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Local structure of analytic transformations of two complex variables, II

By

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This is the continuation of our previous paper of the same title [I]. We continue the investigation of semi-attractive and semi-repulsive transformations of type $(1, b)_1$.

Let us briefly recall some definitions and results in [I]. By an analytic transformation of two complex variables we mean (the germ of) a holomorphic mapping of a neighborhood of $O = (0, 0) \in \mathbf{C}^2$ into $\mathbf{C}^2$ such that $T(O) = O$. We say that T is of type $(1, b)$ if the eigenvalues of the linear part of T are 1 and b. When $b \neq 1$, we can choose local coordinates (x, y) around O so that $T\colon (x, y) \mapsto (x_1, y_1)$ takes the form

$$(0.1) \qquad \begin{cases} x_1 = x + \sum\limits_{i+j \geqq 2} a_{ij} x^i y^j \\ y_1 = by + \sum\limits_{i+j \geqq 2} b_{ij} x^i y^j. \end{cases}$$

In [I, Sec. 6] we showed that every transformation T of type $(1, b)$ with $|b| \neq 0, 1$ is equivalent to a transformation $(z, w) \mapsto (z_1, w_1)$ of a neighborhood of $(\infty, 0) \in \hat{\mathbf{C}} \times \mathbf{C}$ into $\hat{\mathbf{C}} \times \mathbf{C}$ of the form

$$(0.2) \qquad \begin{cases} z_1 = z + a_0 + \dfrac{a_1}{z} + \dfrac{a_2(w)}{z^2} + \cdots \\ w_1 = bw + \dfrac{b_1 w}{z} + \dfrac{b_2(w)}{z^2} + \cdots \end{cases}$$

where a_0, a_1, b_1 are constants and $a_j(w)$ $(i = 2, 3, \ldots)$, $b_j(w)$ $(j = 2, 3, \ldots)$ are holomorphic functions of one complex variable w in a neighborhood of $w = 0$. This transformation will be regarded as an expression of T with respect to the "local coordinate system" (z, w) around $O = (\infty, 0)$ and denoted also by T.

T is said to be of type $(1, b)_1$ if $a_0 \neq 0$ in the expression (0.2). This is equivalent to the condition $a_{20} \neq 0$ in (0.1).

In what follows, we assume that $0 < |b| < 1$, so that T is semi-attractive and its inverse T^{-1} is semi-repulsive. Further, for the simplicity of the argument, we assume that a_0 is a real positive number. The result for the general case

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will be derived by rotating the coordinate z.

11. Invariant asymptotic curve.

In this section we investigate the simply convergent points for the semi-repuisive transformation T^{-1}. We shall show that the set of these points can be obtained as the image of a holomorphic mapping of a domain in $\mathbf{C}$. Thus we will call this set the invariant asymptotic curve.

11.1. Let T be an analytic transformation of the form (0.2) with $0<|b|<1$, $a_0>0$, defined in a neighborhood of $O=(\infty, 0)$. To fix the ideas, let V_1, V_2 be neighborhoods of O such that T is a biholomorphic map of V_1 onto V_2. We choose and fix a relatively compact neighborhoood V of O of the form

$$V=\{(z, w)\in\hat{\mathbf{C}}\times\mathbf{C}\,|\,R'<|z|\leqq\infty,\ |w|<\rho\}, \tag{11.1}$$

contained in $V_1\cap V_2$. Let

$$C=\{(\infty, w)\in\hat{\mathbf{C}}\times\mathbf{C}\,|\quad |w|<\rho\}.$$

For any real number L, we denote by Σ_L the half-plane

$$\Sigma_L=\{\sigma\in\mathbf{C}\,|\,\mathrm{Re}\,\sigma<-L\} \tag{11.2}$$

in $\mathbf{C}$ (which we shall call the σ-plane).

Theorem 11.1. *There are a positive number L and an injective holomorphic mapping*

$$H:\sigma\longmapsto H(\sigma)=(h(\sigma), k(\sigma))$$

of Σ_L into $V-C$ with the properties:

(i) *H satisfies the functional equation*

$$T\circ H(\sigma-a_0)=H(\sigma),\qquad \sigma\in\Sigma_L, \tag{11.3}$$

(ii) $h(\sigma)-\left(\sigma+\dfrac{a_1}{a_0}\log\sigma\right)$ *tends to a constant as* $|\sigma|\to\infty$, $\sigma\in\Sigma_L$, *where* $\log\sigma$ *denotes a single-valued branch of* $\log\sigma$ *on* Σ_L.

(iii) $k(\sigma)$ *tends to* 0 *as* $|\sigma|\to\infty$, $\sigma\in\Sigma_L$.

Remark. If $H(\sigma)$ is as in the theorem and σ_0 is any complex number, then the holomorphic mapping $H(\sigma-\sigma_0)$ ($\sigma\in\Sigma_{L+\mathrm{Re}\,\sigma_0}$) has also the properties of the theorem. A criterion which characterizes these mappings will be given in Proposition 11.5.

The proof of Theorem 11.1 will be reduced to the lemma below. To state the lemma, we consider the following situation. Let $\tilde{\Delta}$ be a domain in $\mathbf{C}^2$ defined by

(11.4) $$\tilde{\Delta} = \{(Z, W) \in \mathbf{C}^2 | \operatorname{Re} Z < -\tilde{R},\ |W| < \rho\} \qquad (\tilde{R}, \rho > 0),$$

and let $\tilde{T}: Q = (Z, W) \mapsto \tilde{T}(Q) = (Z_1, W_1)$ be a holomorphic mapping of $\tilde{\Delta}$ into $\mathbf{C}^2$ of the form

$$\begin{cases} Z_1 = \tilde{f}(Z, W) = Z + a_0 + \alpha(Z, W), \\ W_1 = \tilde{g}(Z, W) = (b + \beta(Z))W + \gamma(Z, W). \end{cases}$$

Here $\alpha(Z, W)$ and $\gamma(Z, W)$ are holomorphic functions on $\tilde{\Delta}$ such that

$$|\alpha(Z, W)| \leqq K/|Z|^{1+\lambda}, \qquad |\gamma(Z, W)| \leqq K/|Z|^{1+\lambda},$$

and $\beta(Z)$ is a holomorphic function of Z on $\{Z \in \mathbf{C} | \operatorname{Re} Z < -\tilde{R}\}$ such that

$$|\beta(Z)| \leqq K/|Z|^{\lambda},$$

for some positive constants K, λ.

Lemma 11.2. *Under these conditions, there exist a positive number L and an injective holomorphic mapping* $\tilde{H}: \sigma \mapsto \tilde{H}(\sigma) = (\tilde{h}(\sigma), \tilde{k}(\sigma))$ *of* Σ_L *into* $\tilde{\Delta}$ *with the properties*:

(i) $\tilde{H}$ *satisfies the functional equation*

(11.3)′ $$\tilde{T} \circ \tilde{H}(\sigma - a_0) = \tilde{H}(\sigma), \qquad \sigma \in \Sigma_L.$$

(ii) $\tilde{h}(\sigma) - \sigma$ *tends to a constant as* $|\sigma| \to \infty$, $\sigma \in \Sigma_L$.

