J. Math. Kyoto Univ. (JMKYAZ)
31-2 (1991) 573-582

The explosion property of holomorphic diffusion processes on a bounded pseudoconvex domain in C^n and its applications

By

Setsuo TANIGUCHI

0. Introduction

Let D be a bounded pseudoconvex domain in $\boldsymbol{C}^n$ and m be an everywhere dense positive Radon measure on D. The aim of this paper is to give sufficient conditions for a transient $C_0^\infty(D)$-regular m-symmetric holomorphic diffusion process to explode in terms of the symmetrizing measure m. As an application, the boundary behaviour of plurisubharmonic (psh in abbreviation) functions along the sample path of the diffusion process will be studied.

Let $\boldsymbol{M}=(Z_t, \zeta, P_z)$ be a $C_0^\infty(D)$-regular m-symmetric holomorphic diffusion process on D with the life time ζ, i.e. a $C_0^\infty(D)$-regular m-symmetric diffusion process such that $h(Z_{t\wedge\tau_K})$ is a martingale under P_z, $\boldsymbol{M}$-q.e. $z\in D$, for every compact $K\subset D$ and holomorphic function h on D, where $\tau_K=\inf\{t>0: Z_t\notin K\}$ and by "$\boldsymbol{M}$-q.e." we have meant "except for a set of zero capacity with respect to the 1-capacity of $\boldsymbol{M}$". For definitions of symmetric diffusion processes and the associated 1-capacity, see [5]. Assume that $\boldsymbol{M}$ is transient:

$$(0.1)\qquad \int_0^\infty T_t f(z)dt<\infty, \qquad m\text{-a.e. for every } f\in L^1(D; m) \text{ with } f\geqq 0,$$

where $\{T_t\}$ is the semigroup associated with $\boldsymbol{M}$. As was seen in [7, Appendix], if, in addition, $m(D)<+\infty$, then $E_z[\zeta]=\int_0^\infty T_t 1(z)dt<+\infty$ m-a.e., where E_z stands for the expectation with respect to P_z. Hence, in this case, $\boldsymbol{M}$ explodes; $P_z[\zeta<+\infty]=1$, $\boldsymbol{M}$-q.e. $z\in D$.

In this paper, we will show that $\boldsymbol{M}$ explodes if D has a defining function φ psh and continuous on a neighbourhood of $\bar{D}$ such that $\int_D|\varphi(z)|m(dz)<\infty$. See Theorem 1.1. Thus, we can weaken the assumption that $m(D)<+\infty$. Moreover, even if D has no global defining function, we will see in Corollary 1.1 that if it has a C^2 boundary, then there is an $\eta_0>0$, depending only on D, such that the explosion of $\boldsymbol{M}$ is sure whenever $\int_D \operatorname{dist}(z; \partial D)^\eta m(dz)<\infty$ for some $\eta<\eta_0$.

We now consider an application of the above observation. In the case when D is

Communicated by Prof. S. Watanabe, September 9, 1989

a unit disk in $\boldsymbol{C}^1$, deeply studied is the boundary behaviour of harmonic functions along the paths of the Brownian motion conditioned to exit D at $\xi\in\partial D$. For example see [4]. Bañuelos and Øksendal ([1]) investigated the similar problem for suitable diffusion processes when D is a unit ball in $\boldsymbol{C}^n$. Moreover, if u is a bounded psh function on D, then the submartingale convergence theorem implies that

$$(0.2)\qquad P_z[\lim_{t\uparrow\zeta}u(Z_t) \text{ exists in } (-\infty, \infty)]=1, \qquad \boldsymbol{M}\text{-q.e. } z,$$

because $u(Z_{t\wedge\tau_K})$ is a bounded submartingale with respect to P_z, $\boldsymbol{M}$-q.e. See [6] and [7].

