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On a conjecture of C. T. C. Wall

By

R. V. GURJAR

Introduction.

In [6] C. T. C. Wall made the following conjecture: 'Let G be a reductive algebraic group/$\boldsymbol{C}$ acting linearly on $\boldsymbol{C}^n$ such that the quotient $\boldsymbol{C}^n/G$ has dimension 2. Then $\boldsymbol{C}^n/G$ is isomorphic as an algebraic variety to $\boldsymbol{C}^2/\Gamma$, where $\Gamma\subset GL(2, \boldsymbol{C})$ is a finite group of automorphisms of $\boldsymbol{C}^2$'.

The purpose of this paper is to prove following more general result:

Theorem. *Let X be a smooth affine variety/$\boldsymbol{C}$ and G a reductive algebraic group/$\boldsymbol{C}$ acting rationally on X. Let $V=X/G$ and $y\in V$ any point. Then there exists a reductive algebraic group $H/\boldsymbol{C}$ acting linearly on some $\boldsymbol{C}^n$ such that the analytic local ring of V at y is isomorphic to the analytic local ring of $\boldsymbol{C}^n/H$ at it's vertex. Further, denoting by φ the quotient morphism $\boldsymbol{C}^n\to\boldsymbol{C}^n/H$, the co-dimension of $\varphi^{-1}(\operatorname{sing}\boldsymbol{C}^n/H)$ in $\boldsymbol{C}^n$ is bigger than* 1.

(Here $\operatorname{sing} Z$ denotes the singular locus of an algebraic variety Z).

Corollary 1. *With the same notations as above, $\pi_1^y(V-\operatorname{sing} V)$ is finite. In particular, if $\dim V=2$, then V has atmost quotient singularities and C. T. C. Wall's conjecture is true.*

(For the definition of $\pi_1^y(V-\operatorname{sing} V)$, see §1.

Corollary 2. *V has only rational singularities.*

In fact, from our proof of the theorem, it is clear that for proving Corollary 2, we only need X to be normal such that the divisor class groups of it's local rings are all torsion. Of course, Boutot's result in [1] is more general, but we have included this result (the main idea in the proof of Corollary 2 being Kempf's) because of the belief that this argument has not been observed before.

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§1. We begin with the definition of $\pi_1^{y}(V-\operatorname{sing} V)$. Let Z be any normal complex space and $z\in Z$. We can find a fundamental system of neighbourhoods $U_1\supset U_2\supset\cdots$ of z in Z satisfying the following conditions:

i) for $i>j$, U_i is a strong deformation retract of U_j

ii) for $i>j$, $U_i-\operatorname{sing} Z$ is a strong deformation retract of $U_j-\operatorname{sing} Z$.

The existence of such a system of neighbourhoods can be proved using the fact that the pair $(Z, \operatorname{sing} Z)$ is triangulable.

Now we define $\pi_1^{z}(Z-\operatorname{sing} Z)=\pi_1(U_1-\operatorname{sing} Z)$.

Now we begin with the proof of the Theorem. Let $y\in V$ be arbitrary. We denote the quotient morphism $X\to V$ by π. We choose a point $x\in\pi^{-1}(y)$ whose orbit is closed. Then by Luna's slice theorem ([3]), the reductive group G_x acts linearly on the tangent space $T_{S,x}$ to the slice S at x such that the analytic local ring of $T_{S,x}/G_x$ at the "vertex" in $T_{S,x}/G_x$ is isomorphic to the analytic local ring of V at y. We can therefore assume that $X=\boldsymbol{C}^m$, G is acting linearly on $\boldsymbol{C}^m$ and y is the "vertex" of V.

Assume now that $\pi^{-1}(\operatorname{sing} V)$ has a co-dimension 1 irreducible component.

The following proposition is one of the key observations in the proof.

Proposition. *Let X be a normal affine variety/$\boldsymbol{C}$ such that the local rings of X at it's closed points all have torsion divisor class groups. Suppose G is a reductive algebraic group acting rationally on X, $V=X/G$ and $\pi: X\to V$ the quotient morphism. Let $S\subset V$ be a closed subvariety of co-dimension$\geqq 2$ in V. Suppose $\pi^{-1}(S)=D\cup E$, where D is the union of all the irreducible components of $\pi^{-1}(S)$ of co-dimension 1 in X. Then $E\neq\phi$ and the induced morphism $X-D/G\to X/G$ is an isomorphism.*

Proof. By assumption on the local rings of X, $X-D$ is affine and G-stable. Write $W=X-D/G$. We have the induced morphism $f: W\to V$. For any $y\in V-S$, let $x\in\pi^{-1}(y)$ be a point with closed G-orbit Gx in X. Clearly $Gx\subset X-D$ and there exists a point $y'\in W$ (which is the image of x under the morphism $g: X-D\to W$) such that $f(y')=y$. Suppose $y''\in W$ is another point such that $f(y'')=y$ and let $x''\in X-D$ be a point with closed orbit Gx'' in $X-D$. Then the closure $\overline{Gx''}$ of Gx'' in X intersects Gx. But since Gx is also closed in $X-D$, Gx'' cannot be closed in $X-D$, a contradiction. This shows that for every $y\in V-S$, there is a unique point in W lying over y. Thus the morphism f is birational. Now we use the following easy result (sometimes attributed to R.W. Richardson).

