| \leq a} \left| \frac{\partial u}{\partial r}(x) \right|^2 dx \leq a \| \nabla u \|^2. \]

Q.E.D.

Now we are in a position to define a selfadjoint operator corresponding to \( H_{\text{formal}} \) in a rigorous way. Consider the quadratic form

\[ h[u, v] = (V u, V v) + (q u, \gamma v), \quad \text{Dom}[h] = H^1(R^3). \]

Since \( q \) is bounded on \( S_a \) by Assumption 1.1, it follows from Lemma 1.1 that \( h \) is a symmetric, lower semibounded, closed form. Therefore, by Kato [9, Chap. VI, Theorem 2.1] we have the following

**Theorem 1.4.** Let \( h \) be the quadratic form defined by (1.9). Then there exists a unique selfadjoint operator \( H \) such that

\[ \text{Dom}(H) \subset \text{Dom}[h], \quad (Hu, v) = h[u, v] \text{ for } u \in \text{Dom}(H) \text{ and } v \in \text{Dom}[h]. \]

We adopt this operator \( H \) as the Schrödinger operator corresponding to \( H_{\text{formal}} \) stated in the Introduction.

**Theorem 1.5.** Let \( A = \min_{x \in S_a} q(x) \). Then

\[ H \geq -A^2. \]

Moreover, we have

\[ H \geq 0 \quad \text{for} \quad -\frac{1}{a} \leq A \]

and

\[ H \geq \frac{4}{a^2} (A + 1) \quad \text{for} \quad -\frac{2}{a} \leq A \leq -\frac{1}{a}. \]

**Proof.** By Theorem 1.4 we have for any \( u \in \text{Dom}(H) \)

\[ (Hu, u) = \| V u \|^2 + \int_{S_a} q(x) |u(x)|^2 dS_x \geq \| V u \|^2 + A \| \gamma u \|^2. \]

If \( A \geq -\frac{1}{a} \), (1.12) follows immediately from (1.14) and (1.4) of Lemma 1.3. Let us assume that \( -\frac{2}{a} \leq A \leq -\frac{1}{a} \). Rewriting (1.14), we have

\[ (Hu, u) \geq \| V u \|^2 - A \left( \frac{2}{Aa} + 1 \right) \| \gamma u \|^2 + A \left( 2 + \frac{2}{Aa} \right) \| \gamma u \|^2. \]
By Lemma 1.3 \((\text{putting } \varepsilon = \frac{a}{2} \text{ in (1.3)})\), we have

\[
(Hu, u) \geq \|Vu\|^2 - A\left(\frac{2}{Aa} + 1\right)a\|Vu\|^2 + A\left(2 + \frac{2}{Aa}\right)\left(\frac{a}{2}\|Vu\|^2 + \frac{2}{a}\|u\|^2\right)
\]

\[
= \frac{4}{a^2}(aA + 1)\|u\|^2.
\]

This implies (1.13). To complete the proof, we have only to show that (1.11) holds when \(A < 0\). In this case, (1.11) follows from (1.14) and Lemma 1.3 with \(\varepsilon = -\frac{1}{A}\).

**Remark 1.6.** The above theorem implies that \(H\) has no negative eigenvalues if \(A \geq \frac{1}{a}\). On the other hand, if \(A < \frac{1}{a}\), \(H\) can have negative eigenvalues. In fact, let \(q(x) = V_0\) (constant) such that \(V_0 < \frac{1}{a}\). Then it is seen that \(H\) has a negative eigenvalue \(-\lambda^2\) \((\lambda > 0)\), where \(\lambda\) is the unique solution of the equation

\[
\frac{1 - e^{-2a\lambda}}{\lambda} = -\frac{2}{V_0},
\]

and a corresponding eigenfunction is \(\frac{1}{|x|}(e^{-\lambda|x| - a} - e^{-\lambda|x| + a})\) (see Dolph et al. [5, pp. 326–327], and cf. Theorem 5.3 below).

Now, we shall characterize the domain of \(H\).

**Theorem 1.7.** \(u \in \text{Dom}(H)\) if and only if

\[
u \in H^1(\mathbb{R}^3), \quad \nu \in H^2(\{|x| < a\}), \quad \nu \in H^2(\{|x| > a\}) \quad \text{and}
\]

\[
q(x)(\nu u)(x) - \left\{\frac{\partial u}{\partial n_+}(x) + \frac{\partial u}{\partial n_-}(x)\right\} |_{s_n} = 0.
\]

In this case, \(Hu = -\Delta u\) in the distribution sense and \(u\) is continuous on \(\mathbb{R}^3\).

**Remark 1.8.** Strictly speaking, \(\frac{\partial u}{\partial n_\pm}\)|\(s_n(x)\) denotes \(\sum_{j=1}^3 \langle n_\pm, e_j \rangle y_\pm \left(\frac{\partial u}{\partial x_j}\right)(x)\), where \(e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)\) and \(\langle x, y \rangle\) means the scalar product of the vectors \(x\) and \(y\).

**Proof of the theorem.** First let \(u \in \text{Dom}(H)\). \(u \in H^1(\mathbb{R}^3)\) is direct from Theorem 1.4. Now, by Theorem 1.4 we have for any \(v \in C^\infty(\{|x| < a\})\)

\[
\int_{|x| < a} (Hu)(x) \overline{v(x)} \, dx = h[u, v] = \int_{|x| < a} (Pu)(x) \overline{(Pv)(x)} \, dx
\]
where and in the sequel it is understood that all derivatives are taken in the distribution sense. (1.18) implies that $Hu = -\Delta u$ in $\{x ; |x| < a\}$ and $u \in H^2(\{x ; |x| < a\})$ (see the note added in proof). Similarly, $Hu = -\Delta u$ in $\{x ; |x| > a\}$ and $u \in H^2(\{x ; |x| > a\})$. Therefore, it makes sense to speak of $\frac{\partial u}{\partial n} |_{S_a}$. Thus, by Theorem 1.4 and Green's Theorem (see e.g. Mizohata [13, Chap. III, § 8]) we obtain for any $v \in C^\infty_0(\mathbb{R}^3)$

$$\begin{align*}
(1.19) \quad (Hu, v) &= h[u, v] = \int_{|x| < a} (\nabla u)(\nabla v)(x) dx \\
&\quad + \int_{|x| > a} (\nabla u)(\nabla v)(x) dx + (q u, v)_{a} \\
&= -\int_{|x| < a} (\Delta u)(x) \overline{v(x)} dx - \left( \frac{\partial u}{\partial n^-} |_{S_a}, \gamma v \right)_{a} \\
&\quad - \int_{|x| > a} (\Delta u)(x) \overline{v(x)} dx - \left( \frac{\partial u}{\partial n^+} |_{S_a}, \gamma v \right)_{a} + (q u, v)_{a} \\
&= (-\Delta u, v) + \left( q u - \left\{ \frac{\partial u}{\partial n^+} + \frac{\partial u}{\partial n^-} \right\} |_{S_a}, \gamma v \right)_{a},
\end{align*}$$

and, since $u \in H^2(\{x ; |x| \neq a\})$ and $Hu = -\Delta u$ as shown above,

$$\left( q u - \left\{ \frac{\partial u}{\partial n^+} + \frac{\partial u}{\partial n^-} \right\} |_{S_a}, \gamma v \right)_{a} = 0 \quad \text{for any } v \in C^\infty_0(\mathbb{R}^3).$$

Since $\{\gamma v = v|_{S_a} ; v \in C^\infty_0(\mathbb{R}^3)\}$ is dense in $L_2(S_a)$, we have

$$q u - \left\{ \frac{\partial u}{\partial n^+} + \frac{\partial u}{\partial n^-} \right\} |_{S_a} = 0.$$

We have thus shown (1.17).

Conversely, let $u$ verify (1.17). Define $w \in L_2(\mathbb{R}^3)$ by $w = -\Delta u$ (except on $S_a$). Then, for any $v \in \text{Dom}(H)$, we have, as we got (1.19),

$$\begin{align*}
(1.20) \quad (Hv, u) &= h[v, u] = (\nabla v, \nabla u) + (q v, u)_{a} \\
&= (v, -\Delta u) + \left( \gamma v, q u - \left\{ \frac{\partial u}{\partial n^+} + \frac{\partial u}{\partial n^-} \right\} |_{S_a} \right)_{a} \\
&= (v, w)
\end{align*}$$

This implies that $u \in \text{Dom}(H^*) = \text{Dom}(H)$.

Finally, let $u \in \text{Dom}(H)$. By what has been shown above, we have $u \in H^2(\{x ; |x| < a\})$, $u \in H^2(\{x ; |x| > a\})$ and $u \in H^1(\mathbb{R}^3)$. Thus, according to
Calderón's extension theorem (e.g. Agmon [1, p.171, Theorem 11.12]), there exist $u_1, u_2 \in H^2(\mathbb{R}^3)$ such that

$$
(1.21) \quad (u_1 |_{|x| < a})(x) = u(x) \quad \text{for a.e. } x \text{ in } \{|x| < a\},
$$

$$
(1.22) \quad (u_2 |_{|x| > a})(x) = u(x) \quad \text{for a.e. } x \text{ in } \{|x| > a\}.
$$

Since $u \in H^1(\mathbb{R}^3)$, we have in view of Lemma 1.2

$$
(1.23) \quad \gamma_-(u_1 |_{|x| < a})(x) = \gamma u = \gamma_+(u_2 |_{|x| > a}).
$$

On the other hand, Sobolev's lemma (e.g. Reed-Simon [18, p.32, Theorem 3.9]) implies that $u_1$ and $u_2$ are continuous on $\mathbb{R}^3$. Hence we have by (1.22)

$$
(1.24) \quad (u_1 |_{|x| = a})(x) = \gamma_-(u_1 |_{|x| < a})(x) = \gamma_+(u_2 |_{|x| > a})(x) = (u_2 |_{|x| = a})(x) \text{ on } S_a.
$$

From (1.21) and (1.23), it follows that $u$ is continuous on $\mathbb{R}^3$. Q.E.D.

§ 2. Preliminary lemmas

We shall introduce the following integral operators $T_\kappa$ and $\tilde{T}_\kappa$ depending on a complex parameter $\kappa$ defined by

$$
(T_\kappa f)(x) = -\frac{1}{4\pi} \int_{S_a} e^{i\kappa |x - y|} |x - y| q(y) f(y) dS_y \quad (x \in \mathbb{R}^3)
$$

and

$$
(\tilde{T}_\kappa f)(x) = -\frac{1}{4\pi} \int_{S_a} e^{i\kappa |x - y|} |x - y| q(y) f(y) dS_y \quad (x \in S_a).
$$

Before studying the properties of $T_\kappa$ and $\tilde{T}_\kappa$, we shall state some lemmas. First, by direct computation using polar coordinates, we have

**Lemma 2.1.** Let $\zeta \in \mathbb{C}$. Then we have for any $x \in \mathbb{R}^3$

$$
(2.1) \quad \int_{S_a} \frac{e^{i\zeta |x - y|}}{|x - y|} dS_y = \frac{2\pi a}{\zeta |x|} (e^{i(a + |x|)} - e^{i(a - |x|)}) \quad (\zeta \neq 0),
$$

$$
(2.2) \quad \int_{S_a} \frac{1}{|x - y|} dS_y = \frac{2\pi a}{|x|} (a + |x| - |a - |x||) \quad (\zeta = 0).
$$

**Lemma 2.2.** There exists a constant $C$ such that for any $x, y \in S_a$,

$$
(2.3) \quad \int_{S_a} \frac{1}{|x - z||z - y|} dS_z \leq C(1 + |\log|x - y||),
$$

$$
(2.4) \quad \int_{S_a} \frac{1}{|x - z||\log|z - y||} dS_z \leq C,
$$
Schrödinger operators

and for any \( x \in \mathbb{R}^3 \), \( 0 < r < 3 \) and \( r + s > 3 \)

\[
\int_{\mathbb{R}^3} \frac{dy}{|x - y|^r (1 + |y|^2)^{3/2}} \leq \begin{cases} 
C & (s < 3) \\
\frac{C \log(1 + |x|)}{(1 + |x|)^r} & (s = 3) \\
C & (s > 3).
\end{cases}
\]

(2.4)

For the proof, see e.g. Kellogg [10, pp. 301–303] or Kuroda [12, p. 162].

**Lemma 2.3.** Let \( \operatorname{Im} \kappa > 0 \). Then \( T_\kappa \) is a Hilbert-Schmidt operator from \( L_2(S_a) \) to \( L_2(\mathbb{R}^3) \).

**Proof.** Put \( b = \operatorname{Im} \kappa \). We compute the Hilbert-Schmidt norm of \( T_\kappa \).

\[
\|4 \pi T_\kappa\|_{H.S.}^2 = \int_{S_a} dS_x \int_{\mathbb{R}^3} dx |q(y)|^2 \frac{e^{-2b|x-y|}}{|x-y|^2} = \frac{2\pi}{b} \|q\|_a^2 < +\infty,
\]

from which follows the assertion. Q.E.D.

**Lemma 2.4.** Let \( \kappa \in \mathbb{C} \). Then \( \tilde{T}_\kappa \) is a compact operator from \( L_2(S_a) \) to itself.