We will establish the following criterion for (0.2) to hold, which is applicable to unbounded psh functions: the identity (0.2) holds for psh u if D has a defining function φ psh and continuous on a neighbourhood of $\bar{D}$ and u satisfies that $\inf_{z\in D}u(z)>-\infty$ and

$$(0.3)\qquad \int_D|\varphi|\,dd^cu\wedge\theta<\infty,$$

where θ is the unique closed positive current of bidegree $(n-1, n-1)$ such that the Dirichlet form $\mathfrak{E}$ on $L_2(D; m)$ of $\boldsymbol{M}$ is given by

$$(0.4)\qquad \mathfrak{E}(f, g)=\int_D df\wedge d^cg\wedge\theta, \qquad f, g\in C_0^\infty(D),$$

and $dd^cu\wedge\theta$ is the positive Radon measure on D defined by $\int_D f\,dd^cu\wedge\theta=\int_D u\,dd^cf\wedge\theta$, $f\in C_0^\infty(D)$. See Theorem 2.1. In the case where ∂D is C^2, the condition (0.3) with $\operatorname{dist}(z; \partial D)^\gamma$ for $|\varphi|$ implies the similar conclusion. See also Theorem 2.1. As will be seen in Example 3.4, there are an unbounded psh u and a holomorphic diffusion process $\boldsymbol{M}$ such that (0.3) is satisfied and hence (0.2) holds. Moreover, it will be seen in Example 3.5 that (0.2) does not hold without the assumption (0.3) in general.

The organization of this paper is as follows. In Section 1, sufficient conditions for a holomorphic diffusion process to explode will be given. Section 2 will be devoted to the study of the boundary behaviour of psh functions along the sample path of the holomorphic diffusion process. In Section 3, we present several examples to illustrate our results.

1. The explosion property

In this section, we will give sufficient conditions for holomorphic diffusion processes to explode.

Let D be a bounded pseudoconvex domain in $\boldsymbol{C}^n$ and m be an everywhere dense positive Radon measure on D. We denote by $\mathfrak{H}\mathfrak{D}(D, m)$ the space of $C_0^\infty(D)$-regular m-symmetric holomorphic diffusion processes on D. Our goal will be:

Theorem 1.1. *Let D and m be as above. Assume that $\boldsymbol{M}=(Z_t, \zeta, P_z)\in\mathfrak{H}\mathfrak{D}(D, m)$ is transient. Then, the explosion of $\boldsymbol{M}$ is sure if the following condition is satisfied:*

(C.1) $D=\{z\in\Omega : \varphi(z)<0\}$ *for some open* $\Omega\supset\bar{D}$ *and psh* $\varphi\in C(\Omega)$,

and $\int_D|\varphi(z)|\,m(dz)<\infty$.

The theorem is an immediate consequence of the following lemma.

Lemma 1.1. *Let D and m be as before. Assume that $\boldsymbol{M}=(Z_t, \zeta, P_z)\in\mathfrak{H}\mathfrak{D}(D, m)$ is transient and that there is a continuous function $u:\bar{D}\to(-\infty, 0]$ such that*

(1.1) $$u \text{ is psh on } D,$$

(1.2) $$u(z)=0 \quad \text{for } z\in\partial D \text{ and } u(z)<0 \text{ for } m\text{-a.e. } z\in D,$$

(1.3) $$\int_D |u(z)|m(dz)<\infty.$$

Then, $\boldsymbol{M}$ explodes.

Proof. We first notice that the martingale convergence theorem yields that

(1.4) $$P_z[\lim_{t\uparrow\zeta} Z_t \text{ exists}]=1, \qquad \boldsymbol{M}\text{-q.e. } z\in D,$$

because D is bounded. Combining this with the transience property of $\boldsymbol{M}$, we can conclude that

(1.5) $$P_z[\lim_{t\uparrow\zeta} Z_t\in\partial D]=1, \qquad \boldsymbol{M}\text{-q.e. } z\in D.$$

We next recall that the plurisubharmonicity of u implies the inequality:

(1.6) $$u(z)\leqq E_z[u(Z_{t\wedge\tau_K})], \qquad \boldsymbol{M}\text{-q.e. } z\in D,$$

where $K\subset D$ is compact and $\tau_K=\inf\{t>0: Z_t\notin K\}$. See [7]. Therefore, by virtue of (1.2) and (1.5), letting $K\uparrow D$ in (1.6), we obtain that

(1.7) $$u(z)\leqq E_z[u(Z_t): t<\zeta]=T_t u(z), \qquad \boldsymbol{M}\text{-q.e. } z\in D.$$

Observe that (1.2) and (1.5) also imply that

(1.8) $$\lim_{t\uparrow+\infty} E_z[u(Z_t): t<\zeta]=0.$$

Thus, by Lebesgue's dominated convergence theorem, we can conclude from (1.3), (1.7) and (1.8) that