Lemma. *Let $f: V_1\to V_2$ be a birational morphism between normal affine varieties. If co-dimension of $\overline{V_2-f(V_1)}\geqq 2$ in V_2, then f is an isomorphism.*

Proof. For every irreducible divisor $\Delta\subset V_2$, there exists a divisor $\Delta'\subset V_1$ such that $f(\Delta')$ is Zariski dense in Δ. Since an affine normal domain is the intersection of it's localizations at height 1 primes, the result follows.

The Proof of the Proposition is now complete.

Going back to the situation of the theorem, write $\pi^{-1}(\operatorname{sing} V)=D\cup E$ as in the proposition above.

Now let $X_1=\boldsymbol{C}^m-D$ and $\pi_1: X_1\to V$ the quotient morphism.

Let $x_1\in X_1$ be a point lying over y such that the orbit of x_1 is closed in X_1. In $\boldsymbol{C}^m$, 0 is the only point lying over y with closed orbit. It follows that the isotropy subgroup $G_{x_1}\subsetneqq G$. Again by Luna's slice theorem, G_{x_1} acts on the tangent space T_{S_1,x_1} of the slice S_1 at x_1 such that the analytic local ring of $T_{S_1,x_1/G_{x_1}}$ at it's "vertex" is isomorphic to the analytic local ring of V at y. Further, $\dim T_{S_1,x_1}<\dim X_1$ because $\dim Gx_1+\dim T_{S_1,x_1}=\dim X_1$ and the orbit Gx_1 is positive dimensional. Then we analyse the map $\pi_2: T_{S_1,x_1}\to T_{S_1,x_1/G_{x_1}}$ treating V as $T_{S_1,x_1/G_{x_1}}$ and T_{S_1,x_1} as the new affine space $\boldsymbol{C}^{m_2}$. If co-dim $\pi_2^{-1}(\operatorname{sing} V)\geqq 2$, we are done, otherwise we repeat the argument until we come to a situation as desired in the theorem.

This completes the proof of the theorem.

Proof of Corollary 1. Assume now that $\varphi: \boldsymbol{C}^n\to V$ has the property that co-dim $\varphi^{-1}(\operatorname{sing} V)\geqq 2$. Then $\boldsymbol{C}^n-\pi^{-1}(\operatorname{sing} V)$ is simply-connected and the homomorphism $\pi_1(\boldsymbol{C}^n-\varphi^{-1}(\operatorname{sing} V))\to\pi_1(V-\operatorname{sing} V)$ has image of finite index (if G is connected, the homomorphism is surjective). See [4] Lemma 1.5. Thus $\pi_1(V-\operatorname{sing} V)$ is finite. As V has a good $\boldsymbol{C}^*$-action with y as the vertex, it is easy to see that $\pi_1^y(V-\operatorname{sing} V)\approx\pi_1(V-\operatorname{sing} V)$. For proving C. T. C. Wall's conjecture, we can use well-known properties of normal affine surfaces with a good $\boldsymbol{C}^*$-action to conclude that when G acts linearly on $\boldsymbol{C}^n$ with $\dim\boldsymbol{C}^n/G=2$, then $\boldsymbol{C}^n/G\approx\boldsymbol{C}^2/\Gamma$ as desired. See, for example [5].

Proof of Corollary 2. Here we use Kempf's argument from [2]. By the result of Hochster and Roberts, V is Cohen-Macaulay. As in the theorem, let $\varphi: \boldsymbol{C}^n\to V$ be a morphism such that co-dim $\varphi^{-1}(\operatorname{sing} V)\geqq 2$. Then Kempf's proof shows that any rational d-form on $V(d=\dim V)$, which is regular on V-sing V extends to a regular form on any desingularization of V, proving that V has rational singularities.

Example. Let $\boldsymbol{C}^*$ act on $\boldsymbol{C}[X, Y, Z, W]$ by $\rho_t(X))=tX$, $\rho_t(Y)=tY$, $\rho_t(Z)=t^{-1}Z$, $\rho_t(W)=t^{-1}W$. The ring of invariants is $R=\boldsymbol{C}[XZ, XW, YZ, YW]$. Write $\bar{U}=XZ$, $\bar{V}=XW$, $\bar{S}=YZ$, $\bar{T}=YW$. Then $R\approx\boldsymbol{C}[U, V, S, T]/(UT-VS)$, $\bar{U}$ being the image of U in R etc. Let $V=\operatorname{Spec} R$. Now let $\boldsymbol{C}^*$ act on R by $\sigma_\lambda(\bar{U})=\bar{U}$, $\sigma_\lambda(\bar{T})=\lambda\bar{T}$, $\sigma_\lambda(\bar{V})=\bar{V}$ and $\sigma_\lambda(\bar{S})=\lambda\bar{S}$ for $\lambda\in\boldsymbol{C}^*$. The ring of invariants, $R^{\boldsymbol{C}^*}$, is $\boldsymbol{C}[\bar{U}, \bar{V}]$ which is isomorphic to a polynomial ring in 2-variables. If $\varphi: V\to\operatorname{Spec} R^{\boldsymbol{C}^*}$ is the quotient morphism, then the inverse image of the "origin" in $\operatorname{Spec} R^{\boldsymbol{C}^*}$ given by the maximal ideal $(\bar{U}, \bar{V})$ is the irreducible divisor $D=\{\bar{U}=0=\bar{V}\}$ in V. It is easy to see that D has infinite order in the divisor class group of V.

This example shows that the condition on the local rings of X in the proposition cannot be dropped.

SCHOOL OF MATHEMATICS
TATA INSTITUTE OF FUNDAMENTAL RESEARCH
HOMI BHABHA ROAD
BOMBAY 4000 005
INDIA

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