(iii) $\tilde{k}(\sigma)$ *tends to* 0 *as* $|\sigma| \to \infty$, $\sigma \in \Sigma_L$

The proof of this lemma is given in the next paragraph.

Proof of Theorem 11.1. First let us note that, if $a_1 = 0$, then the Lemma is directly applied by setting $Z = z$, $W = w$, $\alpha(z, w) = a_2(w)/z^2 + \cdots$, $\gamma(z, w) = b_2(w)/z^2 + \cdots$, and $\beta(z) = b_1/z$.

To prove the general case, we take a domain of the form

$$\Delta_0 = \{(z, w) \in \mathbf{C}^2 | \pi/2 - \varepsilon < \arg(z + R_0) < 3\pi/2 + \varepsilon\}$$

where $0 < \varepsilon < \pi/2$ and R_0 is a positive number such that $\Delta_0 \subset V$. Let

$$\begin{cases} Z = z - \dfrac{a_1}{a_0} \log z \\ W = w \end{cases}$$

on Δ_0. Here $\log z$ denotes a single-valued branch of $\log z$ on the simply connected domain Δ_0. When R_0 is sufficiently large, (Z, W) can be regarded as a coordinate system on Δ_0; and with respect to this coordinate system Δ_0 contains $\tilde{\Delta}$ defined by (11.4) with sufficiently large $\tilde{R}$. This fact can be easily verified (c.f. [I, Sec. 7.3]).

We denote by $\tilde{T}$ the restriction of T to $\tilde{\Delta}$ expressed in terms of (Z, W). Then $\tilde{T}: (Z, W) \to (Z_1, W_1)$ takes the form:

$$Z_1 = z_1 - \frac{a_1}{a_0}\log z_1 = \left(z + a_0 + \frac{a_1}{z} + \cdots\right) - \frac{a_1}{a_0}\log\left(z + a_0 + \frac{a_1}{z} + \cdots\right)$$
$$= z - \frac{a_1}{a_0}\log z + a_0 + \left(\frac{a_1}{z} + \cdots\right) - \frac{a_1}{a_0}\log\left(1 + \frac{a_0}{z} + \cdots\right)$$
$$= Z + a_0 + O(|z|^{-2}).$$
$$W_1 = \left(b + \frac{b_1}{z}\right)W + O(|z|^{-2}).$$

Since $|z|^{-1} = O(|Z|^{-\lambda})$ for any λ $(0 < \lambda < 1)$, the transformation $\tilde{T}$ satisfies the condition of the lemma. Let $\tilde{H}$ be the holomorphic mapping in the lemma and express $\tilde{H}$ in terms of the coordinate system (z, w). Then we obtain the mapping H with the desired properties. Thus Theorem 11.1 is reduced to Lemma 11.2.

11.2. Proof of Lemma 11.2. The proof is divided into the steps $(\alpha) \sim (\varepsilon)$. To simplify the notation we omit $\sim$ in the proof.
(α) We assume that $a_0 = 1$, since the lemma is reduced to this case by replacing Z and σ by $a_0 Z$ and $a_0\sigma$ respectively.
(β) To show the lemma we can replace Δ by a smaller domain of the same form contained in Δ. Replacing ρ by a smaller number and R by a greater number, and estimating Cauchy's integral representations, we assume that

$$|f_Z - 1| \leqq \frac{K'}{|Z|^{1+\lambda}}, \qquad |f_W| \leqq \frac{K'}{|Z|^{1+\lambda}}, \tag{11.5}$$
$$|g_Z| \leqq \frac{K'}{|Z|^{\lambda}}, \qquad |g_W - b| \leqq \frac{K'}{|Z|^{\lambda}},$$

on Δ, where K' is some constant. Further, replacing R by a greater number we assume that

$$|g(Z, W)| \leqq \left(|b| + \frac{K}{R^{\lambda}}\right)\rho + \frac{K}{R^{1+\lambda}} < \rho \tag{11.6}$$

on Δ, and that

$$\sup_{\Delta}|g_Z| + \sup_{\Delta}|g_W| \leqq |b| + \frac{2K'}{R^{\lambda}} < 1. \tag{11.7}$$

(γ) We set

$$M = \sum_{\nu=1}^{\infty} \frac{K}{(R+\nu)^{1+\lambda}}, \tag{11.8}$$

and choose L such that $L \geqq R + M$.

We want to define a sequence of holomorphic mappings

$$H_n(\sigma) = (h_n(\sigma), k_n(\sigma)), \qquad n = 0, 1, 2, \ldots.$$

of Σ_L into Δ by

$$H_n(\sigma) = T^n(\sigma - n, 0). \tag{11.9}$$

Then these mappings will satisfy the recurrence relations:

$$H_{n+1}(\sigma) = T \circ H_n(\sigma - 1). \tag{11.10}$$

We shall show that the sequence $\{H_n(\sigma)\}$ is well-defined and uniformly convergent on Σ_L. Then by (11.10) the limit $H(\sigma) = \lim_{n\to\infty} H_n(\sigma)$ will satisfy the required equation (11.3)$'$.

(δ) To show that $H_n(\sigma)$, $n = 1, 2, \ldots$, are well-defined, we prove the following assertions $*_n$ ($n = 0, 1, 2, \ldots$) inductively:

$*_n$ $H_n(\sigma)$ is well-defined and the inequalities

$$\mathrm{Re}\, h_n(\sigma) \leqq \mathrm{Re}\,\sigma + \sum_{\nu=1}^{n} \frac{K}{|\mathrm{Re}\,\sigma + M - \nu|^{1+\lambda}}, \tag*{(11.11)$_n$}$$

$$|k_n(\sigma)| < \rho \tag*{(11.12)$_n$}$$

are satisfied on Σ_L.

Clearly the assertion $*_0$ is true. Suppose that $*_n$ is already proved. Noting that $\mathrm{Re}\,\sigma < -L \leqq -M - R$, we have

$$\mathrm{Re}\, h_n(\sigma) \leqq \mathrm{Re}\,\sigma + M < -R \qquad (\sigma \in \Sigma_L) \tag{11.13}$$

by (11.11)$_n$ and (11.8). Hence $H_n(\Sigma_L) \subset \Delta$ and so

$$H_{n+1}(\sigma) = T \circ H_n(\sigma - 1) = T(h_n(\sigma - 1), k_n(\sigma - 1))$$

is well-defined.