(1.9) $$\lim_{t\uparrow+\infty}\int_D T_t u(z)m(dz)=0.$$

On the other hand, since the semigroup $\{T_t\}$ is symmetric, we have that

(1.10) $$\int_D T_t u(z)m(dz)=\int_D u(z)P_z[t<\zeta]m(dz).$$

Letting $t\uparrow+\infty$, it follows from (1.9) that

(1.11) $$\int_D u(z)P_z[\zeta=+\infty]m(dz)=0.$$

This and the assumption (1.2) yield that

(1.12) $$P_z[\zeta=+\infty]=0, \qquad m\text{-a.e. } z\in D.$$

By [5: Lemma 4.2.5], this implies that $\boldsymbol{M}$ explodes.

Lemma 1.1 yields another sufficient condition for $\boldsymbol{M}$ to explode:

Corollary 1.1. *Let D be a bounded domain in $\boldsymbol{C}^n$ and m be an everywhere dense positive Radon measure on D. Assume that $\boldsymbol{M}\in\mathfrak{H}\mathfrak{D}(D, m)$ is transient. Then $\boldsymbol{M}$ explodes if the following condition* (C.2) *is fulfilled:*

(C.2) *the boundary ∂D is C^2 (i.e. for each $z\in\partial D$, there are an open set U and a $\phi\in C^2(U)$ such that $D\cap U=\{\phi<0\}$ and $d\phi\neq 0$ on $U\cap\partial D$) and $\int_D \operatorname{dist}(z\,;\partial D)^{\eta} m(dz)<\infty$ for some $\eta<(4ML+1)^{-2}$, where*

$$L=\sup\{|z|: z\in D\} \tag{1.13}$$

$$M=\sup\{(\textstyle\sum_\alpha\{\sum_\beta(\partial^2\sigma(z)/\partial z^\alpha\partial\bar{z}^\beta)\xi^\beta\}^2)^{1/2}: |\xi|=1,\ z\in\partial D\}, \tag{1.14}$$

and $\sigma(z)=-\operatorname{dist}(z\,;\partial D)$ if $z\in D$ and $=\operatorname{dist}(z\,;\partial D)$ if $z\notin D$.

The above σ is C^2 near the boundary since ∂D is C^2.

Proof of Corollary 1.1. Suppose that (C.2) is fulfilled. On account of the above lemma, the proof will be completed once we have shown the existence of a continuous function $u:\bar{D}\to(-\infty, 0]$ such that the assumptions (1.1) and (1.2) are satisfied and

$$|u(z)|\leqq C\operatorname{dist}(z\,;\partial D)^{\eta} \qquad \text{for } z\in D\cap G, \tag{1.15}$$

for some $C>0$ and open $G\subset\boldsymbol{C}^n$ such that $\partial D\subset G$. To do this, we recall that Diederich and Fornaess [3] showed the existence of a continuous function $v:\bar{D}\to(-\infty, 0]$ satisfying (1.1), (1.2) and that

$$|v(z)|\leqq C'\operatorname{dist}(z\,;\partial D)^{\delta} \qquad \text{for } z\in D\cap G',$$

for some $C'>0$, $\delta>0$ and open $G'\subset\boldsymbol{C}^n$ with $\partial D\subset G'$. Thus, we will construct our u by repeating the argument due to Diederich and Fornaess carefully.

Take the function σ stated in Corollary 1.1. Then, $\sigma\in C^2(G_1)$ for some open set $G_1\subset\boldsymbol{C}^n$ with $\partial D\subset G_1$. Since $-\log(-\sigma(z))$ is psh, we can conclude from the identity

$$\mathfrak{L}_{-\log(-\sigma)}(z:\xi)=|\sigma(z)|^{-2}\{|\sigma(z)|\mathfrak{L}_\sigma(z:\xi)+|\langle\partial\sigma(z), \xi\rangle|^2\}$$

that

$$\mathfrak{L}_\sigma(z:\xi)\geqq 0 \quad \text{if } \langle\partial\sigma(z), \xi\rangle=0 \text{ and } z\in D\cap G_1,$$

