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Mutually disjoint irreducible decompositions of the regular representation of a restricted direct product group

Dedicated to Professor Nobuhiko Tatsuma on his 60th birthday

By

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Introduction

It is well-known that the central factor decomposition of a unitary representation is unique for any locally compact group. But decompositions to a direct integral of irreducible representations are not unique when non-type I factor representations appear in the central decomposition. In 1951 Yoshizawa and Mackey have firstly found independently this phenomenon in case of the regular representation of the free group with two generators [11], [6]. In this paper we study the regular representation of a restricted direct product $G=\prod'_{\alpha\in A}G_\alpha$ of a countably infinite family of finite groups G_α, $\alpha\in A$, with discrete topology. We can construct continuously many, mutually disjoint irreducible decomposition, for the left regular representation L of G, in case an infinite number of G_α are not abelian.

The diversity of irreducible decompositions of regular representations of non type I groups has been investigated by many authors [4], [5], [6], [9], [11], but our method for decompositions, explained below, is quite simple.

Firstly, we decompose L into factor representations as follows. Take an irreducible representation ξ_α of G_α on a representation space $V(\xi_\alpha)$ of a finite dimension $d(\xi_\alpha)$. We denote by $\boldsymbol{B}(\xi_\alpha)$ the space of all linear operators on $V(\xi_\alpha)$. The space $\boldsymbol{B}(\xi_\alpha)$ has a natural inner product $\langle u, v\rangle=(1/(d(\xi_\alpha))\operatorname{Tr}(uv^*)$ $(u, v\in\boldsymbol{B}(\xi_\alpha))$. Then we consider a representation on the Hilbert space $\boldsymbol{B}(\xi_\alpha)$ by left multiplications of $\xi_\alpha(g)$, $g\in G_\alpha$. Each component of the central decomposition of L is an infinite tensor product of these finite dimensional representations. The reference vector of this infinite tensor product is a sequence of identity operators in $\boldsymbol{B}(\xi_\alpha)$'s. So the factors generated by the tensor product representations are of type II_1 in case $\dim\boldsymbol{B}(\xi_\alpha)\geqq 2$ for infinitely many $\alpha\in A$, or of type I otherwise.

Secondly, we decompose these factor representations to a direct integral of irreducible representations using orthonormal bases of $V(\xi_\alpha)$'s. The irreducible components of our decompositions are infinite tensor products of $\{\xi_\alpha, V(\xi_\alpha)\}$'s and the reference vectors are sequences of elements of fixed orthonormal bases of $V(\xi_\alpha)$'s. This irreducible decomposition depends essentially on the choice of the system of orthonormal bases of $V(\xi_\alpha)$'s. Roughly speaking, since there exist continuously many

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different ways to choose an orthonormal basis for a Hilbert space of dimension$\geqq 2$, we have continuously many different irreducible decompositions of L.

The contents of this paper is as follows. §1 is a preparatory one. In section 2 we give the central factor decomposition of L by taking irreducible decomposition of the double regular representation on $L^2(G)$. In section 3, decomposing each factor representation to a direct integral of irreducible ones, we get an irreducible decomposition of L. In the final section, §4, we study the dependence of this irreducible decomposition on the choice of the system of orthonormal bases of $V(\xi_\alpha)$'s. We show that, in case G_α's are not commutative for infinitely many $\alpha\in A$, there exist continuously many, mutually disjoint decompositions of the left regular representation L of G.

§1. The regular representation of a restricted direct product group

Let G_α, $\alpha\in A$, be a countably infinite family of finite groups, and G be a restricted direct product group of G_α's:

$$G:=\prod'_{\alpha\in A}G_\alpha\,.$$

By definition, for an element $h=(h_\alpha)_{\alpha\in A}$ of G, h_α is the unit element e_α of G_α for almost all (i.e., except a finite number of) $\alpha\in A$. We consider G as a discrete group. Our problem is to discuss the diversity of irreducible decompositions of the regular representation of G. In case an infinite number of G_α's are not abelian, G is not of type I. The diversity of irreducible decompositions of regular representations of non type I groups has been treated by many mathematicians [4], [5], [6], [9], [11].

We denote by L (resp. by R) the left (resp. right) regular representation of G on $L^2(G)$, the Hilbert space of square integrable functions f on G: $\|f\|^2=\sum_{h\in G}|f(h)|^2<\infty$. Further denote by $L\cdot R$ the double representation of G on $L^2(G)$. The actions of L, R and $L\cdot R$ are given as follows: for $g, g_1, g_2\in G$, and $f\in L^2(G)$, $x\in G$,

$$L(g)f(x)=f(g^{-1}x)\,,$$

$$R(g)f(x)=f(xg)\,,$$

$$(L\cdot R)(g_1, g_2)f(x)=(L(g_1)R(g_2))f(x)=f(g_1^{-1}xg_2)\,,$$

The set $\{\delta_h\,|\,h\in G\}$ of characteristic functions of points gives an orthonormal basis of $L^2(G)$, where $\delta_h(x)=1$ for $x=h$ and $\delta_h(x)=0$ otherwise. For $g\in G$,