Now (11.12)$_{n+1}$ follows from (11.6). To show (11.11)$_{n+1}$ we note that

$$\begin{aligned} h_{n+1}(\sigma) &= f(h_n(\sigma - 1), k_n(\sigma - 1)) \\ &= h_n(\sigma - 1) + 1 + \alpha(h_n(\sigma - 1), k_n(\sigma - 1)), \end{aligned}$$

and that

$$|\alpha(h_n(\sigma - 1), k_n(\sigma - 1))| \leqq \frac{K}{|h_n(\sigma - 1)|^{1+\lambda}} \leqq \frac{K}{|\mathrm{Re}(\sigma - 1) + M|^{1+\lambda}}$$

by (11.13). From this inequality and (11.11)$_n$ it follows that

$$\mathrm{Re}\, h_{n+1}(\sigma) \leqq \mathrm{Re}\,\sigma + \sum_{\nu=1}^{n} \frac{K}{|\mathrm{Re}(\sigma - 1) + M - \nu|^{1+\lambda}} + \frac{K}{|\mathrm{Re}(\sigma - 1) + M|^{1+\lambda}}.$$

Hence (11.11)$_{n+1}$ holds. Thus $*_{n+1}$ is proved.

(ε) For points $Q = (Z, W)$ and $Q' = (Z', W')$ in Δ we define their distance by $d(Q, Q') = \max\{|Z - Z'|, |W - W'|\}$. We want to prove the inequality

(11.14)
$$d(H_n(\sigma),\ H_{n-1}(\sigma)) \leqq \frac{M'}{|\sigma - n|^{1+\lambda}}, \qquad \sigma \in \Sigma_L,\ n = 1,2,\dots,$$

where M' is some constant.

To show (11.14) we fix σ and n; and use the notations

$$Q_{(\nu)} = (Z_{(\nu)}, W_{(\nu)}) = T^{\nu}(\sigma - n, 0) = H_\nu(\sigma - n + \nu),$$
$$Q'_{(\nu)} = (Z'_{(\nu)}, W'_{(\nu)}) = T^{\nu-1}(\sigma - n + 1, 0) = H_{\nu-1}(\sigma - n + \nu)$$

for $\nu = 1, 2, \dots, n$. In particular,

$$Q_{(1)} = (f(\sigma - n, 0), g(\sigma - n, 0)), \qquad Q_{(n)} = H_n(\sigma),$$
$$Q'_{(1)} = (\sigma - n + 1, 0), \qquad Q'_{(n)} = H_{n-1}(\sigma).$$

First, $d(Q_{(1)}, Q'_{(1)})$ is estimated as follows:

$$|Z_{(1)} - Z'_{(1)}| = |f(\sigma - n, 0) - (\sigma - n + 1)| = |\alpha(\sigma - n, 0)| \leqq \frac{K}{|\sigma - n|^{1+\lambda}},$$

$$|W_{(1)} - W'_{(1)}| = |g(\sigma - n, 0) - 0| = |\gamma(\sigma - n, 0)| \leqq \frac{K}{|\sigma - n|^{1+\lambda}}.$$

Hence

(11.15)
$$d(Q_{(1)}, Q'_{(1)}) \leqq \frac{K}{|\sigma - n|^{1+\lambda}}.$$

Next $d(Q_{(\nu+1)}, Q'_{(\nu+1)})$ is estimated in terms of $d(Q_{(\nu)}, Q'_{(\nu)})$:

$$\begin{aligned}|Z_{(\nu+1)} - Z'_{(\nu+1)}| &\leqq |f(Z_{(\nu)}, W_{(\nu)}) - f(Z'_{(\nu)}, W'_{(\nu)})| \\ &\leqq (\max|f_Z|)|Z_{(\nu)} - Z'_{(\nu)}| + (\max|f_W|)|W_{(\nu)} - W'_{(\nu)}| \\ &\leqq (\max|f_Z| + \max|f_W|)\, d(Q_{(\nu)}, Q'_{(\nu)}),\end{aligned}$$

where the maximums are taken over the segment which joins $Q_{(\nu)}$ and $Q'_{(\nu)}$. By (11.13) these points $Q_{(\nu)} = H_\nu(\sigma - n + \nu)$ and $Q'_{(\nu)} = H_{\nu-1}(\sigma - n + \nu)$ are in the domain $\operatorname{Re} Z \leqq -R - n + \nu$ and hence this segment is also. Therefore by (11.5)

$$|Z_{(\nu+1)} - Z'_{(\nu+1)}| \leqq \left\{1 + \frac{2K'}{(R + n - \nu)^{1+\lambda}}\right\} d(Q_{(\nu)}, Q'_{(\nu)}).$$

Similarly by (11.7)

$$|W_{(\nu+1)} - W'_{(\nu+1)}| \leqq (\max|g_Z| + \max|g_W|)\, d(Q_{(\nu)}, Q'_{(\nu)}) \leqq d(Q_{(\nu)}, Q'_{(\nu)}).$$

Thus we obtain

$$d(Q_{(\nu+1)}, Q'_{(\nu+1)}) \leqq \left\{1 + \frac{2K'}{(R + n - \nu)^{1+\lambda}}\right\} d(Q_{(\nu)}, Q'_{(\nu)})$$

for $\nu = 1,\ldots,n-1$; hence

$$d(Q_{(n)}, Q'_{(n)}) \leqq \prod_{\nu=1}^{n-1}\left\{1 + \frac{2K'}{(R+\nu)^{1+\lambda}}\right\} d(Q_{(1)}, Q'_{(1)}).$$

Now the required inequality (11.14) follows from this and (11.15) by setting

$$M' = K\prod_{\nu=1}^{\infty}\left\{1 + \frac{2K'}{(R+\nu)^{1+\lambda}}\right\}.$$

(ζ) Now, for $\sigma \in \Sigma_L$, we have

$$\sum_{n=1}^{\infty} \frac{1}{|\sigma - n|^{1+\lambda}} < \int_0^{\infty} \frac{dx}{|x-\sigma|^{1+\lambda}} < \int_0^{\infty} \frac{dx}{|x+L|^{1+\lambda}}.$$

This integral is convergent. Hence, in view of the estimate (11.14), the sequence $\{H_n(\sigma)\}$ is uniformly convergent on Σ_L. Let $H(\sigma) = \lim H_n(\sigma)$. Then $H(\sigma)$ satisfies the required functional equation (11.3)$'$ as is verified letting $n \to \infty$ in (11.10).

Further we have

$$d(H(\sigma), H_0(\sigma)) < \int_0^{\infty} \frac{dx}{|x-\sigma|^{1+\lambda}}.$$

Since $1/|x-\sigma|^{1+\lambda} \to 0$ ($|\sigma| \to \infty$, $\sigma \in \Sigma_L$), the integral on the right tends to 0 ($|\sigma| \to \infty$, $\sigma \in \Sigma_L$), by the Lebesgue dominated convergence theorem. Thus the properties (ii), (iii) of the lemma is proved.

This completes the proof of Lemma 11.2.