where $\mathfrak{L}_f(z:\xi, \theta)=\sum_{\alpha,\beta=1}^n(\partial^2 f(z)/\partial z^\alpha\partial\bar{z}^\beta)\xi^\alpha\bar{\theta}^\beta$, $\mathfrak{L}_f(z:\xi)=\mathfrak{L}_f(z:\xi, \xi)$ and $\langle\cdot,\cdot\rangle$ is the standard inner product in $\boldsymbol{C}^n$. This implies that if we decompose ξ as $\xi=\xi'+\xi''$, $\xi'=|\partial\sigma(z)|^{-2}\overline{\langle\partial\sigma(z), \xi\rangle}\partial\sigma(z)$, then

$$\mathfrak{L}_\sigma(z:\xi)\geqq\mathfrak{L}_\sigma(z:\xi, \xi')+\mathfrak{L}_\sigma(z:\xi', \xi''). \tag{1.16}$$

Take $\varepsilon>0$ such that

$$\eta=\left\{4\left(\frac{1+\varepsilon}{1-\varepsilon}\right)ML+1\right\}^{-2},$$

where L and M are defined by (1.13) and (1.14). Note that $|\partial\sigma(z)|=1/2$ if $z\in\partial D$ and hence that for some open set G_2 with $\partial D\subset G_2\subset G_1$

$$\sup\{|\mathfrak{L}_\sigma(z:\xi,\hat{\xi})|:|\xi|=|\hat{\xi}|=1,\ z\in G_2\}\leqq(1+\varepsilon)M$$

$$\inf\{|\partial\sigma(z)|:z\in G_2\}\geqq\frac{1-\varepsilon}{2}$$

Therefore, it follows from (1.16) that

$$\mathfrak{L}_\sigma(z:\xi)\geqq-4\Big(\frac{1-\varepsilon}{1+\varepsilon}\Big)M|\xi||\langle\partial\sigma(z),\xi\rangle|\qquad\text{for } z\in D\cap G_2 \text{ and } \xi\in\boldsymbol{C}^n. \tag{1.17}$$

Define

$$\alpha=\Big(2+\frac{1-\varepsilon}{2(1+\varepsilon)ML}\Big)^{-1}$$
$$K=\alpha\eta^{-1}L^{-2}$$
$$\rho(z)=-\{(-\sigma(z))\exp(-K|z|^2)\}^\eta,\qquad z\in\bar{D}.$$

Then, it holds that

$$1-\eta-\Big\{2\Big(\frac{1+\varepsilon}{1-\varepsilon}\Big)M+\eta KL\Big\}^2\{K(1-\alpha)\}^{-1}=0. \tag{1.18}$$

By a straightforward computation, we obtain that

$$\begin{aligned}\mathfrak{L}_\rho(z:\xi)=\eta(-\sigma(z))^{\eta-2}e^{-\eta K|z|^2}[&K\sigma(z)^2(|\xi|^2-\eta K|\langle z,\xi\rangle|^2)\\&+|\sigma(z)|\{\mathfrak{L}_\sigma(z:\xi)-2\eta K\mathrm{Re}\langle\partial\sigma(z),\xi\rangle\overline{\langle z,\xi\rangle}\}+(1-\eta)|\langle\partial\sigma(z),\xi\rangle|^2].\end{aligned}$$

Plugging (1.17) and (1.18) into this identity, we can conclude that

$$\begin{aligned}\mathfrak{L}_\rho(z:\xi)&\geqq\eta(-\sigma(z))^{\eta-2}e^{-\eta K|z|^2}(1-\alpha)K\\&\quad\times\Big[|\sigma(z)||\xi|-\Big\{2\Big(\frac{1+\varepsilon}{1-\varepsilon}\Big)M+\eta KL\Big\}\{K(1-\alpha)\}^{-1}|\langle\partial\sigma(z),\xi\rangle|\Big]^2\\&\geqq0\end{aligned}$$

for $z\in D\cap G_2$ and $\xi\in\boldsymbol{C}^n$. Thus ρ is psh on $D\in G_2$.

Take $a<0$ such that $\overline{\{z\in D\cap G_2:\rho(z)>a\}}\subset D\cap G_2$. It is easily seen that

$$u(z)=\begin{cases}\max\{\rho(z),a\} & \text{if } z\in\bar{D}\cap G_2\\ a & \text{if } z\in D\setminus G_2\end{cases}$$

is a continuous function on $\bar{D}$ with values in $(-\infty,0]$ satisfying the assumptions (1.1), (1.2) and (1.15). The proof is completed.