**Proof.** Define the integral operator \( G_\kappa^{(e)} \) by

\[
(G_\kappa^{(e)} f)(x) = -\int_{S_a} \chi_{\{y \in S_a; |x-y| > \varepsilon \}}(y) \frac{e^{i\kappa |x-y|}}{4\pi |x-y|} q(y) f(y) dS_y \quad (x \in S_a, \varepsilon > 0),
\]

where \( \chi_A(x) \) denotes the characteristic function of the set \( A \). Since we have

\[
\int_{S_a \times S_a} dS_x dS_y \chi_{\{y \in S_a; |x-y| > \varepsilon \}}(y) \frac{e^{i\kappa |x-y|}}{4\pi |x-y|} q(y) \left| \frac{e^{i\kappa |x-y|}}{4\pi |x-y|} \right|^2 \leq \left( \frac{e^{2|\operatorname{Im} \kappa|a}}{4\pi \varepsilon} \max_{y \in S_a} |q(y)| \right)^2 (4\pi a^2)^2 < +\infty,
\]

\( G_\kappa^{(e)} \) is a Hilbert-Schmidt, and a fortiori, compact operator from \( L_2(S_a) \) to itself for each \( \varepsilon > 0 \). To prove the lemma, we have only to show that \( G_\kappa^{(e)} \) converges to \( \tilde{T}_\kappa \) in the operator norm topology when \( \varepsilon \downarrow 0 \). In fact, using Schwarz' inequality we have for any \( f \in L_2(S_a) \) and \( x \in S_a \)

\[
|G_\kappa^{(e)} f)(x) - (\tilde{T}_\kappa f)(x)|^2 \leq \left( \int_{S_a \cap \{y; |x-y| \leq \varepsilon \}} \frac{e^{i|\operatorname{Im} \kappa||x-y|}}{4\pi |x-y|} |q(y)||f(y)| dS_y \right)^2 \\
\leq (\max_{y \in S_a} |q(y)|)^2 \left( \int_{S_a \cap \{y; |x-y| \leq \varepsilon \}} \frac{e^{i|\operatorname{Im} \kappa||x-y|}}{4\pi |x-y|} dS_y \right) \left( \int_{S_a} \frac{e^{i|\operatorname{Im} \kappa||x-y|}}{4\pi |x-y|} |f(y)|^2 dS_y \right) \\
\leq (\max_{y \in S_a} |q(y)|)^2 \frac{e^{i|\operatorname{Im} \kappa| \varepsilon}}{2} \int_{S_a} \frac{e^{i|\operatorname{Im} \kappa||x-y|}}{4\pi |x-y|} |f(y)|^2 dS_y \quad (\text{if} \ \varepsilon \leq a/2),
\]

(2.5)
where we have used the equality
\[
(2.6) \quad \int_{S_a \cap \{y:|x-y|<\varepsilon\}} \frac{1}{4\pi|x-y|} dS_y = \frac{a}{2|x|} (\varepsilon - |a - |x||)
\]
if \(x \in \mathbb{R}^3\), \(|a - |x|| < \varepsilon \leq a/2\). Integrating the both sides of (2.5) over \(S_a\) yields by Lemma 2.1 and Fubini’s theorem
\[
(2.7) \quad \|G_\kappa^{(0)} f - \tilde{T}_\kappa f\|_a^2 \leq (\max_{y \in S_a} |q(y)|)^2 e^{||\text{Im}\kappa||}\frac{\varepsilon}{2} e^{2\pi|\text{Im}\kappa|} a \|f\|_a^2.
\]
From (2.7), the claim follows immediately. Q.E.D.

Define the Fourier transform \(\mathcal{F}_{S_a}\) on \(L_2(S_a)\) by
\[
(2.8) \quad (\mathcal{F}_{S_a} f)(\xi) = (2\pi)^{-3/2} \int_{S_a} e^{-ix \cdot \xi} f(x) dS_x \quad (\xi \in \mathbb{R}^3).
\]
Then, as is well known (e.g. Mochizuki [14, p. 16]), we have

**Proposition 2.5.** Let \(s > 1/2\). Then \(\mathcal{F}_{S_a}\) is a bounded operator from \(L_2(S_a)\) to \(L_2^{-s}(\mathbb{R}^3)\), i.e. there exists a constant \(C\) such that
\[
(2.9) \quad \|\mathcal{F}_{S_a} f\|_{0,-s} \leq C \|f\|_a \quad \text{for any } f \in L_2(S_a).
\]

**Lemma 2.6.** Let \(\text{Im} \kappa > 0\). Then \(T_\kappa\) is a bounded operator from \(L_2(S_a)\) to \(H^1(\mathbb{R}^3)\).

**Proof.** For any \(f \in L_2(S_a)\) we have by Fubini’s theorem
\[
(2.10) \quad (\mathcal{F} T_\kappa f)(\xi) = -\int_{S_a} dS_q q(y) f(y) (2\pi)^{-3/2} \int_{\mathbb{R}^3} dx \frac{e^{-ix \cdot \xi} e^{i|\kappa| |x-y|}}{4\pi |x-y|} \\
= -(2\pi)^{-3/2} \int_{S_a} dS_q \frac{e^{-ix \cdot \xi} q(y) f(y)}{|\xi|^2 - \kappa^2} \\
= -\frac{1}{|\xi|^2 - \kappa^2} (\mathcal{F}_{S_a} (q f))(\xi),
\]
where we used (2.8) and the fact that
\[
(2.11) \quad \mathcal{F} \left( \frac{e^{i|\kappa| |y|}}{4\pi |\cdot - y|} \right)(\xi) = (2\pi)^{-3/2} \frac{e^{-i\xi \cdot y}}{|\xi|^2 - \kappa^2}.
\]
Take \(s\) such that \(1/2 < s < 1\). Then, by Proposition 2.5 we can estimate the \(H^1\) norm \(\|T_\kappa f\|_{H^1}\) of \(T_\kappa f\) as follows.
(2.12) \[ \| T_\kappa f \|_{L^2}^2 = \int_{\mathbb{R}^3} d\xi (1 + |\xi|^2) (|\mathcal{F}T_\kappa f(\xi)|^2 \]
\[ = \int_{\mathbb{R}^3} d\xi (1 + |\xi|^2) \left| \frac{-1}{|\xi|^2 - \kappa^2} (\mathcal{F}S_\alpha (qf))(\xi) \right|^2 \]
\[ \leq \sup_{\xi \in \mathbb{R}^3} \left\{ \frac{(1 + |\xi|^2)^{1+s}}{|\xi|^2 - \kappa^2|^2} \right\} \int_{\mathbb{R}^3} d\xi (1 + |\xi|^2)^{-s} (|\mathcal{F}S_\alpha (qf))(\xi)|^2 \]
\[ = \sup_{\xi \in \mathbb{R}^3} \left\{ \frac{(1 + |\xi|^2)^{1+s}}{|\xi|^2 - \kappa^2|^2} \right\} \| \mathcal{F}S_\alpha (qf) \|_{0,-s}^2 \]
\[ \leq \sup_{\xi \in \mathbb{R}^3} \left\{ \frac{(1 + |\xi|^2)^{1+s}}{|\xi|^2 - \kappa^2|^2} \right\} C^2 \max_{x \in S_\alpha} |q(x)|^2 \| f \|_{H^s}^2, \]

which implies the required result. Q.E.D.

By the above lemma, \( \gamma T_\kappa (\text{Im} \kappa > 0) \) is a well-defined bounded operator from \( L_2(S_\alpha) \) to itself. Furthermore, we have

**Lemma 2.7.** Let \( \text{Im} \kappa > 0 \). Then \( \gamma T_\kappa = \tilde{T}_\kappa \).

**Proof.** Since \( \gamma T_\kappa \) and \( \tilde{T}_\kappa \) are bounded operators on \( L_2(S_\alpha) \) and the set of continuous functions on \( S_\alpha \) is dense in \( L_2(S_\alpha) \), it suffices to prove that \( \gamma T_\kappa = \tilde{T}_\kappa \) on this set. Assume that \( f \) is continuous on \( S_\alpha \). Then it follows in a standard way (e.g. Colton-Kress [4, p.47, Theorem 2.12]) that \( (T_\kappa f)(x) \) is continuous on \( \mathbb{R}^3 \). On the other hand, we have for a.e. \( x \in S_\alpha \)

\[ (\gamma T_\kappa f)(x) = \lim_{y \to x} (T_\kappa f)(y) \quad (y \text{ approaches } x \text{ along } n_x). \]

Therefore, \( (\gamma T_\kappa f)(x) = (T_\kappa f)(x) = (\tilde{T}_\kappa f)(x) \) for a.e. \( x \in S_\alpha \). Thus the lemma has been proven. Q.E.D.

**Lemma 2.8.** Let \( \kappa \in C \). Then \( (\tilde{T}_\kappa)^2 \) is a Hilbert-Schmidt operator from \( L_2(S_\alpha) \) to itself.

**Proof.** The kernel of \( (\tilde{T}_\kappa)^2 \) is

\[ \left( \frac{1}{4\pi} \right)^2 \int_{S_\alpha} dS_z \frac{e^{ik(|x-z| + |z-y|)}}{|x-z||z-y|} q(z) q(y). \]

Introducing polar coordinates, we have by Lemma 2.2

\[ \int_{S_\alpha \times S_\alpha} dS_x dS_y \left| \int_{S_\alpha} dS_z \frac{e^{ik(|x-z| + |z-y|)}}{|x-z||z-y|} q(z) q(y) \right|^2 \]
\[ \leq e^{8a|\text{max}|} (\max_{z \in S_\alpha} |q(z)|)^4 \int_{S_\alpha \times S_\alpha} dS_x dS_y \left( \int_{S_\alpha} dS_z \frac{1}{|x-z||z-y|} \right)^2 \]
\[ \leq e^{\max_{z \in S_a} |q(z)|} \int_{S_a \times S_a} dS_x dS_y C^2 (1 + |\log |x - y||)^2 < + \infty, \]

which proves the lemma. Q.E.D.

**Lemma 2.9.** Let \( s > 1/2 \). Then \( T_\kappa \) is a \( B(L_2(S_a), L_2^s(R^3)) \)-valued continuous function of \( \kappa \) for \( \Im \kappa \geq 0 \).

**Proof.** For any \( f \in L_2(S_a) \), we consider the difference

\[ (T_\kappa f)(x) - (T_\kappa f)(y) = \frac{1}{4\pi} \int_{S_a} \frac{e^{i\kappa |x - y|} - e^{i\kappa' |x - y|}}{|x - y|} q(y) f(y). \]

In view of the inequality

\[ |e^{i\kappa |x - y|} - e^{i\kappa' |x - y|}| \leq |\kappa - \kappa'| \mu |x - y|^\mu e^{-\mu \max \{\Im \kappa, \Im \kappa\} |x - y|} \times \]

\[ \times (e^{-\max \{\Im \kappa, \Im \kappa\} |x - y|} - e^{-\max \{\Im \kappa, \Im \kappa\} |x - y|})^{1 - \mu} \quad (0 \leq \mu \leq 1), \]

we have

\[ |(T_\kappa f)(x) - (T_\kappa f)(y)| \leq \frac{2^{1 - \mu}}{4\pi} |\kappa - \kappa'| \mu \max_{y \in S_a} |q(y)| \int_{S_a} dS_y \frac{|f(y)|}{|x - y|^{1 - \mu}}. \]

Taking \( \mu \) such that \( 0 < \mu < \min(s - 1/2, 1) \), we get by Schwarz' inequality and Fubini's theorem

\[ \| T_\kappa f - T_\kappa' f \|_{0, -s}^2 \]

\[ \leq \int_{R^3} dx (1 + |x|^2)^{-s} \left( \frac{2^{1 - \mu}}{4\pi} |\kappa - \kappa'| \mu \max_{y \in S_a} |q(y)| \right)^2 \int_{S_a} dS_y \frac{|f(y)|}{|x - y|^{1 - \mu}}^2 \times \]

\[ \times \int_{S_a} dS_y \frac{1}{|x - y|^{2 - 2\mu}} \int_{S_a} dS_y |f(y)|^2 \]

\[ = \left( \frac{2^{1 - \mu}}{4\pi} |\kappa - \kappa'| \mu \max_{y \in S_a} |q(y)| \right)^2 \times \]

\[ \times \int_{S_a} dS_y \int_{R^3} dx \frac{1}{|x - y|^{2 - 2\mu}(1 + |x|^2)^s} \| f \|_a^2. \]

(2.16) together with Lemma 2.2, (2.4) yields the required result. Q.E.D.

**Lemma 2.10.** \( \tilde{T}_\kappa \) is a \( B(L_2(S_a)) \)-valued continuous function of \( \kappa \) in \( C \).

**Proof.** Using (2.15) \( (\mu = 1) \), we have for \( f \in L_2(S_a) \) and \( x \in S_a \).
Integrating the both sides of (2.18) over \( S_a \) and making use of Schwarz' inequality, we obtain

\[
\| \tilde{T}_\kappa f - \tilde{T}_{\kappa'} f \|_a^2 \leq \left( \frac{|\kappa - \kappa'| e^{2a(|I_m| + |I_m'|)}}{4\pi} \max_{y \in \mathcal{S}_a} |q(y)| \right)^2 \left( \int_{S_a} |dS_y| \| f(y) \| \right)^2.
\]

which completes the proof. \( \text{Q.E.D.} \)

**Lemma 2.11.** Let \( \kappa \in \mathcal{C} \) and let \( u \in L_2(S_a) \). Then, for any \( w \in \mathcal{C}^\infty_0(R^3) \) we have

\[
(2.20) \quad \int_{R^3} (T_\kappa u)(x)( - \Delta - \kappa^2)w(x)dx = - \int_{S_a} q(x)u(x)w(x)ds_x.
\]

If \( \text{Im} \kappa \geq 0 \), (2.20) holds for any \( w \in \mathcal{S} \), where \( \mathcal{S} = \mathcal{S}(R^3) \) denotes the set of functions which together with all their derivatives fall off faster than the inverse of any polynomial.

**Proof.** By Fubini's theorem we have for \( w \in \mathcal{C}^\infty_0(R^3) \)

\[
(2.21) \quad \int_{R^3} (T_\kappa u)(x)( - \Delta - \kappa^2)w(x)dx = - \int_{S_a} q(x)u(x)w(x)ds_x.
\]

On the other hand, we have by Green's theorem

\[
(2.22) \quad \int_{R^3} dx \frac{e^{i|x-y|}}{|x-y|} (- \Delta - \kappa^2)w(x) = 4\pi w(y)
\]

for \( w \in \mathcal{C}^\infty_0(R^3) \). The first part of the lemma follows immediately from (2.21) and (2.22). The proof of the second half is similar. \( \text{Q.E.D.} \)

**Lemma 2.12.** Let \( \text{Im} \kappa > 0 \). Suppose that \( u \) is a non-trivial solution of the homogeneous equation \( u = \tilde{T}_\kappa u \) in \( L_2(S_a) \). Then \( v \equiv T_\kappa u \) is a non-trivial eigenvector of \( H \) corresponding to eigenvalue \( \kappa^2 \). Conversely, if \( \gamma \) is a non-zero eigenvector of \( H \) corresponding to eigenvalue \( \kappa^2 \), \( \gamma v \) is a non-zero vector in \( L_2(S_a) \) and satisfies the equation \( \gamma v = \tilde{T}_\kappa(\gamma v) \).