11.3. Let us recall some definitions [I, Sec. 4 and 7]. We choose a neighborhood V of O as the domain of definition of a transformation T^{-1}. A point $P \in V$ is said to be stable (relative to T^{-1}) if $T^{-n}(P)$, $n = 1, 2,\ldots$, are all in V (and hence inductively well-defined). A stable point P is said to be simply convergent (relative to T^{-1}) if the sequence $T^{-n}(P)$ ($n = 1, 2,\ldots$) converges to O. The set of all simply convergent points relative to T^{-1} will be denoted by K. We should note that these definitions depend on the choice of V.

A subset $E \subset V$ is called a base of simple convergence for T^{-1} if the following conditions are satisfied: (i) $E \subseteq K$; (ii) For every $P \in K$, there is a sufficiently large number n_0 such that $T^{-n_0}(P) \in E$; (iii) $T^{-1}(E) \subseteq E$. This definition is independent of the choice of V. (See [I, Lemma 7.1]).

Now, when T is a semi-attractive transformation of type $(1, b)_1$, we can present a base of simple convergence for the inverse T^{-1}. Let T, V and H: $\Sigma_L \to V$ be as in Theorem 11.1.

Theorem 11.3. *The set $H(\Sigma_L) \cup \{O\}$ is a base of simple convergence for T^{-1}.*

The proof will be divided into several steps.

(α) The conditions (iii) and (i) are easily verified by the equation $T \circ H(\sigma - a_0)$

$= H(\sigma)$ and by the fact $H(\sigma) \to O$ as $\mathrm{Re}\,\sigma \to -\infty$.

(β) We note that O is the only simply convergent point (relative to T^{-1}) on $C = \{(\infty, w)|\ \ |w| < \rho\}$. Hence, to show (ii), it suffices to prove that, for every $P \in K \cap (V - C)$, there is an n_0 such that $T^{-n_0}(P) \in H(\Sigma_L)$.

We fix real numbers θ and ρ_0 such that $0 < \theta < \pi/2$, $0 < \rho_0 < \rho$. Let

$$\mathscr{B}_R = \{z \in \mathbf{C} | \mathrm{Re}(e^{-i\theta}z) < -R, \quad \text{and} \quad \mathrm{Re}(e^{i\theta}z) < -R\},$$

and

$$\Delta_R = \mathscr{B}_R \times \{w \in \mathbf{C}|\ \ |w| < \rho_0\}$$

with positive R such that $\Delta_R \subset V$. If R is sufficiently large, then Δ_R has the following property: For every point $P \in K \cap (V - C)$, there is an n_1 such that $T^{-n}(P) \in \Delta_R$ for all $n \geqq n_1$. This can be proved by the same argument as in [I, Sec. 7.2].

(γ) We denote $T^{-1}(P) = (z_{-1}. w_{-1})$ for $P = (z, w) \in V$. The following assertion is easily shown: For any positive number ε, there is a sufficiently large number R such that

$$|w_{-1} - w/b| < \varepsilon \qquad \text{for every } P = (z, w) \in \Delta_R.$$

(δ) Consider the holomorphic mapping $H(\sigma) = (h(\sigma), k(\sigma))$ of Σ_L into V. Clearly, we may replace L by a larger number in the proof of the condition (ii). By the same argument as in [I, Sec. 8.3], we can prove that, if L is sufficiently large, then h is injective and that $h(\Sigma_L)$ contains a domain of the form $\mathscr{B}_R$ with some R. Then $H(\Sigma_L) \cap \Delta_R$ is an analytic subset of Δ_R and represented as the graph:

$$H(\Sigma_L) \cap \Delta_R = \{(z, \ell(z)) | z \in \mathscr{B}_R\},$$

where $\ell(z) = k \circ h^{-1}(z)$. Further, for any positive number ε, there is a sufficiently large R such that $|\ell(z)| < \varepsilon$ on $\mathscr{B}_R$ by Theorem 11.1 (iii).

(ε) We choose the numbers ε, L, and R in the following way: Choose ε so that $0 < \varepsilon < \rho_0$, and that

$$c := \frac{\rho_0 + \varepsilon}{\rho_0/|b| - 2\varepsilon} < 1.$$

Next, choose L and R so that the conditions in (δ) are satisfied. Then, replace R by a larger number so that the conditions in (β), (γ) are satisfied.

(ζ) In view of (β), the property (ii) will be derived from the following assertion: If P is a point in Δ_R such that $T^{-1}(P)$, $T^{-2}(P), \dots$ are all contained in Δ_R, then P is in $H(\Sigma_L)$.

For any point $P = (z, w)$ in Δ_R, let $T^{-1}(P) = (z_{-1}, w_{-1})$ and define

$$q(P) = (w - \ell(z))/(w_{-1} - \ell(z_{-1})).$$

If $w_{-1} - \ell(z_{-1}) = 0$, then $T^{-1}(P) = H(\sigma)$ for some $\sigma \in \Sigma_{L+a_0}$; hence $P = H(\sigma + a_0)$ $(\sigma + a_0 \in \Sigma_L)$ and $w = \ell(z)$. Thus $q(P)$ is a holomophic function on Δ_R. Further we want to show the inequality

$$(11.16) \qquad |q(P)| \leqq c \qquad \text{for } P \in \Delta_R.$$

For $P = (z, w)$ with $z \in \mathscr{B}_R$, $|w| = \rho_0$, we have by (γ), (δ),

$$|q(P)| \leqq \frac{|w| + |\ell(z)|}{|w_{-1}| - |\ell(z_{-1})|} \leqq \frac{\rho_0 + \varepsilon}{\rho_0/|b| - 2\varepsilon} = c.$$

The inequality (11.16) follows by the maximum principle.

Now suppose that P, $T^{-1}(P)$, $T^{-2}(P)$,... are all in Δ_R. Then, we have, for any n,

$$\begin{aligned} |w - \ell(z)| &\leqq |q(P)q(T^{-1}(P))\cdots q(T^{-n+1}(P))|\,|w_{-n} - \ell(z_{-n})| \\ &\leqq c^n(\rho_0 + \varepsilon). \end{aligned}$$

This implies that $w = \ell(z)$ and hence $P \in H(\Sigma_L)$. Thus the assertion is proved and the proof of the theorem is completed.

11.4. The set K of all (simply) convergent points relative to T^{-1} in V can be described as the analytic continuation $H(\Sigma)$ of $H(\Sigma_L)$, which we call the invariant asymptotic curve:

Proposition 11.4. *The mapping $H: \Sigma_L \to V - C$ is extended to an injective holomorphic mapping* (*denoted by the same letter*) *$H: \hat{\Sigma} \to V - C$ such that $H(\hat{\Sigma}) \cup \{O\} = K$, where $\hat{\Sigma}$ is some domain in* $\mathbf{C}$ *containing Σ_L.*

Proof. Let $\hat{\Sigma} = \{\sigma \in \mathbf{C} |$ there exists a positive integer n such that $\sigma - na_0 \in \Sigma_L$ and $T^\nu \circ H(\sigma - na_0) \in V$ for $\nu = 1, \ldots, n\}$ and define $H(\sigma) = T^n \circ H(\sigma - na_0)$ for $\sigma \in \hat{\Sigma}$, $\sigma - na_0 \in \Sigma_L$. This gives an extension of H. We can easily verify that $H(\hat{\Sigma}) \cup \{O\} = K$ using Theorem 11.3. q.e.d.