2. The boundary behaviour of psh functions

Consider a bounded pseucoconvex domain D in $\boldsymbol{C}^n$ and a positive Radon measure m on D with $\mathrm{supp}[m]=D$. Let $\boldsymbol{M}=(Z_t,\zeta,P_z)\in\mathfrak{H}\mathfrak{D}(D,m)$ be transient and u be a locally bounded psh function on D. In this section, we aim at giving sufficient conditions in order that (0.2) holds:

$$P_z[\lim_{t\uparrow\zeta}u(Z_t)\text{ exists in }(-\infty,\infty)]=1,\qquad \boldsymbol{M}\text{-q.e.}$$

We denote by θ the unique closed positive current associated with $\boldsymbol{M}$ by the relation

(0.4). We will establish that

Theorem 2.1. *Let D, m, $\boldsymbol{M}$ and θ be as above. Then* (0.2) *holds for a psh u with $\inf_{z\in D} u(z)>-\infty$ if either of the following conditions holds:*

(C.3) $D=\{z\in\Omega : \varphi(z)<0\}$ *for some open $\Omega\supset\bar{D}$ and psh $\varphi\in C(\Omega)$, and*

$$\int_D |\varphi| dd^c u\wedge\theta<\infty, \tag{2.1}$$

(C.4) ∂D *is* C^2 *and*

$$\int_D \mathrm{dist}(\cdot : \partial D)^{\eta} dd^c u\wedge\theta<\infty, \tag{2.2}$$

for some $\eta<(4ML+1)^{-2}$, where M and L are defined by (1.13) *and* (1.14), *respectively.*

Before the proof of Theorem 2.1, we will state its application to the boundary behaviour of holomorphic functions. We will see that

Corollary 2.1. *Let D, m and $\boldsymbol{M}$ be as before. Let h be a holomorphic function on D. Then, it holds that*

$$P_z[\lim_{t\uparrow\zeta} h(Z_t) \text{ exists in } \boldsymbol{C}^1]=1, \qquad \boldsymbol{M}\text{-q.e.}$$

if either (i) $D=\{\varphi<0\}$ *for some open $\Omega\supset\bar{D}$ and psh $\varphi\in C(\Omega)$ and* $\int_D |\varphi| dh\wedge d^c\bar{h}\wedge\theta<\infty$ *or* (ii) ∂D *is C^2 and* $\int_D \mathrm{dist}(\cdot ; \partial D)^{\eta} dh\wedge d^c\bar{h}\wedge\theta<\infty$ *for some* $\eta<(4ML+1)^{-2}$.

Proof. Note that $dd^c|h|^2\wedge\theta=dh\wedge d^c\bar{h}\wedge\theta$. Then, by applying Theorem 2.1 with $u=|h|^2$, we obtain that

$$P_z[\lim_{t\uparrow\zeta}|h(Z_t)|^2<\infty]=1, \qquad \boldsymbol{M}\text{-q.e.} \tag{2.3}$$

By a standard time change argument, $h(Z_t)$ is represented as

$$h(Z_t)-h(Z_0)=\xi(\langle h(Z.), \bar{h}(Z.)\rangle_t), \qquad t<\zeta,$$

for some $\boldsymbol{C}^1$-valued Brownian motion $\xi(t)$ with $\xi(0)=0$. Since $\limsup_{t\uparrow\infty}|\xi(t)|=\infty$, combining this with (2.3), we can conclude that $\langle h(Z.), \bar{h}(Z.)\rangle_{\zeta-}<\infty$ and hence the desired conclusion follows.