**Proof.** Assume that \( u \in L_2(S_a) \), \( u \neq 0 \) and \( u = \tilde{T}_\kappa u \). By Lemma 2.11, we have for any \( w \in \mathcal{C}^\infty_0(R^3) \)
Teruo Ikebe and Shin-ichi Shimada

(2.23) \[ \int_{\mathbb{R}^3} (T_k u)(x)(-\Delta - \kappa^2) \bar{w(x)} \, dx = - \int_{S_n} q(x)u(x) \bar{w(x)} \, dS_x. \]

Since \(v = T_k u\) belongs to \(H^1(\mathbb{R}^3)\) by Lemma 2.6 and hence \(\gamma v = \gamma T_k u = \bar{T}_k u = u\) by Lemma 2.7, we have on integration by parts

(2.24) \[ (V v, V w) - \kappa^2(v, w) + (q v, \gamma w)_n = 0 \quad \text{for any } w \in C_0^\infty(\mathbb{R}^3). \]

Since \(C_0^\infty(\mathbb{R}^3)\) is dense in \(H^1(\mathbb{R}^3)\), \(C_0^\infty(\mathbb{R}^3)\) is a form core of \(h\) by Kato [9, Chap. VI, Theorem 1.21] and Lemma 1.3. So we have by (2.24)

\[ h[v, w] = \kappa^2(v, w) \quad \text{for any } w \in H^1(\mathbb{R}^3). \]

Therefore, we obtain by Theorem 1.4

\[ (v, (H - \kappa^2)w) = 0 \quad \text{for any } w \in \text{Dom}(H). \]

which implies that \(v \in \text{Dom}(H)\) and \((H - \kappa^2)v = 0\). If \(v = 0\), we have by Lemma 2.7 \(u = \bar{T}_k u = \gamma T_k u = \gamma v = 0\), which is a contradiction. Thus \(v\) is non-trivial, and is an eigenvector with eigenvalue \(\kappa^2\).

Conversely, let \(v\) verify that \(v \in \text{Dom}(H)\), \(v \neq 0\) and \((H - \kappa^2)v = 0\). Since \(v \in H^1(\mathbb{R}^3)\) by Theorem 1.7 and hence \(\gamma v \in L_2(S_n)\), we have by Theorem 1.4.

(2.25) \[ (V v, V w) - \kappa^2(v, w) + (q v, \gamma w)_n \]

\[ = h[v, w] - \kappa^2(v, w) = ((H - \kappa^2)v, w) \]

\[ = 0 \quad \text{for any } w \in H^1(\mathbb{R}^3). \]

On the other hand, as we got (2.24) from (2.23), we obtain by Lemma 2.11 and in view of \(\gamma v \in L_2(S_n)\) for any \(w \in \mathcal{F}\)

(2.26) \[ (V(T_k \gamma v), V w) - \kappa^2(T_k \gamma v, w) + (q \gamma v, \gamma w)_n = 0, \]

(note that \(T_k (\gamma v) \in H^1(\mathbb{R}^3)\) by Lemma 2.6). Therefore, from (2.25) and (2.26) it follows that

(2.27) \[ (V(T_k \gamma v - v), V w) - \kappa^2(T_k \gamma v - v, w) = 0 \quad \text{for any } w \in \mathcal{F}. \]

By Parseval’s identity we can rewrite (2.27) as

(2.28) \[ (\mathcal{F}(T_k \gamma v - v), (|\cdot|^2 - \kappa^2)\mathcal{F} w) = 0 \quad \text{for any } w \in \mathcal{F}. \]

Put \(w(x) = \mathcal{F}^{-1} \left( \frac{h}{|\cdot|^2 - \kappa^2} \right)(x)\) for \(h \in \mathcal{F}\). Since \(w\) belongs to \(\mathcal{F}\), we obtain by (2.28)

\[ (\mathcal{F}(T_k \gamma v - v), h) = 0 \quad \text{for any } h \in \mathcal{F}, \]

and hence

(2.29) \[ T_k \gamma v - v = 0 \quad \text{in } L_2(\mathbb{R}^3). \]
If \( \gamma v = 0, v = 0 \) by (2.29), which is a contradiction. Thus \( \gamma v \) is a non-zero vector and \( \gamma v = \gamma T_\kappa(yv) = \tilde{T}_\kappa(yv) \) by (2.29). We have thus completed the proof of the lemma. Q.E.D.

**Lemma 2.13.** Let \( \text{Im} \, \kappa > 0 \). Then

\[
(2.30) \quad T_\kappa^* = -q\gamma R_0(\tilde{\kappa}^2),
\]

which maps from \( L_2(R^3) \) to \( L_2(S_a) \).

**Proof.** By Fubini’s theorem we have for \( u \in L_2(S_a) \) and \( v \in L_2(R^3) \)

\[
(2.31) \quad (T_\kappa u, v) = \int_{R^3} dx \left( -\frac{1}{4\pi} \int_{S_a} dS_y \frac{e^{i\kappa|x-y|}}{|x-y|} q(y)u(y) \right) \overline{v(x)}
\]

\[
= \int_{S_a} dS_y u(y) \left( -\frac{q(y)}{4\pi} \int_{R^3} \frac{e^{i(-\tilde{\kappa})|x-y|}}{|x-y|} v(x) \right)
\]

\[
= (u, -q\gamma R_0(\tilde{\kappa}^2)v),
\]

where we have used the reality and boundedness of \( q \) and \( (\gamma R_0(z)v)(x) = (R_0(z)v)|_{S_a(x)} \in L_2(S_a) \) for \( z \in [0, \infty) \) as is seen by Sobolev’s lemma in view of \( \text{Ran}(R_0(z)) = H^2(R^3) \). The lemma follows from (2.31) immediately. Q.E.D.

Define the integral operators \( T^{(1)}_\kappa \) and \( \tilde{T}^{(1)}_\kappa \) with a complex parameter \( \kappa \) by

\[
(T^{(1)}_\kappa f)(x) = \frac{-1}{4\pi} \int_{S_a} dS_y \frac{e^{i\kappa|x-y|}}{|x-y|} f(y) \quad (x \in R^3)
\]

and

\[
(\tilde{T}^{(1)}_\kappa f)(x) = \frac{-1}{4\pi} \int_{S_a} dS_y \frac{e^{i\kappa|x-y|}}{|x-y|} f(y) \quad (x \in S_a).
\]

We remark that if \( q(x) \equiv 1 \), then \( T_\kappa = T^{(1)}_\kappa \) and \( \tilde{T}_\kappa = \tilde{T}^{(1)}_\kappa \), respectively.

**Lemma 2.14.** Let \( \kappa \in C \). Then

\[
(2.32) \quad (\tilde{T}_\kappa)^* = q \tilde{T}^{(1)}_\kappa,
\]

which maps from \( L_2(S_a) \) to itself.

**Proof.** By Fubini’s theorem we have for \( u, v \in L_2(S_a) \)

\[
(\tilde{T}_\kappa u, v)_a = \int_{S_a} dS_x \left( -\frac{1}{4\pi} \int_{S_a} dS_y \frac{e^{i\kappa|x-y|}}{|x-y|} q(y)u(y) \right) \overline{v(x)}
\]

\[
= \int_{S_a} dS_y u(y) \left( -\frac{q(y)}{4\pi} \int_{S_a} dS_x \frac{e^{i(-\tilde{\kappa})|x-y|}}{|x-y|} v(x) \right)
\]

\[
= (u, q \tilde{T}^{(1)}_\kappa v)_a,
\]
§ 3. The resolvent equation

In this section, we shall study the resolvent $R(z)$ of $H$. As remarked in the proof of Lemma 2.13, $\gamma R_0(z)$ is a bounded operator from $L_2(\mathbb{R}^3)$ to $L_2(S_0)$. More precisely, combining Lemmas 2.3 and 2.13 ($q(x) \equiv 1$), we have

**Lemma 3.1.** Let $z \in \mathbb{R} \cup [0, \infty)$. Then $\gamma R_0(z)$ is a Hilbert-Schmidt operator from $L_2(\mathbb{R}^3)$ to $L_2(S_0)$.

**Theorem 3.2.** Let $z \in \rho(H) \cap \rho(H_0)$, where $\rho$ denotes the resolvent set. Then $\gamma R(z)$ is a bounded operator from $L_2(\mathbb{R}^3)$ to $L_2(S_0)$ and the following resolvent equation holds:

$$ (3.1) \quad R(z) - R_0(z) = T_{\sqrt{z}} \gamma R(z), $$

where and in the sequel, by $\sqrt{z}$ is meant the branch of the square root of $z$ with $\text{Im} \sqrt{z} \geq 0$.

**Proof.** To prove the first part of the theorem, we have only to show that $R(z)$ is a bounded operator from $L_2(\mathbb{R}^3)$ to $H^1(\mathbb{R}^3)$. From Theorem 1.4, it follows that $\text{Ran} R(z) = \text{Dom}(H) \subset H^1(\mathbb{R}^3)$ and $\text{Dom}(R(z)) = L_2(\mathbb{R}^3)$. Let $\{u_n\}$ be such that for some $u \in L_2(\mathbb{R}^3)$ and $v \in H^1(\mathbb{R}^3)$, $u_n \to u$ in $L_2(\mathbb{R}^3)$ and $R(z)u_n \to v$ in $H^1(\mathbb{R}^3)$ as $n \to \infty$. Then, since $R(z)$ is a bounded operator from $L_2(\mathbb{R}^3)$ to itself, we have

$$ R(z)u = \lim_{n \to \infty} R(z)u_n = v \quad \text{in} \quad L_2(\mathbb{R}^3), $$

and hence $R(z)$ is a closed operator from $L_2(\mathbb{R}^3)$ to $H^1(\mathbb{R}^3)$. Therefore, from the closed graph theorem it follows that $R(z)$ belongs to $\mathcal{B}(L_2(\mathbb{R}^3), H^1(\mathbb{R}^3))$.

Finally, let us show the resolvent equation. Let $u \in \text{Dom}(H)$ and $v \in \text{Dom}(H_0)$. In view of Theorem 1.4 and $\text{Dom}(H_0) = H^2(\mathbb{R}^3)$, we have

$$ (3.2) \quad ((H - z)u, v) = h[u, v] - (u, \bar{z}v) = (u, (H_0 - \bar{z})v) + (q \gamma u, \gamma v), $$

and hence, on putting $u = R(z)\varphi$ and $v = R_0(\bar{z})\psi = R_0(z)^*\psi$, we obtain

$$ (3.3) \quad (R_0(z)\varphi, \psi) = (R(z)\varphi, \psi) + (q \gamma R(z)\varphi, \gamma R_0(\bar{z})\psi), $$

$$ = (R(z)\varphi - T_{\sqrt{z}} \gamma R(z)\varphi, \psi), $$

where we have used Lemma 2.13. The required resolvent equation follows from (3.3) immediately.

**Q.E.D.**

§ 4. The wave operators

The wave operators $W_{\pm}$ which intertwine $H$ and $H_0$ are defined as
if they exist. In this section we shall prove the following

**Theorem 4.1.**  $W_\pm$ exist and are complete.

The proof of the above theorem will be given after proving the next

**Lemma 4.2.**  $\gamma R(- b^2)$ is a Hilbert-Schmidt operator from $L_2(\mathbb{R}^3)$ to $L_2(S_a)$ for a sufficiently large $b > 0$.

**Proof.**  On operating $\gamma$ from left on the resolvent equation (3.1) ($z = - b^2$), we have, using Lemma 2.7,

\[ (1 - \tilde{T}_b) \gamma R(- b^2) = \gamma R_0(- b^2). \]

If we show that $1 - \tilde{T}_b$ has a bounded inverse for a suitable $b > 0$, the the lemma follows, for $\gamma R_0(- b^2)$ is a Hilbert-Schmidt operator by Lemma 3.1. Using Schwarz’ inequality, Fubini’s theorem and Lemma 2.1, we have for any $u \in L_2(S_a)$

\[ \| \tilde{T}_b u \|_a^2 = \int_{S_a} dS_x \int_{S_a} dS_y \frac{e^{-b|x-y|}}{-4\pi|x-y|} q(y)u(y) \]

\[ \leq \left( \frac{1}{4\pi} \max_{y \in S_a} |q(y)| \right)^2 \int_{S_a} dS_x \int_{S_a} dS_y \frac{e^{-2b|x-y|}}{|x-y|} \|

\[ = \left( \frac{1}{4\pi} \max_{y \in S_a} |q(y)| \right)^2 \int_{S_a} dS_x \frac{\pi}{b} (1 - e^{-4ab}) \int_{S_a} dS_y \frac{|u(y)|^2}{|x-y|} \]

\[ = \left( \frac{1}{4\pi} \max_{y \in S_a} |q(y)| \right)^2 \frac{\pi}{b} (1 - e^{-4ab}) \int_{S_a} dS_y |u(y)|^2 \int_{S_a} dS_x \frac{1}{|x-y|} \]

\[ = (\max_{y \in S_a} |q(y)|)^2 \frac{a}{4b} (1 - e^{-4ab}) \| u \|_a^2. \]

Therefore, we obtain

\[ \| \tilde{T}_b u \| \leq (\max_{y \in S_a} |q(y)|) \left( \frac{a}{4b} (1 - e^{-4ab}) \right)^{1/2}, \]

and hence, the operator norm of $\tilde{T}_b$ is less than unity for sufficiently large $b > 0$, which makes possible the Neumann series inversion of $1 - \tilde{T}_b$. Q.E.D.

**Proof of Theorem 4.1.**  It is known that the wave operators exist and are complete if the difference of the resolvents is a trace-class operator (Kato [9, Chap.X, Theorem 4.8]). On the other hand, as is well known, an operator is in the trace-class if and only if it is a product of two Hilbert-Schmidt operators (e.g. Kato [9, p.521]). Thus, from Lemma 2.3, Theorem 3.2 and Lemma 4.2 it follows that $R(z) - R_0(z)$ is in the trace-class. The proof is now complete. Q.E.D.
§ 5. The spectrum of $H$

As is mentioned in the previous section, the difference of the resolvents of $H$ and $H_0$ is a trace-class operator (see the proof of Theorem 4.1). Thus, concerning the essential spectrum $\sigma_{\text{ess}}(H)$ of $H$, we have by Weyl's theorem (e.g. Reed-Simon [19, p.112, Theorem XIII.14])

**Theorem 5.1.** $\sigma_{\text{ess}}(H) = \sigma_{\text{ess}}(H_0) = [0, \infty)$.

As to the point spectrum of $H$, we get the following result.