Now we can give a characterization of the mapping H mentioned after Theorem 11.1.

Proposition 11.5. *Let Σ' be a domain in* $\mathbf{C}$ *such that* (i) *for every $\sigma \in \mathbf{C}$ there is a positive integer n such that $\sigma - na_0 \in \Sigma'$ and that* (ii) *$\sigma \in \Sigma'$ implies $\sigma - a_0 \in \Sigma'$. Let $H': \Sigma' \to V - C$ be an injective holomorphic mapping satisfying the equation $T \circ H'(\sigma) = H'(\sigma - a_0)$ $(\sigma \in \Sigma')$ and such that $H'(\sigma - na_0) \to 0$ as $n \to \infty$. Then $H'(\sigma) = H(\sigma - \sigma_0)$ for some constant σ_0.*

Proof. The image $H'(\Sigma')$ is contained in the set $K - \{O\}$. Therefore the composition $\eta = H^{-1} \circ H': \Sigma' \to \mathbf{C}$ can be defined. This satisfies the equation

$$(11.17) \qquad \eta(\sigma - a_0) = \eta(\sigma) - a_0 \qquad (\sigma \in \Sigma').$$

By this equation, η can be extended to an entire function (denoted by the same letter) $\eta: \mathbf{C} \to \mathbf{C}$ satisfying (11.17). Since η is injective on Σ', it is injective on $\mathbf{C}$. Therefore η is a linear function. In view of (11.17), η is of the form $\eta(\sigma) = \sigma - \sigma_0$. Hence $H'(\sigma) = H \circ \eta(\sigma) = H(\sigma - \sigma_0)$. q.e.d.

11.5. We consider the case of global transformations as in [I, Sec. 10]. Let $\mathfrak{M}$ be a complex manifold of dimension 2 and let T be a holomorphic automorphism of $\mathfrak{M}$. Suppose that there is a fixed point $0 \in \mathfrak{M}$ of T and that T is semi-attractive of type $(1, b)_1$ at 0. In the expression (0.2) we can assume that $a_0 = 1$. By applying the results of this section, we easily obtain the following theorem.

Theorem 11.6. (i) *There is an injective holomorphic mapping* $H: \mathbf{C} \to \mathfrak{M}$ *which satisfies the equation* $T \circ H(\sigma) = H(\sigma + 1)$ $(\sigma \in \mathbf{C})$, *and such that* $H(\sigma) \to O$ *as* $\operatorname{Re} \sigma \to -\infty$.

(ii) *Such a mapping* H *is unique up to the translation of the variable* σ.

(iii) *The set* $H(\mathbf{C}) \cup \{O\}$ *is the set of all* (*simply*) *convergent points with respect to* T^{-1}.

12. Analytic continuation of the invariant asymptotic curve and coordinates on the domain of uniform convergence for T.

In this section we continue considering local transformations of type $(1, b)_1$. We show that the invariant asymptotic curve $H(\hat{\Sigma})$ and the set U of all uniformly convergent points with respect to T have non-empty intersection. We obtain a canonically determined coordinate system on U using the invariant asymptotic curve. This supplements the result of [I, Sec. 9].

12.1. We defined, in the σ-plane, a half-plane Σ_L by (11.2), which will be denoted by Σ for simplicity. In addition we define half-planes by

$$\Sigma^+ = \Sigma^+_{L'} = \{\sigma \in \mathbf{C} \mid \operatorname{Im} \sigma > L'\},$$
$$\Sigma^- = \Sigma^-_{L'} = \{\sigma \in \mathbf{C} \mid \operatorname{Im} \sigma < -L'\},$$

where L' is a positive number. We set

$$\Sigma^* = \Sigma^*_{L,L'} = \Sigma \cup \Sigma^+ \cup \Sigma^-$$

Theorem 12.1. *If* L' *is sufficiently large, then the mapping* $H: \Sigma \to V$ *can be extended to a holomorphic mapping* (*denoted by the same letter*) $H: \Sigma^* \to V$. *The points in* $H(\Sigma^+)$, $H(\Sigma^-)$ *are all uniformly convergent with respect to* T.

Let us first prepare a lemma. We denote

$$D_{\theta,R} = \{(z, w) \mid \operatorname{Re}(e^{-i\theta} z) > R,\ |w| < \rho\} \subset V.$$

We have shown in [I, Proposition 7.2] that, for every θ with $-\pi/2 < \theta < \pi/2$, there is a sufficiently large R such that $D_{\theta,R}$ is a base of uniform convergence for T. We fix θ $(0 < \theta < \pi/2)$ and R such that $D_{\theta,R}$ and $D_{-\theta,R}$ are both bases of uniform convergence for T. Then $D := D_{\theta,R} \cup D_{-\theta,R}$ is also a base of uniform convergence.

Further we consider domains in the σ-plane:

$$A^+ = A^+_{L,L'} = \{\sigma\in\mathbf{C}\,|\,-L-a_0 \leqq \operatorname{Re}\sigma < -L,\ \operatorname{Im}\sigma > L'\},$$
$$A^- = A^-_{L,L'} = \{\sigma\in\mathbf{C}\,|\,-L-a_0 \leqq \operatorname{Re}\sigma < -L,\ \operatorname{Im}\sigma < -L'\}.$$

Lemma 12.2. *If L' is sufficiently large, then $H(A^+)$ is contained in $D_{\theta,R}$ and $H(A^-)$ is contained in $D_{-\theta,R}$.*

Proof. Choose ε so that $0<\varepsilon<\sin\theta$. By Theorem 11.1, (ii) we can choose L'_0 so that $|h(\sigma)-\sigma|<\varepsilon\operatorname{Im}\sigma$ for $\sigma\in A^+_{L,L_0'}$. Then we have $|\operatorname{Re}(e^{-i\theta}(h(\sigma)-\sigma))| < \varepsilon\operatorname{Im}\sigma$, and hence

$$\begin{aligned}\operatorname{Re}(e^{-i\theta}h(\sigma)) &> \operatorname{Re}(e^{-i\theta}\sigma)-\varepsilon\operatorname{Im}\sigma\\ &= \cos\theta\cdot\operatorname{Re}\sigma+(\sin\theta-\varepsilon)\operatorname{Im}\sigma\\ &\geqq -\cos\theta\cdot(L+a_0)+(\sin\theta-\varepsilon)\operatorname{Im}\sigma.\end{aligned}$$

Thus, if $L'(\geqq L'_0)$ is sufficiently large, we have

$$\operatorname{Re}(e^{-i\theta}h(\sigma)) > -\cos\theta\cdot(L+a_0)+(\sin\theta-\varepsilon)L'>0$$

for $\sigma\in A^+_{L,L'}$. This implies $H(A^+_{L,L'})\subset D_{\theta,R}$. Similarly we have $H(A^-_{L,L'})\subset D_{-\theta,R}$.
q.e.d.