Let D_1 be a unit ball in $\boldsymbol{C}^n$. We denote by $P_{0,\xi}$ the distribution of the absorbing boundary Brownian motion $\boldsymbol{M}_1$ conditioned to exit at $\xi\in\partial D$. Applying the corollary to $\boldsymbol{M}_1$, we can conclude that if a holomorphic function h satisfies that $\int_{D_1}|\partial h(z)|^2(1-|z|^2)V(dz)<\infty$, V being the Lebesgue measure on D_1, then the boundary limits of h along Z_t exist $P_{0,\xi}$-a.s. for l-a.e. ξ, where l is the Lebesgue measure on ∂D_1. Bañuelos and Øksendal [1] have studied the boundary behaviour of harmonic functions along the sample paths of more general diffusion processes on D_1. In particular, in the case where $n=1$, they also obtained the similar result under the stronger assumption that $\int_{D_1}|\partial h(z)|^2(1-|z|)^{\alpha}V(dz)<\infty$ for some α, $0\leqq\alpha<1$. In this case, our corollary covers their result. However, it should be remarked that their exceptional set in ∂D_1 is not only of zero l-measure but also of zero capacity with respect to the potential of order α if $\alpha>0$ and the logarithmic potential if $\alpha=0$.

We now proceed to the proof of Theorem 2.1.

Proof of Theorem 2.1. Assume that (C.3) holds. Take an arbitrary bounded positive $f\in L_1(D; m)\cap C(D)$ and fix it. Define a positive Radon measure μ on D by

$$d\mu=dd^c u\wedge\theta+f\,dm. \tag{2.4}$$

Then μ charges no set of zero capacity with respect to the 1-capacity of $\boldsymbol{M}$ (cf. [6, 7]). Let A_t be the positive continuous additive functional associated with μ and $\tilde{D}$ be its support (for definitions, see [5]). If we denote by $\mathrm{Cap}(E)$ the 1-capacity of $E\subset D$ with respect to $\boldsymbol{M}$, then we observe that

$$\mathrm{Cap}(D\setminus\tilde{D})=0. \tag{2.5}$$

Indeed, if we set $R=\inf\{t>0: A_t>0\}$, then, by [5, Lemma 5.5.1], $E_z[e^{-R}]=1$ μ-a.e. z. By the definition (2.4), this implies that $E_z[e^{-R}]=1$ m-a.e. Hence, by [5, Lemma 4.2.5], we see that $E_z[e^{-R}]=1$ $\boldsymbol{M}$-q.e., which means that (2.5) holds.

Define

$$\tau_t=\inf\{s>0: A_s>t\},$$
$$\tilde{Z}_t=Z_{\tau_t},\qquad \eta=A_\zeta,$$
$$\tilde{\boldsymbol{M}}=(\tilde{Z}_t,\ \eta,\ P_z).$$

Then, by virtue of (1.8), $\tilde{\boldsymbol{M}}$ becomes a μ-symmetric diffusion process on $\tilde{D}$ if we replace $\tilde{D}$ by an appropriate smaller set such that $D\setminus\tilde{D}$ is properly exceptional with respect to $\boldsymbol{M}$. Hence, by making every $z\in D\setminus\tilde{D}$ a trap, we can extend $\tilde{\boldsymbol{M}}$ to a μ-symmetric diffusion process on D, which is again denoted by $\tilde{\boldsymbol{M}}=(\tilde{Z}_t,\ \eta,\ P_z)$. By [5, Theorem 5.5.1] and [8, Theorem 2.1], we see that $\tilde{\boldsymbol{M}}\in\mathfrak{H}\mathfrak{D}(D, \mu)$, is transient and the corresponding closed positive current is again θ.

We decompose $u(\tilde{Z}_t)-u(\tilde{Z}_0)$ as

$$u(\tilde{Z}_t)-u(\tilde{Z}_0)=\tilde{M}_t+\tilde{N}_t,\qquad t<\eta, \tag{2.6}$$

where $\tilde{M}$ is a local martingale additive functional and $\tilde{N}$ is a continuous additive functional of energy zero. By [6, Lemma 7], we obtain that the Revuz measure of $\tilde{N}$ is $dd^c u\wedge\theta$. Since $0\leqq dd^c u\wedge\theta\leqq\mu$, we have that

$$0\leqq\tilde{N}_s\leqq\tilde{N}_t\leqq t,\qquad 0\leqq s\leqq t<\eta. \tag{2.7}$$

On the other hand, since $f\in L^1(D; m)$ and φ is bounded on D, it follows from (2.1) that

$$\int_D|\varphi(z)|\,\mu(dz)<\infty.$$

Applying Theorem 1.1, we obtain that

$$P_z[\eta<\infty]=1,\qquad \tilde{\boldsymbol{M}}\text{-q.e.} \tag{2.8}$$

Then, it follows from (2.7) and (2.8) that

$$P_z[\lim_{t\uparrow\eta}\tilde{N}_t \text{ exists in } [0, \infty)]=1, \quad \tilde{\boldsymbol{M}}\text{-q.e.} \tag{2.9}$$