**Theorem 5.2.** $\sigma_p(H) \cap (0, \infty) = \emptyset$.

**Proof.** Assume that $\lambda > 0$, $(H - \lambda)u = 0$ and $u \in \text{Dom}(H)$. By Theorem 1.7 $u$ satisfies

\( (A + \lambda)u(x) = 0 \quad \text{in} \quad \{x; |x| < a\} \cup \{x; |x| > a\}. \)

In view of Mizohata [13, Chap. VIII, Lemma 8.4], we have

\( u(x) = 0 \quad \text{in} \quad \{x; |x| > a\}. \)

Thus it follows from (5.2) and Theorem 1.7 that

\( \frac{\partial u}{\partial n_-}|_{S_a}(x) = u|_{S_a}(x) = 0. \)

Now let us define $\tilde{u}(x)$ by

\( \tilde{u}(x) = \begin{cases} u(x) & \text{if} \quad |x| \leq a \\ 0 & \text{if} \quad |x| > a. \end{cases} \)

Then, for any $\varphi \in C_0^\infty(\mathbb{R}^3)$, we have by (5.1), (5.3) and Green’s theorem

\[
\int_{\mathbb{R}^3} \tilde{u}(x)(A + \lambda)\varphi(x)dx = \int_{|x| < a} u(x)A\varphi(x)dx + \lambda \int_{|x| < a} u(x)\varphi(x)dx
\]

\[
= \int_{|x| < a} (A + \lambda)u(x)\varphi(x)dx = 0,
\]

which implies

\( (A + \lambda)\tilde{u}(x) = 0 \quad \text{in} \quad \mathbb{R}^3. \)

Operating the Fourier transform on the both sides of (5.5), we have

\( (\lambda - |\xi|^2)(\mathcal{F}\tilde{u})(\xi) = 0. \)

Since $\mathcal{F}\tilde{u} \in L^2(\mathbb{R}^3)$, we obtain $\mathcal{F}\tilde{u} = 0$, and hence

\( \tilde{u}(x) = 0 \quad \text{for a.e.} \quad x \in \mathbb{R}^3. \)

Therefore, from (5.2), (5.4) and (5.6) it follows that
Schrödinger operators

\[ u(x) = 0 \quad \text{in} \quad L_2(\mathbb{R}^3). \]

Q.E.D.

In contrast to the above theorem the point 0 may or may not belong to \( \sigma_p(H) \). If \( q \) is constant on \( S_a \), however, we get the following criterion.

Theorem 5.3. Let \( q(x) = V_0 \) (constant). Then \( 0 \in \sigma_p(H) \) if and only if there exists a positive integer \( n \) such that \( aV_0 + 2n + 1 = 0 \). In this case, the corresponding eigenspace is spanned by the vectors \( v(|x|)Y_n^m(m = -n, -n + 1, \ldots, n) \), where

\[
v(r) = \begin{cases} r^n & r \leq a \\ a^{2n+1}r^{-n-1} & r \geq a. \end{cases}
\]

and \( Y_n^m(n = 0, 1, \ldots, m = -n, -n + 1, \ldots, n) \) denote the spherical harmonics which provide a basis for \( L_2(S^2) \) (\( S^2 \) the unit sphere in \( \mathbb{R}^3 \)).

Proof. (cf. Colton-Kress [4, pp. 78–79])

Suppose that \( Hu = 0 \) and \( u \in \text{Dom}(H) \). By Theorem 1.7 we have

\[
\Delta u(x) = 0 \quad \text{in} \quad \{ x; |x| < a \} \cup \{ x; |x| > a \}.
\]

Thus \( u(x) \) is a \( C^\infty \)-function in the above region by Weyl's lemma (e.g. Reed-Simon [18, p. 53]). Let \( (r, \theta, \varphi) \) denote the spherical coordinates with \( r = |x| \). For each fixed \( r \) we can expand \( u \) in a uniformly convergent series

\[
u(x) = \sum_{k=0}^{\infty} \sum_{m=-k}^{k} v_{km}(r) Y_k^m(\theta, \varphi),
\]

where

\[
v_{km}(r) = \int_0^{2\pi} \int_0^{\pi} u(r, \theta, \varphi) \overline{Y_k^m(\theta, \varphi)} \sin \theta \, d\theta \, d\varphi.
\]

Since \( u \in C^\infty (\{ x; |x| \neq a \}) \), we can differentiate under the integral and integrate by parts using \( \Delta u = 0 \) to conclude that \( v_{km} \) is a solution of the following equation

\[
\frac{d^2}{dr^2} v_{km} + \frac{2}{r} \frac{d}{dr} v_{km} - \frac{k(k+1)}{r^2} v_{km} = 0,
\]

which has a fundamental system of solutions \( r^k \) and \( r^{-k-1} \). Since \( v_{km} \) is bounded near zero and belongs to \( L_2(0, \infty); r^2 \, dr \) by Theorem 1.7, \( v_{km} \) has the form

\[
v_{00}(r) = \alpha_{00} (r < a), = 0 \quad (r > a),
\]

\[
v_{km}(r) = \begin{cases} \alpha_{km} r^k & (r < a) \\ \beta_{km} r^{-k-1} & (r > a) \end{cases} \quad (k \geq 1),
\]

where \( \alpha_{km} \) and \( \beta_{km} \) are constants. In view of Theorem 1.7, \( v_{km} \) is continuous at \( r = a \) and satisfies the boundary condition

\[
V_0 v_{km}(a) + \left( \frac{d}{dr} v_{km} \right)(a - 0) - \left( \frac{d}{dr} v_{km} \right)(a + 0) = 0,
\]
where $f(a \pm 0)$ denotes $\lim_{\varepsilon \to 0} f(a \pm \varepsilon)$. Therefore, $\alpha_{km}$ and $\beta_{km}$ satisfy the following equations

$$
\alpha_{00} = 0,
$$

$$
\alpha_{km}a^{2k+1} = \beta_{km}, \quad (aV_0 + 2k + 1)\alpha_{km} = 0 \quad (k \geq 1),
$$

from which the required result follows immediately. Q.E.D.

§ 6. Bound states of $H$

Let us define the quadratic form $h_t$ depending on a real parameter $t$ by

$$
h_t[u, v] = (Vu, Vv) + t(q_\gamma u, \gamma v),
$$

$$
\text{Dom}[h_t] = H^1(\mathbb{R}^3).
$$

The form $h_t$ can be seen to be lower semibounded and closed in exactly the same way as for $h (t = 1)$. Therefore, Theorem 1.4 applies to $h_t$. We denote the corresponding unique selfadjoint operator by $H_t (H_1 = H$ (see §1)). Put for $n = 1, 2, \ldots$ and $t \in \mathbb{R}$,

$$
\mu_n(t) = \sup_{\phi_j \in L^2(\mathbb{R}^3)} \inf_{\|u\| = 1} \min(h_t[u], 0),
$$

where $h_t[u] = h_t[u, u]$ and $[\phi_1, \phi_2, \ldots, \phi_{n-1}]^\perp$ is short hand for \{u; \langle u, \phi_j \rangle = 0, j = 1, 2, \ldots, n - 1\}. Then our min-max principle will read as follows:

**Lemma 6.1.** Let $n$ and $t \in \mathbb{R}$ be fixed. Then, either (a) $0 = \mu_n(t) = \mu_{n+1}(t) = \mu_{n+2}(t) = \cdots$ and there are at most $n - 1$ eigenvalues of $H_t$ (counting multiplicity), or (b) there are $n$ eigenvalues of $H_t$ (counting multiplicity) and $\mu_n(t)$ is the $n$-th negative eigenvalue of $H_t$ (counting multiplicity) from below.

**Proof.** (cf. Reed-Simon [19, p.76, Theorem XIII.1])

Let $E_\alpha(\cdot)$ be the spectral measure for $H_t$. First let us show

$$
\dim[\text{Ran}(E_\alpha([-\infty, \alpha]))] < n \quad \text{if } \alpha < \mu_n(t)
$$

$$
\dim[\text{Ran}(E_\alpha([-\infty, \alpha]))] \geq n \quad \text{if } \alpha > \mu_n(t)
$$

Here we remark that $\mu_n(t)$ is finite for each $t \in \mathbb{R}$ and

$$
\text{Ran}(E_\alpha([-\infty, \alpha])) \subset \text{Dom}(H_t) \quad (\subset H^1(\mathbb{R}^3)) \text{ if } \alpha < + \infty,
$$

because of the fact that $H_t$ is bounded from below by Theorem 1.5.

Suppose that (6.3) is false. Then, for any $\varphi_1, \varphi_2, \ldots, \varphi_{n-1}$ we can find $u$ such that $u \in \text{Ran}(E_\alpha([-\infty, \alpha])) \cap [\varphi_1, \varphi_2, \ldots, \varphi_{n-1}]^\perp$ and, by (6.5), $(H_t u, u) \leq \alpha \|u\|^2$. By Theorem 1.4, this implies that
for any \( \varphi_1, \varphi_2, \ldots, \varphi_{n-1} \in L^2(\mathbb{R}^3) \), and hence \( \mu_n(t) \leq \alpha \), which is a contradiction. This proves (6.3).

Since \( \mu_n(t) \leq 0 \) and Theorem 5.1 holds, we have only to prove (6.4) when \( \mu_n(t) < \alpha \leq 0 \). Thus, suppose that (6.4) is false when \( \mu_n(t) < \alpha \leq 0 \). Then we can find \( \varphi_1, \varphi_2, \ldots, \varphi_{n-1} \) such that \( \text{l.h.s.} \{ \varphi_1, \varphi_2, \ldots, \varphi_{n-1} \} = \text{Ran}(E_t((-\infty, \alpha))) \), where l.h.s. denotes the subspace spanned by \( \varphi \). Since any \( u \in [\varphi_1, \varphi_2, \ldots, \varphi_{n-1}]^\perp \cap \text{Dom}(H_t) \) is in \( \text{Ran}(E_t([\alpha, \infty))) \), we have by Theorem 1.4, \( h_t[u] = (H_t u, u) \geq \alpha \| u \|^2 \). Since \( \text{Dom}(H_t) \) is a form core for \( h_t \) (e.g. Reed-Simon [17, p.281]), it follows that

\[
0 < h_t[u] = \inf_{u \in [\varphi_1, \varphi_2, \ldots, \varphi_{n-1}]^\perp \cap H^1(\mathbb{R}^3)} \min_{|u| = 1} (h_t[u], 0) \leq \alpha
\]

and hence \( \mu_n(t) \geq \alpha \), which is a contradiction. This proves (6.4).

First, suppose that

\[
\text{dim} [\text{Ran}(E_t((-\infty, \mu_n(t) + \varepsilon))] = \infty \quad \text{for all } \varepsilon > 0.
\]

Then the situation (a) holds. In fact, by (6.3) we have

\[
\text{dim} [\text{Ran}(E_t((-\infty, \mu_n(t) - \varepsilon))] < n \quad \text{for all } \varepsilon > 0,
\]

and hence

\[
\text{dim} [\text{Ran}(E_t([\alpha, \infty)))] = \infty \quad \text{for all } \varepsilon > 0.
\]

This implies that

\[
\mu_n(t) \in \sigma_{\text{ess}}(H_t).
\]

Since \( \mu_n(t) \leq 0 \) and \( \sigma_{\text{ess}}(H_t) = [0, \infty) \) by Theorem 5.1, it follows that \( \mu_n(t) = 0 \). If \( \mu_{n+1}(t) > \mu_n(t) \), we have by putting \( \alpha = \frac{1}{2}(\mu_{n+1}(t) + \mu_n(t)) \) (\( \alpha < \mu_{n+1}(t) \)) in (6.3)

\[
\text{dim} [\text{Ran}(E_t((-\infty, \frac{1}{2}(\mu_{n+1}(t) + \mu_n(t))])] < n + 1,
\]

which contradicts (6.6). Thus, noting that \( \mu_{n+1}(t) \geq \mu_n(t) \), we obtain \( \mu_n(t) = \mu_{n+1}(t) \cdots \). Finally, if there are \( n \) eigenvalues strictly below \( \mu_n(t) \) and \( \lambda \) is the \( n \)-th eigenvalue, we have
\[
\dim \left[ \text{Ran} \left( E_n \left( \left[ -\infty, \frac{1}{2}(\mu_n(t) + \lambda) \right) \right) \right) \right] \geq n,
\]
which contradicts (6.3) \((\lambda = \frac{1}{2}(\mu_n(t) + \lambda) < \mu_n(t))\). Thus it is seen that there are at most \(n - 1\) eigenvalues of \(H_n\).

Next, assume that (6.6) fails, i.e., for some \(\varepsilon_0 > 0\)
\[
\dim \left[ \text{Ran} \left( E_n \left( \left[ -\infty, \mu_n(t) + \varepsilon_0 \right) \right) \right) \right] < +\infty.
\]
Then the situation (b) arises. In fact, we have by (6.3) an (6.4)
\[
\dim \left[ \text{Ran} \left( E_n \left( \left[ \mu_n(t) - \varepsilon, \mu_n(t) + \varepsilon \right) \right) \right) \right] \geq 1 \quad \text{for any } \varepsilon > 0.
\]
On the other hand, (6.8) implies
\[
\dim \left[ \text{Ran} \left( E_n \left( \left[ -\infty, \mu_n(t) - \varepsilon_0, \mu_n(t) + \varepsilon_0 \right) \right) \right) \right] < +\infty.
\]
Thus it follows from (6.9) an (6.10) that \(\mu_n(t)\) is a discrete eigenvalue of \(H_n\). Take \(\delta > 0\) such that \((\mu_n(t) - \delta, \mu_n(t) + \delta) \cap \sigma(H_n) = \{\mu_n(t)\}\). Then we have by (6.4)
\[
\dim \left[ \text{Ran} \left( E_n \left( \left[ \mu_n(t) - \varepsilon, \mu_n(t) + \varepsilon \right) \right) \right) \right] 
= \dim \left[ \text{Ran} \left( E_n \left( \left[ -\infty, \mu_n(t) + \varepsilon_0 \right) \right) \right) \right] 
\geq n.
\]
Thus there exist at least \(n\) eigenvalues of \(H_n\): \(\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \mu_n(t)\). If \(\lambda_n < \mu_n(t)\), we have by putting \(\alpha = \frac{1}{2}(\mu_n(t) + \lambda_n)\) \((< \mu_n(t))\) in (6.3)
\[
n \leq \dim \left[ \text{Ran} \left( E_n \left( \left[ -\infty, \lambda_n \right) \right) \right) \right] 
\leq \dim \left[ \text{Ran} \left( E_n \left( \left[ -\infty, \alpha \right) \right) \right) \right] < n,
\]
which is a contradiction. Therefore, \(\lambda_n = \mu_n(t)\), i.e. \(\mu_n(t)\) is the \(n\)-th eigenvalue of \(H_n\). The lemma has now been proven. Q.E.D.

**Lemma 6.2.** For each \(n\), \(\mu_n(t)\) is monotone nonincreasing in \(t\) on \([0, \infty)\).

**Proof.** Since \(\min(h_t[u], 0)\) is monotone nonincreasing in \(t\) on \([0, \infty)\), the required result follows immediately. Q.E.D.

**Lemma 6.3.** For each \(n\), \(\mu_n(t)\) is continuous in \(t\) on \(\mathbb{R}\).

**Proof.** (cf. Simon [21, p.71, Theorem II.33]) For each \(u \in H^1(\mathbb{R}^3)\) with \(\|u\| = 1\), we put
\[
f(t; u) = \min(h_t[u], 0).
\]
If we show that \(\{f(\cdot; u) ; u \in H^1(\mathbb{R}^3), \|u\| = 1\}\) is equicontinuous, the conclusion follows.