To prove the theorem, we choose L' as in the lemma. For $\sigma\in\Sigma^+$ [resp. Σ^-] we choose an integer n so that $\sigma-na_0\in A^+$ [resp. A^-]. Then the point $H(\sigma-na_0)$ is a (uniformly) convergent point with respect to T by the lemma. It is also a (simply) convergent point with respect to T^{-1}. Hence $T^n\circ H(\sigma-na_0)$ is in V. We define $H(\sigma)$ by setting $H(\sigma)=T^n\circ H(\sigma-na_0)$. In view of the functional equation (11.3), this gives an extension of H to a holomorphic mapping defined on Σ^*. Thus Theorem 12.1 is proved.

12.2. We have constructed in [I, Sec. 8] a solution φ, called Abel-Fatou function, of the equation

$$\varphi(T(P))=\varphi(P)+a_0 \tag{12.1}$$

defined in the domain of uniform convergence U for T. Since $H(\Sigma^+)$, $H(\Sigma^-)$ are contained in U, we can define functions

$$\begin{aligned} p^+(\sigma)&=\varphi(H(\sigma)) \qquad \sigma\in\Sigma^+,\\ p^-(\sigma)&=\varphi(H(\sigma)) \qquad \sigma\in\Sigma^-.\end{aligned}\tag{12.2}$$

Proposition 12.3. (i) The functions $p^\pm$ satisfy the equation

$$p^\pm(\sigma+a_0)=p^\pm(\sigma)+a_0,\qquad \sigma\in\Sigma^\pm.$$

(ii) They are expressed by series

$$p^+(\sigma)=\sigma+\sum_{\nu=0}^{\infty}p^+_\nu\exp(2\nu\pi i\sigma/a_0)$$

$$p^-(\sigma) = \sigma + \sum_{\nu=0}^{\infty} p_{-\nu}^- \exp(-2\nu\pi i\sigma/a_0).$$

(iii) The constant terms p_0^+ and p_0^- are related by the equation

$$p_0^- - p_0^+ = 2\pi i a_1/a_0.$$

Proof. For $\sigma \in \Sigma^\pm$, we have

$$p^\pm(\sigma + a_0) = \varphi(H(\sigma + a_0)) = \varphi(T(H(\sigma))) = \varphi(H(\sigma)) + a_0 = p^\pm(\sigma) + a_0,$$

which shows (i).

Now, by (i), $p^\pm(\sigma) - \sigma$ are holomorphic functions with period a_0 on $\Sigma^\pm$. Hence they are expanded into Fourier series

$$p^\pm(\sigma) - \sigma = \sum_{\nu=-\infty}^{\infty} p_\nu^\pm \exp(2\nu\pi i\sigma/a_0)$$

convergent on Σ^+ and Σ^- respectively. To prove (ii), we want to show that $p_\nu^+ = 0$ for $\nu < 0$ and that $p_\nu^- = 0$ for $\nu > 0$. In view of the periodicity of $p^\pm(\sigma) - \sigma$, it suffices to show that $p^+(\sigma) - \sigma$ [resp. $p^-(\sigma) - \sigma$] tends to a finite value as $|\sigma| \to \infty$ $\sigma \in A^+$ [resp. $\sigma \in A^-$].

We use the notation $\log^{(1)}\zeta$ [resp. $\log^{(2)}\zeta$] to denote the single-valued branch of $\log\zeta$ on $\mathbf{C}-$(the negative real axis) [resp. $\mathbf{C}-$(the positive real axis)] determined by the condition

$$-\pi < \operatorname{Im}\log^{(1)}\zeta < \pi \qquad [\text{resp. } 0 < \operatorname{Im}\log^{(2)}\zeta < 2\pi].$$

We note that

$$\log^{(2)}\zeta - \log^{(1)}\zeta = \begin{cases} 0 & \text{if } \operatorname{Im}\zeta > 0, \\ 2\pi i & \text{if } \operatorname{Im}\zeta < 0. \end{cases}$$

Now every Abel-Fatou function $\varphi(P)$ has the form

$$\varphi(P) = z(P) - \frac{a_1}{a_0}\log^{(1)} z(P) + A(P),$$

where $A(P) \to A$ (constant) as $|z(P)| \to \infty$, $P \in D = D_{\theta,R} \cup D_{-\theta,R}$ (see [I, Sec. 8.1]).

On the other hand, by Theorem 11.1, the mapping $H(\sigma) = (h(\sigma), k(\sigma))$ has the form

$$h(\sigma) = \sigma + \frac{a_1}{a_0}\log^{(2)}\sigma + B(\sigma)$$

where $B(\sigma) \to B$ (constant) as $|\sigma| \to \infty$, $\sigma \in \Sigma$.

If $\sigma \in A^\pm$, then $H(\sigma) \in D$ and hence

$$\begin{aligned} p^\pm(\sigma) &= \varphi(h(\sigma), k(\sigma)) \\ &= \sigma + \frac{a_1}{a_0}\log^{(2)}\sigma + B(\sigma) - \frac{a_1}{a_0}\log^{(1)} h(\sigma) + A(H(\sigma)). \end{aligned}$$

When $|\sigma| \to \infty$, $\sigma \in A^{\pm}$, we have $h(\sigma)/\sigma \to 1$; hence

$$\log^{(1)} h(\sigma) - \log^{(1)} \sigma \longrightarrow 0.$$

Since $\log^{(2)} \sigma - \log^{(1)} \sigma = 0$ or $2\pi i$ according as $\sigma \in A^+$ or A^-, we conclude that $p^+(\sigma) - \sigma \to A + B$ when $|\sigma| \to \infty$, $\sigma \in A^+$; and $p^-(\sigma) - \sigma \to A + B + 2\pi i a_1/a_0$ when $|\sigma| \to \infty$, $\sigma \in A^-$. Thus (ii) is proved.

Since $p_0^+ = A + B$, and $p_0^- = A + B + 2\pi i a_1/a_0$, the assertion (iii) is also proved.

As a corollary we have the following

Proposition 12.4. *If L' is sufficiently large, then $s = p^+(\sigma)$ [resp. $s = p^-(\sigma)$] maps conformally Σ^+ [resp. Σ^-] onto a domain $\mathscr{S}^+ = p^+(\Sigma^+)$ [resp. $\mathscr{S}^- = p^-(\Sigma^-)$] in $\mathbf{C}$ (the s-plane). The domains $\mathscr{S}^{\pm}$ are invariant under the translation $s \mapsto s + a_0$; The domain $\mathscr{S}^+$ [resp. $\mathscr{S}^-$] contains a half-plane of the form $\{s \in \mathbf{C} | \operatorname{Im} s > L''\}$ [resp. $\{s \in \mathbf{C} | \operatorname{Im} s < -L''\}$].*

12.3. Now let us recall the result of [I, Sec. 9] and make some supplementary observations (which could have been included in Sec. 9).