Plugging (2.9) and the lower boundedness of u into (2.6), we have

$$P_z[\liminf_{t\uparrow\eta}\tilde{M}_t>-\infty]=1, \quad \tilde{\boldsymbol{M}}\text{-q.e.} \tag{2.10}$$

By the standard time change argument, we see that $\tilde{M}_t=B(\langle\tilde{M}\rangle_t)$, $t<\eta$, for some $\boldsymbol{R}^1$-valued Brownian motion with $B(0)=0$, where $\langle\tilde{M}\rangle_t$ is the quadratic variation process of $\tilde{M}_t$. Since $\liminf_{t\uparrow\infty}B(t)=-\infty$, (2.10) implies that

$$P_z[\lim_{t\uparrow\eta}\tilde{M}_t \text{ exists in } (-\infty, \infty)]=1, \quad \tilde{\boldsymbol{M}}\text{-q.e.} \tag{2.11}$$

Substituting (2.9) and (2.11) to (2.6), we obtain that

$$P_z[\lim_{t\uparrow\eta}u(\tilde{Z}_t) \text{ exists in } (-\infty, \infty)]=1, \quad \tilde{\boldsymbol{M}}\text{-q.e.} \tag{2.12}$$

Since $\boldsymbol{M}$ and $\tilde{\boldsymbol{M}}$ are both transient and $C_0^\infty(D)$-regular, and their Dirichlet forms coincide on $C_0^\infty(D)$, a set E is of zero capacity with respect to the 1-capacity of $\tilde{\boldsymbol{M}}$ if and only if so is to that of $\boldsymbol{M}$. See [5]. Therefore, it follows from (2.12) that (0.2) holds:

$$P_z[\lim_{t\uparrow\zeta}u(Z_t) \text{ exists in } (-\infty, \infty)]=1 \quad \boldsymbol{M}\text{-q.e.}$$

The proof in the case where (C.3) holds is complete.

That (0.2) holds if (C.4) is satisfied will be verified in exactly the same manner as above. We omit the details.

3. Examples

In this section, we will present several examples to illustrate our theorems. We start with the following two examples on transience.

Example 3.1. Let D be a bounded domain in $\boldsymbol{C}^n$ and m be a positive Radon measure on D with $\mathrm{supp}[m]=D$. It was seen in [9] that $\boldsymbol{M}\in\mathfrak{H}\mathfrak{D}(D, m)$ is transient if it is irreducible, i.e., if the defining function $\mathbf{1}_E$ of $E\subset D$ is locally in $\mathfrak{F}$, where $\mathfrak{F}$ is the domain of the Dirichlet form of $\boldsymbol{M}$, then either $m(E)=0$ or $m(D\setminus E)=0$. For the sake of completeness, we repeat the proof given in [9]. Let h be a bounded holomorphic function on D. By the martingale convergence theorem, we have

$$P_z[\lim_{t\uparrow\zeta}h(Z_t) \text{ exists in } \boldsymbol{C}^1]=1, \quad \boldsymbol{M}\text{-q.e.} \tag{3.1}$$

If $\boldsymbol{M}$ is not transient, then it is recurrent and hence (3.1) never holds, which is a contradiction. Thus $\boldsymbol{M}$ is transient.

Example 3.2. Let D be a bounded domain in $\boldsymbol{C}^1$. Take $a\in D$. We denote by V (resp. δ_a) the Lebesgue measure on $\boldsymbol{C}^1$ (resp. the Dirac measure concentrated at a) and put

$$m(dz)=V(dz)+\delta_a(dz).$$

Then there is a unique $\boldsymbol{M}=(Z_t, \zeta, P_z)\in\mathfrak{H}\mathfrak{D}(D, m)$ with the associated (0, 0)-current 1.