Given \(t_0 \in \mathbb{R}^1\), we have by Lemma 1.3 \((\varepsilon = \left[ 2(\max_{x \in S} |q(x)| + 1)(|t_0| + 1) \right]^{-1})\)
\[
(qy_x, \gamma u)_x \leq \max_{x \in S} |q(x)| \cdot \|\gamma u\|^2 \leq \frac{\max_{x \in S} |q(x)|}{2(\max_{x \in S} |q(x)| + 1)(|t_0| + 1)} \|F u\|^2 +
\]

\[ + 2 (\max |q(x)|)(\max |q(x)| + 1)(|t_0| + 1)\|u\|^2 \leq \frac{1}{2(|t_0| + 1)} \|Vu\|^2 + b_{t_0}. \]

where we put \( b_{t_0} = 2 (\max |q(x)| + 1)^2 (|t_0| + 1). \) Suppose that \( h_t[u] \leq 0 \) for some \( t \) such that \(|t - t_0| < 1\). Then we have

\[ f(t; u) = h_t[u] = \|Vu\|^2 + t(q\gamma u, \gamma u)_a \leq 0, \]

which implies

(6.12) \[ \|Vu\|^2 \leq -t(q\gamma u, \gamma u)_a \leq |t||q\gamma u, \gamma u)_a| \leq (|t_0| + 1)|q\gamma u, \gamma u)_a|. \]

Therefore, from (6.11) and (6.12) it follows that

\[ |q\gamma u, \gamma u)_a| \leq \frac{1}{2} |q\gamma u, \gamma u)_a| + b_{t_0}, \]

and hence

(6.13) \[ |q\gamma u, \gamma u)_a| \leq 2b_{t_0} \]

if \( h_t[u] \leq 0 \) for some \( t \) such that \(|t - t_0| < 1\).

Now, for given \( \varepsilon > 0 \), let \( \delta = \min \left( \frac{\varepsilon}{2b_{t_0}}, 1 \right) \). Let \(|t - t_0| < \delta\). If \( h_t[u] \leq 0 \) or \( h_{t_0}[u] \leq 0 \), then we have by (6.13)

\[ |f(t; u) - f(t_0; u)| \leq |h_t[u] - h_{t_0}[u]| \leq |t - t_0||q\gamma u, \gamma u)_a| < \delta \cdot 2b_{t_0} \leq \varepsilon. \]

The above inequality is trivially satisfied if \( h_t[u] > 0 \) and \( h_{t_0}[u] > 0 \). We have thus obtained the required equicontinuity. Q.E.D.

**Lemma 6.4.** For each \( n \), \( \mu_n(t) \) is strictly monotone decreasing on \([t_1, + \infty)\) once \( \mu_n(t_1) < 0 \) for some \( t_1 > 0 \).

**Proof.** Let \( t_1 \) be such that \( E = \mu_n(t_1) < 0 \) and \( t_1 > 0 \). Assume that there exists \( t_2 \) such that \( t_1 \leq t_2 \) and \( \mu_n(t_1) = \mu_n(t_2) = E < 0 \). Then, for any \( t \in [t_1, t_2] \), \( \mu_n(t) = E \) holds by Lemma 6.2. Therefore, by Lemma 6.1 we can find \( u_t \) for each \( t \in [t_1, t_2] \) which satisfies

(6.14) \[ u_t \in \text{Dom}(H_t), \ u_t \neq 0 \text{ and } (H_t - E)u_t = 0. \]

In view of Lemma 2.12, we have

(6.15) \[ \gamma u_t = t \cdot \tilde{T}_{i\gamma - E}(\gamma u_t) \text{ and } \gamma u_t \neq 0 \text{ in } L_2(S_a). \]

This implies that for every \( t \in [t_1, t_2] \) \( t^{-1} \) is an eigenvalue of \( \tilde{T}_{i\gamma - E} \), which is a contradiction, for \( \tilde{T}_{i\gamma - E} \) is a compact operator by Lemma 2.4. Therefore, we must have the lemma in view of Lemma 6.2. Q.E.D.

Now, as an analogue of the Birman-Schwinger bound (e.g. Reed-Simon [19, p.98, Theorem XIII.10]), we shall give a bound on the total number of bound states of \( H \). Let \( E < 0 \) and define \( N(E) \) by
where \( \#A \) denotes the cardinality of the set \( A \). Then we have the following

**Theorem 6.5.** Let \( E < 0 \). Then

\[
N(E) \leq \| (\tilde{T}_{i\sqrt{-E}})^2 \|_{l.s.}^2 \leq M < + \infty ,
\]

where \( M \) is a constant independent of \( E < 0 \). In particular, the total number of negative eigenvalues of \( H \) is finite.

**Proof.** Since \( \mu_n(0) = 0 \) for every \( n \) and \( \mu_n(t) \) is continuous by Lemma 6.3, it follows from the intermediate value theorem and Lemma 6.4 that \( \mu_n(1) < E \) if and only if \( \mu_n(t) = E \) for exactly one \( t \in (0, 1) \). Using Lemma 2.12 repeatedly, it is seen that \( t^{-2} \) satisfying the equation \( \mu_n(t) = E \) is an eigenvalue of \( (\tilde{T}_{i\sqrt{-E}})^2 \). Further, since \( (\tilde{T}_{i\sqrt{-E}})^2 \) is a Hilbert-Schmidt operator by Lemma 2.8, we have

\[
N(E) = \# \{ n; \mu_n(t) = E \text{ for some } t \in (0, 1) \}
\]

\[
\leq \sum_{\{ t \in (0,1) : \mu_n(t) = E, k = 1,2,\ldots,N(E) \}} t^{-4}
\]

\[
\leq \sum_{\{ t \in (0,1) : \mu_n(t) = E, k = 1,2,\ldots \}} t^{-4}
\]

\[
\leq \sum_{\{ t \in (0,1) : t^{-2} \text{ is an eigenvalue of } (\tilde{T}_{i\sqrt{-E}})^2 \}} t^{-4}
\]

\[
\leq \| (\tilde{T}_{i\sqrt{-E}})^2 \|_{l.s.}^2
\]

\[
\leq C^2 (\max_{z \in S_n} |q(z)|)^4 \int_{S_n \times S_n} ds_x ds_y (1 + |\log |x - y||)^2 \equiv M < + \infty ,
\]

where \( C \) is a constant which is independent of \( E \) (see the proof of Lemma 2.8). The above inequality shows the theorem. Q.E.D.

§ 7. The limiting absorption principle for \( H \)

In this section we shall prove the limiting absorption principle for \( H \).

**Theorem 7.1.** Let \( s > \frac{1}{2} \). Then \( R(z) \) can be extended to a \( B(L^2_1(R^3), L^2_2(R^3)) \)-valued continuous function of \( z \) on \( \Pi \setminus (\sigma_p(H) \cup \{0\}) \).

**Proof.** Let us recall the resolvent equation

\[
R(z) - R_0(z) = T_{i\sqrt{-E}} \gamma R(z).
\]

If we assume that \( (1 - \tilde{T}_{i\sqrt{-E}})^{-1} \) exists, we have on operating \( \gamma \) from left on the both sides of (7.1) and solving for \( R(z) \),

\[
R(z) = R_0(z) + T_{i\sqrt{-E}} (1 - \tilde{T}_{i\sqrt{-E}})^{-1} \gamma R_0(z)
\]
for \( z \in \rho(H) \cap \rho(H_0) \). Here we have used Lemma 2.7. By Lemma 2.9 \( T_{\sqrt{z}} \) is a \( B(L_2(S_a), L_2^{-s}(R^3)) \)-valued continuous function of \( z \) on \( \text{Im} \sqrt{z} \geq 0 \) if \( s > 1/2 \). Thus \((T_{\sqrt{z}})^* \) is a \( B(L_2^s(R^3), L_2(S_a)) \)-valued continuous function of \( z \) on \( \text{Im} \sqrt{z} \geq 0 \) if \( s > 1/2 \). On the other hand, we have by Lemma 2.13 \((q(x) \equiv 1)\)

\[
\gamma R_0(z) = -(T_{\sqrt{z}})^* \quad \text{if } \text{Im} \sqrt{z} > 0.
\]

Thus, since \( T_\kappa = T_\kappa^{(1)} \) if \( q(x) \equiv 1 \), \( \gamma R_0(z) \) can be extended to a \( B(L_2^s(R^3), L_2(S_a)) \)-valued continuous function of \( z \) on \( \Pi \) if \( s > 1/2 \). Therefore, in view of the well-known limiting absorption principle for \( H_0 \) (see e.g. Agmon [2]), the proof of the above theorem is reduced to the next

**Lemma 7.2.** Let \( z \in \Pi \setminus (\sigma_p(H) \cup \{0\}) \). Then \((1 - \tilde{T}_{\sqrt{z}})^{-1} \) exists and belongs to \( B(L_2(S_a)) \). In this case, \((1 - \tilde{T}_{\sqrt{z}})^{-1} \) is a \( B(L_2(S_a)) \)-valued continuous function of \( z \) on \( \Pi \setminus (\sigma_p(H) \cup \{0\}) \), where \( B(X) \) denotes \( B(X, X) \).

We will show this lemma after proving a series of lemmas. First, we have by Lemma 2.12

**Lemma 7.3.** Let \( \text{Im} \sqrt{z} > 0 \). Then \( 1 \in \sigma_p(\tilde{T}_{\sqrt{z}}) \) if and only if \( z \in \sigma_p(H) \).

**Lemma 7.4.** Let \( \zeta \in C \) and let \( u \in L_2(S_a) \) satisfy the homogeneous equation \( u = \tilde{T}_\zeta u \) in \( L_2(S_a) \). Then \( u \) is bounded on \( S_a \).

**Proof.** Let \( k(x, y) \) be the integral kernel of \((\tilde{T}_\zeta)^3 \). It follows from Lemma 2.2 that \( k(x, y) \) is bounded on \( S_a \times S_a \). Thus we have by Schwarz' inequality

\[
|u(x)| = |(\tilde{T}_\zeta)^3 u(x)| = \left| \int_{S_a} dS_y k(x, y) u(y) \right| \\
\leq \sup_{(x, y) \in S_a \times S_a} |k(x, y)| \int_{S_a} dS_y |u(y)| \\
\leq \sup_{(x, y) \in S_a \times S_a} |k(x, y)|(4\pi a^2)^{1/2} \|u\|_a < +\infty,
\]

which proves the lemma. Q.E.D.

**Lemma 7.5.** Under the conditions of Lemma 7.4 \( u(x) \) is Hölder continuous on \( S_a \).

**Proof.** We consider the difference

\[
(7.3) \quad u(x) - u(x') = \frac{1}{4\pi} \int_{S_a} e^{i\sqrt{|x-y|}} - e^{i\sqrt{|x'-y|}} \frac{q(y)u(y)dS_y}{|x-y|} \\
+ \frac{1}{4\pi} \int_{S_a} \left( \frac{1}{|x-y|} - \frac{1}{|x'-y|} \right) e^{i\sqrt{|x'-y|}} q(y)u(y)dS_y \\
= J_1 + J_2.
\]
We shall estimate $J_1$ and $J_2$ as follows. Considering the inequality

$$|e^{i\zeta|x-x'|} - e^{i\zeta|x'-y|}| \leq |\zeta||x-x'|e^{\operatorname{Im} \zeta |x-x'|},$$

we have for $x, x' \in S_a$

$$|J_1| \leq \frac{1}{4\pi} A |\zeta| |x-x'| \int_{S_a} \frac{1}{|x-y|} dS_y$$

$$= Aa |\zeta| e^{2|\operatorname{Im} \zeta|} |x-x'|,$$

where $A = \sup_{y \in S_a} |q(y)u(y)| < +\infty$ by Assumption 1.1, Lemma 7.4 and Lemma 2.1. We proceed to estimate $J_2$. In view of the inequality

$$\left| \frac{1}{|x-y|} - \frac{1}{|x'-y|} \right| \leq \frac{|x-x'|}{|x-y||x'-y|},$$

we have for $x, x' \in S_a$

$$|J_2| \leq \frac{1}{4\pi} A e^{2|\operatorname{Im} \zeta|} |x-x'| \int_{S_a} \frac{1}{|x-y||x'-y|} dS_y$$

$$\leq \frac{1}{4\pi} A e^{2|\operatorname{Im} \zeta|} C |x-x'| (1 + |\log |x-x'||),$$

where we used Lemma 2.2. The conclusion follows from (7.3), (7.4) and (7.5).

Q.E.D.

**Lemma 7.6.** Let $\mu \in \mathbb{R}$ and $u \in L_2(S_a)$. Put $U(x) \equiv (T_\mu u)(x)$. Then $U(x)$ has the following asymptotic behavior

$$U(x) = -\frac{1}{4\pi} e^{i|x|} \int_{S_a} e^{i\mu x \cdot y} q(y) u(y) dS_y + O\left(\frac{1}{|x|^2}\right)$$

as $|x| \to \infty$, where $\omega_x$ denotes the unit vector with the direction of $x$. Further, $U(x)$ satisfies the following radiation condition

$$\partial U \over \partial |x| (x) - i\mu U(x) = O\left(\frac{1}{|x|^2}\right) \quad \text{as} \quad |x| \to +\infty.$$

**Proof.** In view of the relation

$$\frac{e^{i|u| |x-y|}}{|x-y|} = \frac{1}{|x|} e^{i|u| \omega_x \cdot y + i|u| |x| \eta_1} + \frac{\eta_2}{|x|} e^{i|u| |x-y|}$$

$$+ \frac{1}{|x|^3} \omega_x \cdot ye^{i|u| |x-y|} \quad (x \in \mathbb{R}^3, |y| < R < +\infty),$$

where $\eta_1$ and $\eta_2$ are real valued functions satisfying $\eta_1 = O\left(\frac{|y|}{|x|^2}\right)$, and $\eta_2$...
\[ U(x) = -\frac{1}{4\pi} \frac{e^{i\mu|x|}}{|x|} \int_{S_a} e^{-i\mu a \cdot y} q(y)u(y)dy \]
\[ -\frac{1}{4\pi} \frac{e^{i\mu|x|}}{|x|} \int_{S_a} \left( e^{-i\mu a \cdot y} (e^{-i\mu |x|} - 1)q(y)u(y)dy \right) \]
\[ -\frac{1}{4\pi} \frac{1}{|x|} \int_{S_a} \eta_2 e^{i\mu |x-y|} q(y)u(y)dy \]
\[ -\frac{1}{4\pi} \frac{1}{|x|^2} \int_{S_a} \omega_\alpha \cdot y e^{i\mu |x-y|} q(y)u(y)dy \]
\[ = -\frac{1}{4\pi} \frac{e^{i\mu|x|}}{|x|} \int_{S_a} e^{-i\mu a \cdot y} q(y)u(y)dy + I_1 + I_2 + I_3. \]

\[ I_i \ (i = 1, 2, 3) \] are estimated as follows:

\[ |I_1| \leq \frac{1}{4\pi} \max_{y \in S_a} |q(y)| \frac{1}{|x|} \int_{S_a} |e^{-i\mu |x|} - 1||u(y)|dy \]
\[ \leq \frac{1}{4\pi} \max_{y \in S_a} |q(y)| \frac{1}{|x|} \int_{S_a} C\mu \frac{a}{|x|^3} |u(y)|dy \]
\[ \leq \text{const.} \frac{1}{|x|^2} \int_{S_a} |u(y)|dy \]
\[ \leq \text{const.} \frac{1}{|x|^2} \|u\|_a, \]
\[ |I_2| \leq \text{const.} \frac{1}{|x|^3} \|u\|_a, \quad |I_3| \leq \text{const.} \frac{1}{|x|^2} \|u\|_a. \]

These estimates prove (7.6).