As in [I, Sec. 8.2], let

$$\mathscr{B} = \{s \in \mathbf{C} | \operatorname{Re} s > R_1\}$$

and

$$D[\mathscr{B}] = \{P \in U | \varphi(P) \in \mathscr{B}\}.$$

When R_1 is sufficiently large, the mapping $P \mapsto (s, v)$ defined by $s = \varphi(P)$, $v = w(P)$ maps $D[\mathscr{B}]$ biholomorphically onto $\mathscr{B} \times \{|v| < \rho\}$. (In (8.11), we have $\alpha = \arg a_1 = 0$; and set $\theta' = -\pi/2$, $\theta'' = \pi/2$.)

We constructed a holomorphic function $\psi(P)$ on $D[\mathscr{B}]$ which satisfies the equation

$$\psi(T(P)) = \psi(P) + \kappa(\varphi(P)). \qquad P \in D[\mathscr{B}], \tag{12.3}$$

where $\kappa(s)$ is a holomorphic function of one complex variable $s \in \mathscr{B}$.

We want to obtain, by modifying $\psi(P)$, a holomorphic function $\psi_*(P)$ which is invariant under T, i.e., $\psi_*(T(P)) = \psi_*(P)$. For this purpose we will construct a holomorphic function $\lambda(s)$ on B satisfying the difference equation

$$\lambda(s + a_0) = \lambda(s) + \kappa(s) \qquad s \in \mathscr{B}. \tag{12.4}$$

We prove the existence of a solution $\lambda(s)$ of (12.4) in a slightly general situation. We denote

$$\mathscr{B}(R_1, R_2) = \{s \in \mathbf{C} | R_1 < \operatorname{Re} s < R_2\} \qquad \text{for } -\infty \leqq R_1 < R_2 \leqq +\infty.$$

Lemma 12.5. *Let $\kappa(s)$ be a holomorphic function on $\mathscr{B}(R_1, R_2)$ and let a_0 be a positive number. Then there exists a holomorphic function $\lambda(s)$ on $\mathscr{B}(R_1, R_2 + a_0)$ which satisfies the equation* (12.4) *for $s \in \mathscr{B}(R_1, R_2)$.*

Proof. Choose a real number R_0 with $R_1 < R_0 < R_2$, and a positive number δ so that $0 < \delta < a_0/4$, $R_1 < R_0 - \delta$, $R_0 + \delta < R_2$. Consider the holomorphic mapping $\varepsilon: \mathbf{C} \to \mathbf{C}^*$ defined by $s \mapsto \zeta = \varepsilon(s) = \exp(2\pi i (s - R_0)/a_0)$. This function ε maps the domains $\mathscr{B}_1 = \mathscr{B}(R_0 - \delta, R_0 + a_0/2 + \delta)$, $\mathscr{B}_2 = \mathscr{B}(R_0 + a_0/2 - \delta, R_0 + a_0 + \delta)$ conformally onto

$$\mathscr{A}_1 = \{\zeta \in \mathbf{C} \mid -\delta' < \arg\zeta < \pi + \delta'\}$$
$$\mathscr{A}_2 = \{\zeta \in \mathbf{C} \mid \pi - \delta' < \arg\zeta < 2\pi + \delta'\},$$

respectively, where $\delta' = 2\pi\delta/a_0 (< \pi/2)$.

$\{\mathscr{A}_1, \mathscr{A}_2\}$ is an open covering of $\mathbf{C}^*$ and the intersection of $\mathscr{A}_1$ and $\mathscr{A}_2$ consists of two connected components:

$$\mathscr{A}' = \{\zeta \in \mathbf{C} \mid -\delta' < \arg\zeta < \delta'\}$$
$$\mathscr{A}'' = \{\zeta \in \mathbf{C} \mid \pi - \delta' < \arg\zeta < \pi + \delta'\}.$$

We note that ε maps $\mathscr{B}' = \mathscr{B}(R_0 - \delta, R_0 + \delta)$ onto $\mathscr{A}'$. Let

$$\tilde{\kappa}(\zeta) = \kappa \circ (\varepsilon|\mathscr{B}')^{-1}(\zeta) \qquad \text{for } \zeta \in \mathscr{A}'.$$

Since the first cohomology group $H^1(\mathbf{C}^*, \mathscr{O})$ with coefficients in holomorphic functions vanishes, there exist holomorphic functions $\tilde{\lambda}_1(\zeta)$ on $\mathscr{A}_1$ and $\tilde{\lambda}_2(\zeta)$ on $\mathscr{A}_2$ such that

$$\tilde{\lambda}_2(\zeta) - \tilde{\lambda}_1(\zeta) = \begin{cases} \tilde{\kappa}(\zeta) & \text{on } \mathscr{A}' \\ 0 & \text{on } \mathscr{A}''. \end{cases} \tag{12.5}$$

We define

$$\lambda(s) = \begin{cases} \tilde{\lambda}_1(\varepsilon(s)) & \text{on } \mathscr{B}_1 \\ \tilde{\lambda}_2(\varepsilon(s)) & \text{on } \mathscr{B}_2 \end{cases}$$

We know by the second equality of (12.5) that $\lambda(s)$ is well-defined, and by the first equality that $\lambda(s)$ satisfy (12.4) for $s \in \mathscr{B}(R_0 - \delta, R_0 + \delta)$. Using this equation, $\lambda(s)$ can be analytically continued to $\mathscr{B}(R_1, R_2 + a_0)$. q.e.d.

Now returing to our problem, let $\lambda(s)$, $s \in B$, be a solution of (12.4). We set

$$\psi_*(P) = \psi(P) - \lambda(\varphi(P)), \qquad P \in D[\mathscr{B}]. \tag{12.6}$$

This function is invariant under T, as can be verified by substituting $s = \varphi(P)$ into (12.4) and subtracting (12.4) from (12.3).

As was mentioned in [I, Sec. 9.3], we can regard $s = \varphi(P)$, $u = \psi(P)$ as coordinates on $D[\mathscr{B}]$. Hence, setting $t = \psi_*(p)$, we can regard $(s, t) = (s, u - \lambda(s))$ as a coordinate system on $D[\mathscr{B}]$. With respect to this coordinate system, the transformation T restricted to $D[\mathscr{B}]$ is represented by the translation $(s, t) \mapsto (s + a_0, t)$.

12.4. We want to choose a solution $\lambda(s)$ of (12.4) so that $\psi_*(P)$ defined by

(12.6) have some good property. The clue for the choice is the invariant asymptotic curve.