See [6]. If we denote by $\{B_t\}$ a $\boldsymbol{C}^1$-valued Brownian motion starting at 0 on a probability space $(\Omega, \mathfrak{B}, P)$, then it is easily seen that

$$\text{the law of } \{Z_t\} \text{ under } P_z = \text{the law of } \{z+B_{t\wedge\sigma_z}\} \text{ under } P$$
$$P_a[Z_t = a \text{ for any } t \geqq 0] = 1,$$

where $\sigma_z = \inf\{t>0 : z+B_t \notin D\}$. Thus, it holds that

$$P_z[\zeta<+\infty]=1 \quad \text{if } z\neq a \text{ and} = 0 \text{ if } z=a$$

and $\boldsymbol{M}$ is not transient. Since $m(\{a\})=1$, this implies that $\boldsymbol{M}$ does not explode.

In the following example, we consider the assumption on integrability stated in Theorem 1.1.

Example 3.3. Let $D=\{z\in\boldsymbol{C}^1 : |z|<1\}$. Define

$$m^\alpha(dz)=(1-|z|^2)^{-\alpha}V(dz).$$

There exists a unique $\boldsymbol{M}^\alpha=(Z_t, \zeta, P_z^\alpha)\in\mathfrak{H}\mathfrak{D}(D, m^\alpha)$ with the associated (0, 0)-current 1. We will then observe that $\boldsymbol{M}^\alpha$ explodes if $\alpha<2$ and does not explode if $\alpha\geqq 2$.

Indeed, since $\boldsymbol{M}^\alpha$ is generated by $(1-|z|^2)^\alpha\{(\partial/\partial x)^2+(\partial/\partial y)^2\}$, where $z=x+iy\in\boldsymbol{C}^1$, it is elementary to see that the law of $\{|Z_t|^2\}$ under P_z^α is the one of the 1-dimensional diffusion process $\{\xi_t\}$ on (0, 1) generated by $4(1-\xi)^\alpha\xi(d/d\xi(^2+4(1-\xi)^\alpha(d/d\xi)$. Thus, by applying Feller's test, we obtain the desired conclusion.

The above observation yields that one cannot replace in Theorem 1.1 the integrability assumption that $\int_D|\varphi|dm<+\infty$ by the weaker one that $\int_D|\varphi|^{1+\varepsilon}dm<+\infty$ for some $\varepsilon>0$.

We finally give two examples concerning Theorem 2.1.

Example 3.4. Let $\varphi(z)=\max\{|z^1|^2, |z^2|^2\}-1$ for $z=(z^1, z^2)\in\boldsymbol{C}^2$ and $D=\{\varphi<0\}$. We denote by $\boldsymbol{M}=(Z_t, \zeta, P_z)$ the absorbing boundary Brownian motion on D. Then $\boldsymbol{M}\in\mathfrak{H}\mathfrak{D}(D, V)$, is transient and the associated current is $dd^c|z|^2$, where V is the Lebesgue measure on D. Set

$$u(z)=-\log(2-|z|^2).$$

It is easily seen that

$$\int_D|\varphi|dd^cu\wedge dd^c|z|^2<\infty.$$

Hence, our theorem implies that (0.2) holds for such u and $\boldsymbol{M}$ although u is not bounded.

Example 3.5. Let φ and D be as in Example 3.4. Define

$$\theta_0=dd^c\{-\log(1-|z^1|^2)-\log(1-|z^2|^2)\}.$$

Then there exists a unique $\boldsymbol{M}=(Z_t, \zeta, P_z)\in\mathfrak{H}\mathfrak{D}(D, V)$ whose Dirichlet form on $C_0^\infty(D)$ is given by (0.4) with $\theta=\theta_0$.

Combining the observations in [2] and [9], we see that if we set

$$S=\{|z^1|=|z^2|=1\}$$

then,

$$P_z[\lim_{t\uparrow\zeta} Z_t \in S]=1, \qquad \boldsymbol{M}\text{-q. e.}$$

In particular, if we take the same psh function u as in Example 3.4, then it holds that

$$P_z[\lim_{t\uparrow\zeta} u(Z_t)=\infty]=1, \qquad \boldsymbol{M}\text{-q. e.}$$

It is straightforward to see that $\int_D |\varphi| dd^c u \wedge \theta_0 = +\infty$. Therefore (0.2) does not hold without the integrability assumption (2.1) in general.

Acknowledgements. The author would like to thank the anonymous referee for his valuable comments.

DEPARTMENT OF APPLIED SCIENCE
FACULTY OF ENGINEERING
KYUSHU UNIVERSITY

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