Let us show (7.7). By differentiation under the integral sign, we have

\[ \frac{\partial U}{\partial |x|}(x) - i\mu U(x) \]
\[ = -\frac{i\mu}{4\pi} \int_{S_a} e^{i\mu |x-y|} \left( \frac{|x|^2 - x \cdot y}{|x-y|^2} - 1 \right) q(y)u(y)dy \]
\[ + \frac{1}{4\pi} \int_{S_a} \frac{|x|^2 - x \cdot y}{|x||x-y|^2} e^{i\mu |x-y|} q(y)u(y)dy \]
\[ = J_1 + J_2. \]

Considering \(|y| = a\), we have
\[ |x - y|^{-1} = |x|^{-1} \left( 1 + \omega_x \cdot \left( \frac{y}{|x|} \right) + O\left( \frac{1}{|x|^2} \right) \right) \]

as \(|x| \to +\infty\), and hence

\[ \frac{1}{|x - y|} \left( \frac{|x|^2 - x \cdot y}{|x||x - y|} - 1 \right) = O\left( \frac{1}{|x|^3} \right), \]

\[ \frac{|x|^2 - x \cdot y}{|x||x - y|^3} = O\left( \frac{1}{|x|^3} \right) \quad \text{as } |x| \to +\infty. \]

Therefore, we have

\[ |J_1| \leq \text{const.} \frac{\int_{S_a} |u(y)| \, dS_y}{|x|^3} \leq \text{const.} \frac{\|u\|_a}{|x|^3}, \]

\[ |J_2| \leq \text{const.} \frac{\|u\|_a}{|x|^2}. \]

Thus (7.7) follows from (7.8) and (7.9) immediately. Q.E.D.

**Lemma 7.7.** Let \( \mu \in R \setminus \{0\} \) and let \( u \in L_2(S_a) \) satisfy \( u = \tilde{T}_\mu u \) in \( L_2(S_a) \). Then, for an arbitrary unit vector \( \omega \) we have

\[ \int_{S_a} e^{-i\mu \omega \cdot y} q(y)u(y) \, dS_y = 0. \]

**Proof.** Put \( U(x) \equiv (T_\mu u)(x) \). Since \( u(x) \) is continuous on \( S_a \) by Lemma 7.5, \( U(x) \) is continuous on \( R^3 \) (see e.g. Colton-Kress \([4, \text{p.} 47, \text{Theorem} \ 2.12]\)), and hence

\[ (U|_{S_a})(x) = u(x). \]

On the other hand, by Lemma 2.11 \(((T_\mu u)(x) \equiv U(x))\), \( U(x) \) satisfies the reduced wave equation

\[ (A + \mu^2)U(x) = 0 \quad \text{on } \{x; |x| < a\} \cup \{x; |x| > a\}. \]

Further, \( (\frac{\partial U}{\partial n_+})(x) \) can be continuously extended from \( \{x; |x| < a\} \) to \( \{x; |x| \leq a\} \) and from \( \{x; |x| > a\} \) to \( \{x; |x| \geq a\} \) with the limiting values

\[ (\frac{\partial U}{\partial n_+})\left( x \right) = \pm \frac{1}{2} q(x)u(x) + W(x) \quad (x \in S_a), \]

respectively. Here

\[ W(x) = -\frac{1}{4\pi} \int_{S_a} \left( \frac{\partial}{\partial n_+} \right) \left( e^{i\mu |x - y|} \right) \frac{q(y)u(y)}{|x - y|} \, dS_y \] (the integral exists as an improper integral) and \( (\frac{\partial U}{\partial n_+})(x) \) are the limits of \( (\frac{\partial U}{\partial n_+})(x) \) obtained by approaching \( S_a \) from \( \{x; |x| > a\} \) and \( \{x; |x| < a\} \), respectively, that is,
\[
\left( \frac{\partial U}{\partial n_+} \right)^{(+)\text{a}}(x) = \lim_{y \to x; \|y\| > a} \left( \frac{\partial U}{\partial n_+} \right)(y), \\
\left( \frac{\partial U}{\partial n_+} \right)^{(-)\text{a}}(x) = \lim_{y \to x; \|y\| < a} \left( \frac{\partial U}{\partial n_+} \right)(y), \quad x \in S_a
\]

(see e.g. Colton-Kress [4, p.47]). Using (7.12), (7.13) and Green's theorem, we have

(7.14) \[ 0 = \int_{|x| < a} \{(\Delta + \mu^2)U(x) \cdot \overline{U(x)} - (\Delta + \mu^2)^2U(x) \cdot U(x)\} \, dx \]

\[ = \int_{|x| < a} \{(\Delta U)(x)\overline{U(x)} - (\Delta U)(x) \, U(x)\} \, dx \]

\[ = \int_{S_a} \left\{ \left( \frac{\partial U}{\partial n_+} \right)^{(-)}(x) \overline{U(x)} - \left( \frac{\partial U}{\partial n_+} \right)^{(-)}(y) \, U(y) \right\} dS_x \]

\[ = \int_{S_a} (W(x) \overline{U(x)} - \overline{W(x)} \, U(x)) dS_x, \]

where we have used the fact that \( \mu \) and \( q(x) \) are real-valued. Similarly, for any \( b \) such that \( b > a \) we have

(7.15) \[ 0 = \int_{a < |x| < b} \{(\Delta + \mu^2)U(x) \cdot \overline{U(x)} - (\Delta + \mu^2)^2U(x) \cdot U(x)\} \, dx \]

\[ = -\int_{S_a} \left\{ \left( \frac{\partial U}{\partial n_+} \right)^{(+)\text{a}}(x) \overline{U(x)} - \left( \frac{\partial U}{\partial n_+} \right)^{(+)\text{a}}(x) \, U(x) \right\} dS_x \]

\[ + \int_{S_a} \left\{ \left( \frac{\partial U}{\partial n_+} \right)(x) \overline{U(x)} - \left( \frac{\partial U}{\partial n_+} \right)(x) \, U(x) \right\} dS_x \]

\[ = -\int_{S_a} (W(x) \overline{U(x)} - \overline{W(x)} \, U(x)) dS_x \]

\[ + \int_{S_a} \left\{ \left( \frac{\partial U}{\partial n_+} \right)(x) \overline{U(x)} - \left( \frac{\partial U}{\partial n_+} \right)(x) \, U(x) \right\} dS_x. \]

Thus we obtain by (7.14) and (7.15)

(7.16) \[ \int_{|x| = b} \left\{ \left( \frac{\partial U}{\partial n_+} \right)(x) \overline{U(x)} - \left( \frac{\partial U}{\partial n_+} \right)(x) \, U(x) \right\} dS_x = 0, \]
Teruo Ikebe and Shin-ichi Shimada

for any $b$ such that $b > a$. Once Lemma 7.6 and (7.16) are shown, an argument similar to Povzner [16, Chap. II, Lemma 5] gives

\[(7.17) \int_{S_a} e^{-i\omega \cdot y} q(y) (U|_{S_a})(y) dS_y = 0 \quad (\omega \in S^2),\]

which implies (7.10) by (7.11).

**Q.E.D.**

**Lemma 7.8.** Let $\lambda > 0$ and let $u \in L_2(S_a)$ satisfy $u = \tilde{T}_{\sqrt{\lambda} + i \theta} u$ (or $u = \tilde{T}_{\sqrt{\lambda} - i \theta} u$) in $L_2(S_a)$. Then $u = 0$ in $L_2(S_a)$.

**Proof.** Put $U(x) \equiv (T_{\sqrt{\lambda} + i \theta})(u) (= T_{\sqrt{\lambda}} u)(x)$. Then, by Lemmas 7.6, 7.7 and 2.11, we have

\[(4 + \lambda) U(x) = 0 \quad \text{on} \quad \{x; \ |x| > a\}, \quad U(x) = O\left(\frac{1}{|x|^2}\right) \quad \text{as} \quad |x| \to +\infty.\]

Thus, in view of Mizohata [13, Chap. VIII §5, Lemma 8.4], we have

\[U(x) \equiv 0 \quad \text{on} \quad \{x; \ |x| > a\}.\]

Since $U(x)$ is continuous on $R^3$ as mentioned in the proof of Lemma 7.7, we obtain

\[U(x) \equiv 0 \quad \text{on} \quad \{x; \ |x| \geq a\},\]

and hence

\[u(x) = (U|_{S_a})(x) = 0.\]

Similarly, the case that $u = \tilde{T}_{\sqrt{\lambda} - i \theta} u$ can be proven.

**Q.E.D.**

We are now in a position to make use of the Fredholm-Riesz theory of compact operators in a Hilbert space, according to which, if $T$ is a compact operator in a Hilbert space $X$, $1 - T$ is injective if and only if $(1 - T)^{-1}$ exists and belongs to $B(X)$ (see e.g. Riesz-Nagy [20, Chap. IV]). Thus, by Lemmas 2.4, 7.3 and 7.8 we have the following

**Lemma 7.9.** Let $z \in \Pi \setminus (\sigma_p(H) \cup \{0\})$. Then $(1 - T_{\sqrt{z}})^{-1}$ exists and belongs to $B(L_2(S_a))$.

**Lemma 7.10.** $(1 - T_{\sqrt{z}})^{-1}$ is a $B(L_2(S_a))$-valued continuous function of $z$ on $\Pi \setminus (\sigma_p(H) \cup \{0\})$.

**Proof.** The conclusion follows from Lemma 2.10 and the standard estimate

\[\|(1 - \tilde{T}_{\sqrt{z}})^{-1} - (1 - \tilde{T}_{\sqrt{z}})^{-1}\| \leq \frac{\|\tilde{T}_{\sqrt{z}} - \tilde{T}_{\sqrt{z}}\| \|1 - (1 - \tilde{T}_{\sqrt{z}})^{-1}\|}{1 - \|\tilde{T}_{\sqrt{z}} - \tilde{T}_{\sqrt{z}}\| \|1 - (1 - \tilde{T}_{\sqrt{z}})^{-1}\|}. \quad \text{Q.E.D.}\]

The above two lemmas imply Lemma 7.2. Therefore, Theorem 7.1 has now been proven.
Once the limiting absorption principle for $H$ is established, the absolute continuity of $H$ on $(0, \infty)$ readily follows from the same argument as Ikebe-Saitô [8]. Thus we have the following

**Theorem 7.11.** $E((0, \infty))H$ is an absolutely continuous operator, where $E(\cdot)$ is the spectral measure associated with $H$.

§8. Eigenfunction expansions

We shall proceed to show the eigenfunction expansion theorem. Our method is based on Kuroda [12] and Ikebe [6, 7].

We shall start with a well-known formula.

**Lemma 8.1.** Let $s > \frac{1}{2}$. Suppose that $u \in L^2_s(R^3)$ and $\mathcal{F}v \in C_0^\infty(R^3 \setminus \{0\})$. Then we have

\[
(u, W_{\pm}v) = \lim_{\varepsilon \to 0} \frac{\varepsilon}{\pi} \int_{-\infty}^{+\infty} (R(\lambda \pm i\varepsilon)u, R_0(\lambda \pm i\varepsilon)v) d\lambda.
\]

For the proof, see e.g. Kuroda [11, §5.4].

**Lemma 8.2.** Let $s > \frac{1}{2}$. Suppose that $u \in L^2_s(R^3)$ and $\mathcal{F}v \in C_0^\infty(R^3 \setminus \{0\})$ such that $\text{supp } \mathcal{F}v \subset \{\xi; \alpha < |\xi|^2 < \beta\}$ ($0 < \alpha < \beta$). Then we have

\[
(u, W_{\pm}v) = \lim_{\varepsilon \to 0} \frac{\varepsilon}{\pi} \int_{\alpha}^{\beta} (R(\lambda \pm i\varepsilon)u, R_0(\lambda \pm i\varepsilon)v) d\lambda.
\]

Here $\text{supp}$ means support.