Let $\lambda(s)$ be any solution of (12.4) and let $\psi_*(P)$ be defined by (12.6). We set

$$q^{\pm}(\sigma) = \psi_*(H(\sigma)) \qquad \sigma \in \Sigma^{\pm}.$$

The functions $q^{\pm}(\sigma)$ are holomorphic and have period a_0, i.e., $q^{\pm}(\sigma + a_0) = q^{\pm}(\sigma)$. Hence they can be expanded into Fourier series

$$q^{\pm}(\sigma) = \sum_{\nu=-\infty}^{\infty} q_\nu^{\pm} \exp(2\nu\pi i\sigma/a_0) \qquad \sigma \in \Sigma^{\pm}.$$

The sets $H(\Sigma^{\pm})$ are analytic subsets of $\{P \in D[\mathscr{B}] \mid \varphi(P) \in \mathscr{S}^{\pm}\}$ and expressed with respect to the coordinate system (s, t) as graphs

$$H(\Sigma^{\pm}) = \{(s, \rho^{\pm}(s)) \mid s \in \mathscr{S}^{\pm}\}$$

where $\rho^{\pm}(s) = q^{\pm}((p^{\pm})^{-1}(s))$ by Proposition 12.4. The functions $\rho^{\pm}(s)$ have period a_0, since $(p^{\pm})^{-1}(s + a_0) = (p^{\pm})^{-1}(s) + a_0$ and $q^{\pm}(\sigma)$ have period a_0. (This is also clear by the fact that $H(\Sigma^{\pm})$ are invariant under T.) Hence $\rho^{\pm}(s)$ can be expanded into Fourier series

$$\rho^{\pm}(s) = \sum_{\nu=-\infty}^{\infty} \rho_\nu^{\pm} \exp(2\nu\pi is/a_0) \qquad \operatorname{Im} s > L'' \text{ or } \operatorname{Im} s < -L''.$$

Proposition 12.6. *We can choose a solution $\lambda(s)$ of* (12.4) (*and hence $\psi_*(P)$*) *in such a way that*

(i) $$q_\nu^+ = 0 \text{ for } \nu < 0 \text{ and } q_\nu^- = 0 \text{ for } \nu > 0.$$

(ii) $$\rho_\nu^+ = 0 \text{ for } \nu < 0 \text{ and } \rho_\nu^- = 0 \text{ for } \nu > 0.$$

We note that (i) is equivalent to the condition: $q^+(\sigma)$ [resp. $q^-(\sigma)$] tends to a finite value when $\operatorname{Im}\sigma \to +\infty$ [resp. $\operatorname{Im}\sigma \to -\infty$]; and that (ii) is equivalent to the condition $\rho^+(s)$ [resp. $\rho^-(s)$] tends to a finite value when $\operatorname{Im} s \to +\infty$ [resp. $\operatorname{Im} s \to -\infty$]. Since $\Sigma^{\pm}$ and $\mathscr{S}^{\pm}$ correspond conformally and $\operatorname{Im}\sigma \to \pm\infty$ if and only if $\operatorname{Im} s \to \pm\infty$, the conditions (i), (ii) are eqivalent to each other.

To prove Proposition 12.6, we start from any solution $\lambda(s)$ which exists by Lemma 12.5. We shall construct a new solution $\hat{\lambda}(s)$ required in the proposition by modifying $\lambda(s)$.

We decompose $\rho^{\pm}(s)$ as follows:

$$\rho^+(s) = \rho_-^+(s) + \rho_0^+ + \rho_+^+(s)$$
$$\rho^-(s) = \rho_-^-(s) + \rho_0^- + \rho_+^-(s)$$

where we set

$$\rho_-^+(s) = \sum_{\nu<0} \rho_\nu^+ \exp(2\nu\pi is/a_0), \qquad \rho_+^+(s) = \sum_{\nu>0} \rho_\nu^+ \exp(2\nu\pi is/a_0),$$

$$\rho_-^-(s) = \sum_{\nu<0} \rho_\nu^- \exp(2\nu\pi is/a_0), \qquad \rho_+^-(s) = \sum_{\nu>0} \rho_\nu^- \exp(2\nu\pi is/a_0).$$

Here, it is important to observe that the series which define $\rho_-^+(s)$ and $\rho_+^-(s)$ are convergent for all $s \in \mathbf{C}$. We regard them as entire functions with period a_0.

We define a new solution of (12.4) by

$$\hat{\lambda}(s) = \lambda(s) + \rho_-^+(s) + \rho_+^-(s) \qquad s \in \mathscr{B}.$$

Let

$$\begin{aligned} \hat{\psi}_*(P) &= \psi(P) - \hat{\lambda}(\psi(P)) && (P \in D[\mathscr{B}]); \\ \hat{q}^\pm(\sigma) &= \hat{\psi}_*(H(\sigma)) && (\sigma \in \Sigma^\pm); \\ \hat{\rho}^\pm(s) &= \hat{q}^\pm((p^\pm)^{-1}(s)) && (s \in \mathscr{S}^\pm). \end{aligned}$$

Then,

$$\hat{q}^\pm(\sigma) = q^\pm(\sigma) - \rho_-^+(\varphi(H(\sigma)) - \rho_+^-(H(\sigma)) \qquad (\sigma \in \Sigma^\pm).$$

Hence

$$\begin{aligned} \hat{q}^+(\sigma) &= \rho^+(p^+(\sigma)) - \rho_-^+(p^+(\sigma)) - \rho_+^-(p^+(\sigma)) \\ &= p_0^+ + \rho_+^+(p^+(\sigma)) - \rho_+^-(p^+(\sigma)) && (\sigma \in \Sigma^+); \\ \hat{\rho}^+(s) &= p_0^+ + \rho_+^+(s) - \rho_+^-(s) \\ &= p_0^+ + \sum_{\nu=1}^{\infty} (\rho_\nu^+ - \rho_\nu^-) \exp(2\nu\pi is/a_0) && (s \in \mathscr{S}^+). \end{aligned}$$

Similarly,

$$\begin{aligned} \hat{q}^-(\sigma) &= \rho^-(p^-(\sigma)) - \rho_-^+(p^-(\sigma)) - \rho_+^-(p^-(\sigma)) \\ &= p_0^- + \rho_-^-(p^-(\sigma)) - \rho_-^+(p^-(\sigma)) && (\sigma \in \Sigma^-); \\ \hat{\rho}^-(s) &= p_0^+ + \rho_-^-(s) - \rho_-^+(s) \\ &= p_0^- + \sum_{\nu=1}^{\infty} (\rho_{-\nu}^- - \rho_{-\nu}^+) \exp(-2\nu\pi is/a_0) && (s \in \mathscr{S}^-). \end{aligned}$$

Thus $\hat{\lambda}(s)$ has the required properties and Proposition 12.6 is proved.

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