**Proof.** (cf. Kuroda [12, p.151, Proposition 5.12]) Let $J = R \setminus [\alpha, \beta]$. By Lemma 8.1 we have only to show

\[
\lim_{\varepsilon \to 0} \frac{\varepsilon}{\pi} \int_{J} (R(\lambda \pm i\varepsilon)u, R_0(\lambda \pm i\varepsilon)v) d\lambda = 0.
\]

By Schwarz' inequality we have

\[
\left| \frac{\varepsilon}{\pi} \int_{J} (R(\lambda \pm i\varepsilon)u, R_0(\lambda \pm i\varepsilon)v) d\lambda \right|
\leq \left( \frac{\varepsilon}{\pi} \int_{J} \| R(\lambda \pm i\varepsilon)u \|^2 d\lambda \right)^{1/2} \left( \frac{\varepsilon}{\pi} \int_{J} \| R_0(\lambda \pm i\varepsilon)v \|^2 d\lambda \right)^{1/2}
\leq I_1(\varepsilon)^{1/2} \cdot I_2(\varepsilon)^{1/2}.
\]

Thus, to prove (8.3) it is sufficient to show

\[
\lim_{\varepsilon \to 0} I_2(\varepsilon) = 0,
\]

(8.4)

\[
I_1(\varepsilon) \leq \|u\|^2 \quad \text{for all } \varepsilon > 0.
\]

(8.5)
Using the spectral representation for $H_0$, we have
\begin{equation}
\frac{e}{\pi} \| R_0(\lambda \pm ie)v \|^2 = \int_{-\infty}^{+\infty} \frac{1}{\pi} \cdot \frac{1}{(\mu - \lambda)^2 + e^2} d(E_0(\mu)v, v),
\end{equation}
where $E_0(\cdot)$ denotes the spectral measure associated with $H_0$. Using the fact that $H_0$ is an absolutely continuous operator, we have
\begin{equation}
\frac{e}{\pi} \| R_0(\lambda \pm ie)v \|^2 = (P_\varepsilon \ast \rho)(\lambda),
\end{equation}
where $P_\varepsilon(\mu) = \frac{e}{\pi(\mu^2 + \varepsilon^2)}$ (the Poisson kernel), $\rho(\mu) = \frac{d}{d\mu}(E_0(\mu)v, v)$ and $\ast$ means convolution. Further, $\rho(\mu)$ belongs to $L_1(R^1)$ and $\rho(\mu) = 0$ for a.e. $\mu \in J$ since $E_0(J)v = 0$. Thus we obtain
\begin{equation}
I_2(e) = \int_J (P_\varepsilon \ast \rho)(\lambda)d\lambda = \int_J ((P_\varepsilon \ast \rho)(\lambda) - \rho(\lambda))d\lambda
\end{equation}
\begin{equation}
\leq \| P_\varepsilon \ast \rho - \rho \|_{L_1(R^1)} \to 0 \text{ as } e \downarrow 0,
\end{equation}
which implies (8.4). Let us show (8.5). As we got (8.6), we have
\begin{equation}
I_1(e) = \frac{e}{\pi} \int_J \| R(\lambda \pm ie)u \|^2 d\lambda = \int_J \int_{-\infty}^{+\infty} P_\varepsilon(\mu - \lambda)d(E(\mu)u, u)
\end{equation}
\begin{equation}
\leq \int_{-\infty}^{+\infty} d(E(\mu)u, u) \int_{-\infty}^{+\infty} P_\varepsilon(\mu - \lambda)d\lambda = \| u \|^2,
\end{equation}
where we used Fubini's theorem and the well-known properties of $P_\varepsilon(\mu)$ that
\begin{equation}
P_\varepsilon(\mu) > 0 \text{ for all } \mu \text{ and } \int_{-\infty}^{+\infty} P_\varepsilon(\mu)d\mu = 1.
\end{equation}
This implies (8.5).

Q.E.D.

Let us define the generalized Fourier transform $\mathcal{F}_\pm$ and the generalized eigenfunctions $\varphi_\pm(x, \xi)$ by
\begin{equation}
\mathcal{F}_\pm = \mathcal{F} W_\pm^*,
\end{equation}
\begin{equation}
\varphi_\pm(x, \xi) = e^{i\xi \cdot x} + [T^{(1)}_{\pm|\xi|}(1 - q \tilde{T}^{(1)}_{\pm|\xi|})^{-1}](e^{i\xi \cdot q})](x)
\end{equation}
for $(x, \xi) \in R^3 \times (R^3 \setminus \{0\})$, respectively. We should note here that by Lemmas 2.14 and 7.2 $(1 - q \tilde{T}_{\pm|\xi|}^{(1)})^{-1}$ exist and satisfy the relations
\begin{equation}
(1 - q \tilde{T}^{(1)}_{\pm|\xi|})^{-1} = [(1 - \tilde{T}_{\pm|\xi|})^{-1}]^* \text{ for } \xi \in R^3 \setminus \{0\}.
\end{equation}
We also remark that $\varphi_\pm(x, \xi)$ are regarded as the generalized eigenfunctions of $H$ in the sense stated in Theorem 8.6. Further, they are seen to be the integral kernels of $\mathcal{F}_\pm$ by the following theorem.
Theorem 8.3. For any \( u \in L^2(\mathbb{R}^3) \), \( \mathcal{T}_\pm \) have the form

\[
(\mathcal{T}_\pm u)(\xi) = \lim_{R \to +\infty} (2\pi)^{-3/2} \int_{|x| \leq R} \frac{\varphi_\pm(x, \xi) u(x) dx}{1 + |x|^2},
\]

where i.i.m. means the limit in the mean.

Proof. Let \( u \in C^\infty_0(\mathbb{R}^3) \) and \( v \in C^\infty_0(\mathbb{R}^3 \setminus \{0\}) \) such that \( \text{supp } v \subset \{x; \alpha < |x|^2 < \beta\} \) \((0 < \alpha < \beta)\). Using (8.7), Lemma 8.2 and (7.2), we have

\[
(\mathcal{T}_\pm u, v) = (u, W_{\pm} \mathcal{F} v)
\]

\[
= \lim_{\epsilon \to 0} \frac{\epsilon}{\pi} \int_0^\beta (R_0(\lambda \pm i\epsilon) u, R_0(\lambda \pm i\epsilon) \mathcal{F} v) d\lambda
\]

\[
+ \lim_{\epsilon \to 0} \frac{\epsilon}{\pi} \int_0^\beta (T_{\sqrt{\lambda \pm i\epsilon}}(1 - \tilde{T}_{\sqrt{\lambda \pm i\epsilon}})^{-1} \gamma R_0(\lambda \pm i\epsilon) u, R_0(\lambda \pm i\epsilon) \mathcal{F} v) d\lambda
\]

\[
= \lim_{\epsilon \to 0} J_1(\epsilon) + \lim_{\epsilon \to 0} J_2(\epsilon).
\]

For the first term of the right hand side of (8.11), as is well known (see e.g. Kuroda [12, p.54]), we have

\[
\lim_{\epsilon \to 0} J_1(\epsilon) = \int_{|\xi|^2 < \beta} d\xi \, (\mathcal{F} u)(\xi) v(x) = (\mathcal{F} u, v).
\]

We shall consider the second term. In view of Parseval's equality and (2.10), we have

\[
(\mathcal{T}_{\sqrt{\lambda \pm i\epsilon}}(1 - \tilde{T}_{\sqrt{\lambda \pm i\epsilon}})^{-1} \gamma R_0(\lambda \pm i\epsilon) u, R_0(\lambda \pm i\epsilon) \mathcal{F} v)
\]

\[
= \left( - \frac{1}{|\cdot|^2 - (\lambda \pm i\epsilon)} \mathcal{F}_u q(1 - \tilde{T}_{\sqrt{\lambda \pm i\epsilon}})^{-1} \gamma R_0(\lambda \pm i\epsilon) u, \frac{v}{|\cdot|^2 - (\lambda \pm i\epsilon)} \right)
\]

\[
= \int_{\mathbb{R}^3} d\xi \, \left\{ \frac{1}{|\xi|^2 - (\lambda \pm i\epsilon)} (2\pi)^{-3/2} \int_{S_n} dS_y e^{-i\xi y} q(y) \times \right. 
\]

\[
\times [(1 - \tilde{T}_{\sqrt{\lambda \pm i\epsilon}})^{-1} \gamma R_0(\lambda \pm i\epsilon) u](y) \} \frac{\overline{v(\xi)}}{|\xi|^2 - (\lambda \pm i\epsilon)}
\]

\[
= - (2\pi)^{-3/2} \int_{\mathbb{R}^3} d\xi \, \left\{ \frac{\overline{v(\xi)}}{|\xi|^2 - (\lambda \pm i\epsilon)} \times 
\right. 
\]

\[
\times [(1 - \tilde{T}_{\sqrt{\lambda \pm i\epsilon}})^{-1} \gamma R_0(\lambda \pm i\epsilon) u, e^{i\xi \cdot q}]a.
\]

Since \( (1 - \tilde{T}_{\sqrt{\lambda \pm i\epsilon}})^{-1} \gamma R_0(\lambda \pm i\epsilon) u, e^{i\xi \cdot q} \) are continuous in \( \lambda \) and \( \epsilon \) on \([\alpha, \beta][0, 1]\) by Lemmas 7.2, 2.9 with \( q(x) \equiv 1 \) and the fact that \( \gamma R_0(z) = -(T_{\frac{1}{2}}) \), we have
\[
\lim_{\epsilon \to 0} \int_\alpha^\beta \frac{d\lambda}{\pi} \frac{e^{\lambda \epsilon}}{(\lambda - |\xi|^2)^2 + \epsilon^2} \left((1 - \widehat{T}_{\lambda+i\epsilon})^{-1} \gamma R_0(\lambda \pm i\epsilon)u, e^{i\xi}q\right)_a
\]
\[
= \left((1 - \widehat{T}_{\pm|\xi|})^{-1} \gamma R_0(|\xi|^2 \pm i0)u, e^{i\xi}q\right)_a,
\]
where we have made use of the well-known relation
\[
\lim_{\epsilon \to 0} \int_\alpha^\beta \frac{d\lambda}{\pi} \frac{e^{\lambda \epsilon}}{(\lambda - a)^2 + \epsilon^2} f(\lambda, \epsilon)
\]
\[
= \begin{cases} 
0 & \text{if } a < \alpha \text{ or } \beta < a \\
(f(a, 0) & \text{if } \alpha < a < \beta.
\end{cases}
\]
in which \(f(\lambda, \epsilon)\) is a continuous function of \((\lambda, \epsilon)\) for \((\lambda, \epsilon) \in [\alpha, \beta] \times [0, 1]\) (see e.g. Titchmarsh [22, p.31]). Further, from (8.9) and the fact that \(\gamma R_0(z) = -(T_{\frac{1}{2}}e)^*\), it follows that
\[
\left((1 - \widehat{T}_{\pm|\xi|})^{-1} \gamma R_0(|\xi|^2 \pm i0)u, e^{i\xi}q\right)_a
\]
\[
= (\gamma R_0(|\xi|^2 \pm i0)u, (1 - q \widehat{T}_{\frac{1}{2}}e)^{-1}(e^{i\xi}q))_a
\]
\[
= -\int_{\mathbb{R}^3} dx \ u(x) \left[T_{\frac{1}{2}}e(1 - q \widehat{T}_{\frac{1}{2}}e)^{-1}(e^{i\xi}q)(x)\right],
\]
where we have used Fubini’s theorem in the last equality. Thus, making use of Fubini’s theorem and the dominated convergence theorem, we see from (8.13), (8.14) and (8.15) that
\[
\lim_{\epsilon \to 0} J_2(\epsilon)
\]
\[
= (2\pi)^{-3/2} \int_{\mathbb{R}^3} d\xi \left(\int_{\mathbb{R}^3} dx \ u(x) \left[T_{\frac{1}{2}}e(1 - q \widehat{T}_{\frac{1}{2}}e)^{-1}(e^{i\xi}q)(x)\right]v(\xi)\right).
\]
Now we have by (8.8), (8.11) (8.12) and (8.16)
\[
(\mathcal{F}_\pm u)(\xi) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} dx \ \overline{\varphi_\pm(x, \xi)} u(x) \quad \text{for any } u \in C_0^\infty(\mathbb{R}^3).
\]
Since \(C_0^\infty(\mathbb{R}^3)\) is dense in \(L_2(\mathbb{R}^3)\), the conclusion follows. \(\text{Q.E.D.}\)

To prove the continuity and boundedness of \(\varphi_\pm(x, \xi)\), we need the following lemma.

**Lemma 8.4.** Let \(K\) be a compact set in \(\mathbb{R}^3\). Let \(f(x, \xi)\) be a continuous function of \((x, \xi) \in S_a \times K\). Then \((T_{\frac{1}{2}}e f(\cdot, \xi))(x)\) is bounded and continuous in \((x, \xi) \in \mathbb{R}^3 \times K\). In particular, \((\widehat{T}_{\frac{1}{2}}e f(\cdot, \xi))(x)\) is continuous in \((x, \xi) \in S_a \times K\).

**Proof.** By Lemma 2.1 we have
\begin{align*}
|T_{\pm\infty}^{(1)} f(\cdot, \bar{\xi})|(x) &\leq \frac{1}{4\pi} \max_{(y, \xi) \in S_{a} \times K} |f(y, \bar{\xi})| \int_{S_{a}} \frac{1}{|x - y|} dS_{y} \\
&= a \max_{(y, \xi) \in S_{a} \times K} \left| f(y, \bar{\xi}) \right| \frac{a + |x| - |a - |x||}{|x|} \leq a \max_{(y, \xi) \in S_{a} \times K} \left| f(y, \bar{\xi}) \right|, 
\end{align*}

which proves the boundeness of \((T_{\pm\infty}^{(1)} f(\cdot, \bar{\xi}))(x)\). Let us show the continuity. Let us introduce the functions \(G_{\pm}^{(1)}(x, \bar{\xi})\) with a real parameter \(\epsilon\) by

\[
G_{\pm}^{(1)}(x, \bar{\xi}) = -\frac{1}{4\pi} \int_{S_{a} \cap \{|y - x| > \epsilon\}} \frac{e^{\pm i|\bar{\xi} - y|}}{|x - y|} f(y, \bar{\xi}) dS_{y},
\]

\((x, \bar{\xi}) \in \mathbb{R}^{3} \times K, \epsilon > 0\). It is easily seen that for each \(\epsilon\), \(G_{\pm}^{(1)}(x, \bar{\xi})\) is continuous in \((x, \bar{\xi})\) in \(\mathbb{R}^{3} \times K\). Further, \(G_{\pm}^{(1)}(x, \bar{\xi})\) uniformly converges to \((T_{\pm\infty}^{(1)} f(\cdot, \bar{\xi}))(x)\) when \(\epsilon \downarrow 0\). In fact, we have for a sufficiently small \(\epsilon\)

\[
|G_{\pm}^{(1)}(x, \bar{\xi}) - (T_{\pm\infty}^{(1)} f(\cdot, \bar{\xi}))(x)| \leq \max_{(y, \xi) \in S_{a} \times K} |f(y, \bar{\xi})| \int_{S_{a} \cap \{|y - x| \leq \epsilon\}} \frac{1}{4\pi |x - y|} dS_{y} \\
= \left\{ \begin{array}{ll}
\max_{(y, \xi) \in S_{a} \times K} |f(y, \bar{\xi})| & \text{if } |a - |x|| < \epsilon \\
0 & \text{if } |a - |x|| \geq \epsilon
\end{array} \right.
\]

where we have used (2.6). Thus the continuity of \((T_{\pm\infty}^{(1)} f(\cdot, \bar{\xi}))(x)\) has been proven. Since \((\bar{T}_{\pm\infty}^{(1)} f(\cdot, \bar{\xi}))(x) = (T_{\pm\infty}^{(1)} f(\cdot, \bar{\xi}))(x) \in S_{a}\), the assertion for \((\bar{T}_{\pm\infty}^{(1)} f(\cdot, \bar{\xi}))(x)\) holds.

**Theorem 8.5.** \(\varphi_{\pm}(x, \bar{\xi})\) is continuous in \((x, \bar{\xi}) \in \mathbb{R}^{3} \times (\mathbb{R}^{3} \setminus \{0\})\) and bounded on \(\mathbb{R}^{3} \times K\), where \(K\) is any compact set in \(\mathbb{R}^{3} \setminus \{0\}\).

**Proof.** Put \(\psi_{\pm}(x, \bar{\xi}) = [(1 - q \bar{T}_{\pm\infty}^{(1)})(e^{i\bar{\xi} q})^{-1}] (x)\). Then \(\psi_{\pm}(\cdot, \bar{\xi})\) is an \(L_{2}(S_{a})\)-valued continuous function of \(\bar{\xi} \in \mathbb{R}^{3} \setminus \{0\}\) by Lemma 7.2 and (8.9). If we show that \(\psi_{\pm}(x, \bar{\xi})\) is a continuous function of \((x, \bar{\xi}) \in S_{a} \times K\), the conclusion follows from (8.8) and Lemma 8.4. Since \(\psi_{\pm}(x, \bar{\xi})\) satisfy the equation

\[
\psi_{\pm}(x, \bar{\xi}) = e^{i\bar{\xi} q x} + (q \bar{T}_{\pm\infty}^{(1)})(\psi_{\pm}(x, \bar{\xi}),
\]

we have, using (8.17) repeatedly,

\[
\psi_{\pm}(x, \bar{\xi}) = e^{i\bar{\xi} q x} + [(q \bar{T}_{\pm\infty}^{(1)})(e^{i\bar{\xi} q})](x) \\
+ [(q \bar{T}_{\pm\infty}^{(1)})^{2}(e^{i\bar{\xi} q})](x) + [(q \bar{T}_{\pm\infty}^{(1)})^{3}(e^{i\bar{\xi} q})](x) \\
+ [(q \bar{T}_{\pm\infty}^{(1)})^{4}\psi_{\pm}(\cdot, \bar{\xi})](x).
\]

\[\text{Q.E.D.}\]
It follows from Lemma 8.4 that the first four terms of the right hand side of (8.18) are continuous in $(x, \xi)\in S_a \times K$. Thus the proof of this theorem is reduced to showing the continuity of $[(qT^{(1)}_{+\xi})^4 \psi_\pm (\cdot, \xi)](x)$. Let $\mathcal{K}_\pm (x, y; \xi)$ be the integral kernel of $(qT^{(1)}_{+\xi})^4$. Then, in the same way as we proved Lemma 7.5, we can show that $\mathcal{K}_\pm (x, y; \xi)$ is continuous in $(x, y)$ on $S_a \times S_a \times K$. We consider the difference

\begin{equation}
[(qT^{(1)}_{+\xi})^4 \psi_\pm (\cdot, \xi)](x) - [(qT^{(1)}_{+\xi}\xi_0)^4 \psi_\pm (\cdot, \xi_0)](x_0)
= \int_{S_a} (\mathcal{K}_\pm (x, y; \xi) - \mathcal{K}_\pm (x_0, y; \xi_0)) \psi_\pm (x, \xi)dS_y
+ \int_{S_a} (\mathcal{K}_\pm (x, y; \xi) - \mathcal{K}_\pm (x_0, y; \xi_0)) \psi_\pm (x, \xi)dS_y
+ \int_{S_a} \mathcal{K}_\pm (x_0, y; \xi_0)(\psi_\pm (x, \xi) - \psi_\pm (x, \xi_0))dS_y
= J_1 + J_2 + J_3.
\end{equation}

$J_i (i = 1, 2, 3)$ are estimated as follows:

\begin{equation}
|J_1| \leq \max_{y \in S_u} |\mathcal{K}_\pm (x, y; \xi) - \mathcal{K}_\pm (x_0, y; \xi_0)| \int_{S_u} |\psi_\pm (y, \xi)|dS_y
\leq \max_{y \in S_u} |\mathcal{K}_\pm (x, y; \xi) - \mathcal{K}_\pm (x_0, y; \xi_0)|(4\pi a^2)^{1/2} \|\psi_\pm (\cdot, \xi)\|_a,
\end{equation}

\begin{equation}
|J_2| \leq \max_{y \in S_u} |\mathcal{K}_\pm (x_0, y; \xi) - \mathcal{K}_\pm (x_0, y; \xi_0)|(4\pi a^2)^{1/2} \|\psi_\pm (\cdot, \xi)\|_a,
\end{equation}

\begin{equation}
|J_3| \leq \max_{y \in S_u} |\mathcal{K}_\pm (x_0, y; \xi_0)|(4\pi a^2)^{1/2} \|\psi_\pm (\cdot, \xi) - \psi_\pm (x, \xi_0)\|_a,
\end{equation}

It follows from (8.19) and (8.20) that $[(qT^{(1)}_{+\xi})^4 \psi_\pm (\cdot, \xi)](x)$ is continuous in $(x, \xi)$ on $S_a \times K$. Thus the theorem follows. Q.E.D.

**Theorem 8.6.** Let $\xi \in R^3 \setminus \{0\}$. Then $\varphi_\pm (x, \xi)$ satisfy the following equations

\begin{equation}
\varphi_\pm (x, \xi) = e^{i\xi \cdot x} - \frac{1}{4\pi} \int_{S_u} e^{\frac{\pi^2 |\xi| (|x - y|)}{|x - y|}} q(y)\varphi_\pm (y, \xi)dS_y
\end{equation}

(the Lippmann-Schwinger equation).

\begin{equation}
\int_{R^3} \varphi_\pm (x, \xi)(-\Delta - |\xi|^2)v(x)dx + \int_{S_u} q(x)\varphi_\pm (x, \xi)v(x)dS_x = 0 \quad \text{for any} \ v \in C_0^\infty (R^3).
\end{equation}

**Proof.** By (8.8) we have
which implies (8.21). Let us show (8.22). In view of (8.21), \( \varphi_\pm(x, \xi) \) can be written as
\[
\varphi_\pm(x, \xi) = e^{i\xi \cdot x} + (T_{\pm|\xi|}(\varphi_\pm|_{S_\xi})(\cdot, \xi))(x),
\]
Therefore, by Lemma 2.11 we have for any \( \psi \in C_0^\infty(\mathbb{R}^3) \)
\[
\int_{\mathbb{R}^3} \varphi_\pm(x, \xi)(-\Delta - |\xi|^2)\psi(x)dx
\]
\[
= \int_{\mathbb{R}^3} (\varphi_\pm(x, \xi) - e^{i\xi \cdot x})(-\Delta - |\xi|^2)\psi(x)dx
\]
\[
= \int_{\mathbb{R}^3} (T_{\pm|\xi|}\varphi_\pm|_{S_\xi})(\cdot, \xi))(x)(-\Delta - |\xi|^2)\psi(x)dx
\]
\[
= -\int_{S_\xi} q(x)\varphi_\pm(x, \xi)\psi(x)ds_x,
\]
which implies (8.22). Q.E.D.

**Theorem 8.7.** \( \mathcal{F}_\pm \) are partially isometric operators with the domain \( E((0, \infty)) L_2(\mathbb{R}^3) \) and the range \( L_2(\mathbb{R}^3) \). Further, \( \mathcal{F}_\pm \) have the following properties: Let \( \Lambda \) be any Borel set on \( \mathbb{R} \). Then,
\[
(8.23) \mathcal{F}_\pm E(\Lambda) = \chi_{\{\xi: \xi^2 \in \Lambda\}} \mathcal{F}_\pm,
\]
where \( \chi_A \) denotes the operator of multiplication by the characteristic function of \( A \). In particular, if \( u \in L_2(\mathbb{R}^3) \), and \( \alpha \) and \( \beta \) are such that \( 0 < \alpha < \beta \), then
\[
(8.24) \| E((\alpha, \beta))u \|^2 = \int_{\xi<\beta} |(\mathcal{F}_\pm u)(\xi)|^2 d\xi,
\]
\[
(8.25) E((\alpha, \beta))u(x) = (2\pi)^{-3/2} \int_{\xi<\beta} (\mathcal{F}_\pm u)(\xi) \varphi_\pm(x, \xi) d\xi.
\]

**Proof.** First, let us recall the well-known relations
\[
(8.26) \mathcal{F} E_0(A)\mathcal{F}^* = \chi_{\{\xi: \xi^2 \in \Lambda\}},
\]
\[
(8.27) E(A)W_\pm = W_\pm E_0(A),
\]
where \( A \) is a Borel set on \( \mathbb{R} \) (see e.g. Kuroda [12, §3.4, Theorem 2]). Putting \( A = (0, \infty) \) in (8.27), we have
\[
E((0, \infty))W_\pm = W_\pm E_0((0, \infty)) = W_\pm,
\]
from which it follows that
\[
\text{Ran}(W_\pm) \subset E((0, \infty))L_2(\mathbb{R}^3).
\]
On the other hand, it follows from Theorem 7.11 that \( E((0, \infty))L_2(\mathbb{R}^3) \) is included in the absolute continuous subspace \( \mathcal{T}_a(H) \) of \( L_2(\mathbb{R}^3) \) relative to \( H \). Therefore, we have by Theorem 4.1.
\[
E((0, \infty))L_2(\mathbb{R}^3) \subset \mathcal{T}_a(H) = \text{Ran}(W_\pm) \subset E((0, \infty))L_2(\mathbb{R}^3),
\]
and hence
\[
\text{Ran}(W_\pm) = E((0, \infty))L_2(\mathbb{R}^3).
\]
This implies that \( W_\pm \) are partially isometric operators with the domain \( L_2(\mathbb{R}^3) \) and the range \( E((0, \infty))L_2(\mathbb{R}^3) \). Thus, it follows from (8.7) that \( \mathcal{T}_\pm \) are partially isometric operators with the domain in \( E((0, \infty))L_2(\mathbb{R}^3) \) and the range \( L_2(\mathbb{R}^3) \). By (8.7), (8.26) and (8.27) we have
\[
\mathcal{T}_\pm E(A) = \mathcal{T} \mathcal{W}_\pm E(A) = \mathcal{W} E_0(A) W_\pm = \mathcal{W} E_0(A) \mathcal{F}_\pm \mathcal{W}_\pm = \chi_{\{ \xi : |\xi|^2 < \beta \}}(\xi) \mathcal{T}_\pm.
\]
This proves (8.23). Let us show (8.24) and (8.25). Since \( \mathcal{T}_\pm \) are partially isometric operators with domain \( E((0, \infty))L_2(\mathbb{R}^3) \), it follows that
\[
\mathcal{T}_\pm \mathcal{T}_\pm = E((0, \infty)).
\]
Therefore, we have by (8.23) \( (A = (\alpha, \beta) \subset (0, \infty)) \)
\[
E((\alpha, \beta)) = \mathcal{T}_\pm \chi_{\{ \xi : |\xi|^2 < \beta \}}(\xi) \mathcal{T}_\pm,
\]
from which (8.24) and (8.25) follow immediately. Q.E.D.

We shall now proceed to the eigenfunction expansion theorem.

**Theorem 8.8.** Let \( \lambda_1, \lambda_2, \cdots \) be the nonpositive eigenvalues of \( H \) (counting multiplicity) and \( \{ \varphi_1, \varphi_2, \cdots \} \) a corresponding orthonormal system of eigenfunctions of \( H \), if any. Then, for any \( u \in L_2(\mathbb{R}^3) \) we have the following expansion formula
\[
(8.28) \quad u(x) = \sum_n (u, \varphi_n)\varphi_n(x) + \text{l.i.m. \; (2\pi)^{-3/2} \int}_{\alpha, \beta, \gamma} d\xi \chi_{\{ \xi : |\xi|^2 < \beta \}}(\xi) \mathcal{T}_\pm u(\xi),
\]
Further, \( u \in \text{Dom}(H) \) if and only if \( |\cdot|^2 \mathcal{T}_\pm u \in L_2(\mathbb{R}^3) \). In this case, we have the following representation of \( H \)
\[
(8.29) \quad Hu(x) = \sum_n \lambda_n (u, \varphi_n)\varphi_n(x) + \text{l.i.m. \; (2\pi)^{-3/2} \int}_{\alpha, \beta, \gamma} d\xi \chi_{\{ \xi : |\xi|^2 < \beta \}}(\xi) \mathcal{T}_\pm u(\xi),
\]
for \( u \in \text{Dom}(H) \).

**Proof.** According to Theorem 5.1 \( E(( - \infty, 0]) L_2(\mathbb{R}^3) \) is spanned by
\{\varphi_1, \varphi_2, \cdots \}. Therefore, we have for any \(u \in L_2(\mathbb{R}^3)\)

\begin{equation}
(8.30) \quad u(x) = \sum_{n} (u, \varphi_n)\varphi_n(x) + \text{i.m.} \ E((\alpha, \beta))u(x). \nonumber
\end{equation}

(8.28) follows from (8.30) and Theorem 8.7, (8.25). Using Theorem 8.7, (8.24), we have

\begin{equation}
(8.31) \quad \int_{0}^{\infty} \lambda^2 \|E(\lambda)u\|^2 = \int_{\mathbb{R}^3} |\xi|^4 |(\mathcal{F}_\pm u)(\xi)|^2 \, d\xi. \nonumber
\end{equation}

On the other hand, it follows from Theorem 6.5 that

\begin{equation}
(8.32) \quad E((-\infty, 0])L_2(\mathbb{R}^3) \subset \text{Dom}(H). \nonumber
\end{equation}

Thus, we have by (8.31) and (8.32)

\begin{equation}
(8.33) \quad \text{Dom}(H) = \{u; u \in L_2(\mathbb{R}^3), \ |\cdot|^2 \mathcal{F}_\pm u \in L_2(\mathbb{R}^3)\}. \nonumber
\end{equation}

Finally, let us show (8.29). From the “intertwining” relation \(W_\pm H_0 \subset HW_\pm\) (see e.g. Kuroda [11, §3.4]) and (8.33), it follows that

\begin{equation}
(8.34) \quad \mathcal{F}_\pm H = |\cdot|^2 \mathcal{F}_\pm. \nonumber
\end{equation}

Therefore, if we replace \(u\) by \(Hu\) in (8.28), (8.29) follows rom (8.34) and the fact that \((Hu, \varphi_n) = \lambda_n(u, \varphi_n)\). Thus the theorem has been proven. \( \Box \)

\textbf{Added in proof.} The proof of the assertion \(u \in H^2(\mathbb{R}^3 \setminus S_0)\) is incomplete. But this can be proven by the standard argument for showing the global regularity for solutions to elliptic boundary-value problems (see e.g. Mizohata [13, Chap 3, §12]) if one takes into account the already known facts that \(\Delta u \in L_2(\mathbb{R}^3)\), \(u \in H^1(\mathbb{R}^3) = \text{Dom}[h]\) and that \(h[u, v] = (Hu, v)\) for any \(v \in H^1(\mathbb{R}^3)\).

\textbf{References}

[5] C. L. Dolph, J. B. McLeod and D. Thoe, The analytic continuation of the resolvent kernel and...