$$L(g)\delta_h=\delta_{gh}\,,\qquad R(g)\delta_h=\delta_{hg^{-1}}\,.$$

§2. The irreducible decomposition of the double regular representation

Let Ξ_α be a complete family of representatives of $\hat{G}_\alpha$, the set of equivalence classes of irreducible unitary representations of G_α. For a representation $\xi_\alpha\in\Xi_\alpha$, denote by $V(\xi_\alpha)$ the representation space of ξ_α and by $\boldsymbol{B}(\xi_\alpha)=\boldsymbol{B}(V(\xi_\alpha))$ the space of all bounded linear operators of $V(\xi_\alpha)$, and put $d(\xi_\alpha)=\dim V(\xi_\alpha)$. Since G_α is assumed to be finite, $V(\xi_\alpha)$ is finite-dimensional. So $\boldsymbol{B}(\xi_\alpha)$ is nothing but the space of all linear operators on $V(\xi_\alpha)$. We consider $\boldsymbol{B}(\xi_\alpha)$ as a Hilbert space with an inner product

$$\langle u, v\rangle = \langle u, v\rangle_{B(\xi_\alpha)} = \frac{1}{d(\xi_\alpha)} \mathrm{Tr}(uv^*) \qquad (u, v \in \boldsymbol{B}(\xi_\alpha)),$$

then the identity operator $1_{V(\xi_\alpha)}$ is a unit vector.

Let μ_α be a normalized measure on Ξ_α such that

$$\mu_\alpha(\{\xi_\alpha\}) = \frac{1}{|G_\alpha|} d(\xi_\alpha)^2 \qquad (\xi_\alpha \in \Xi_\alpha),$$

where $|G_\alpha|$ denotes the number of elements of G_α. Let Ξ be the direct product of Ξ_α, $\alpha \in A$ (not a restricted direct product) and μ be the product measure of μ_α's on it:

$$\Xi = \prod_{\alpha \in A} \Xi_\alpha, \qquad \mu = \prod_{\alpha \in A} \mu_\alpha .$$

Let $\xi = (\xi_\alpha) \in \Xi$. Taking the reference vector $1(\xi) = (1_{V(\xi_\alpha)})_{\alpha \in A}$, we define the infinite tensor product $\mathcal{H}(\xi)$ of Hilbert spaces $B(\xi_\alpha)$ (for the definition, cf. [3], [7] or [8]):

$$\mathcal{H}(\xi) := \otimes_{\alpha \in A} \{\boldsymbol{B}(\xi_\alpha), 1_{V(\xi_\alpha)}\} = \otimes_{\alpha \in A}^{1(\xi)} \boldsymbol{B}(\xi_\alpha).$$

We denote by ρ_ξ, $\hat{\rho}_\xi$, and $\rho_\xi \cdot \hat{\rho}_\xi$ the representations on $\mathcal{H}(\xi)$ defined as follows: for $g = (g_\alpha) \in G$ and a decomposable element $\phi = \otimes_{\alpha \in A}^{1(\xi)} \phi_{\xi_\alpha} \in \mathcal{H}(\xi)$,

$$\rho_\xi(g)\phi = \otimes_{\alpha \in A}^{1(\xi)} (\xi_\alpha(g_\alpha) \cdot \phi_{\xi_\alpha}),$$

$$\hat{\rho}_\xi(g)\phi = \otimes_{\alpha \in A}^{1(\xi)} (\phi_{\xi_\alpha} \cdot \xi_\alpha(g_\alpha^{-1})),$$

$$(\rho_\xi \cdot \hat{\rho}_\xi)(g_1, g_2) = \rho_\xi(g_1) \hat{\rho}_\xi(g_2).$$

Since $g_\alpha = e_\alpha$ for almost all α, these actions are well-defined and can be extended to the whole $\mathcal{H}(\xi)$. Because of the irreducibility of each α-component, the double representation $\rho_\xi \cdot \hat{\rho}_\xi$ is irreducible.

Let ϕ_h be a vector field on the measure space (Ξ, μ) corresponding to $h = (h_\alpha) \in G$ given as

$$\phi_h(\xi) = \otimes_{\alpha \in A}^{1(\xi)} \xi_\alpha(h_\alpha) \in \mathcal{H}(\xi), \qquad \text{for } \xi \in \Xi .$$

Since $\xi_\alpha(h_\alpha) = 1_{V(\xi_\alpha)}$ for almost all α, the value $\phi_h(\xi)$ belongs to $\mathcal{H}(\xi)$ as is wanted.

Lemma 2.1. *The family $\{\mathcal{H}(\xi) | \xi \in \Xi\}$ is a measurable field of Hilbert spaces on the measure space (Ξ, μ), with respect to the fundamental sequence $\{\phi_h | h \in G\}$ in the sense of Dixmier* [3].

This means that (i) the functions $\langle \phi_h(\xi), \phi_{h'}(\xi)\rangle_{\mathcal{H}(\xi)}$ are measurable on Ξ for fixed $h, h' \in G$, and (ii) the set of vectors $\{\phi_h(\xi) | h \in G\}$ is total in $\mathcal{H}(\xi)$ for each $\xi \in \Xi$.

Proof. For (i), let $h = (h_\alpha)$, $h' = (h'_\alpha)$, and B be a finite subset of A such that $h_\alpha = h'_\alpha = e_\alpha$ for $\alpha \in A \setminus B$. Then the function $\langle \phi_h(\xi), \phi_{h'}(\xi)\rangle_{\mathcal{H}(\xi)}$ is given as

$$\langle \phi_h(\xi), \phi_{h'}(\xi)\rangle_{\mathcal{H}(\xi)} = \prod_{\alpha \in A} \langle \xi_\alpha(h_\alpha), \xi_\alpha(h'_\alpha)\rangle_{B(\xi_\alpha)}$$

$$= \prod_{\alpha \in A} \frac{1}{d(\xi_\alpha)} \mathrm{Tr}(\xi_\alpha(h_\alpha) \xi_\alpha(h'_\alpha)^*)$$

$$=\prod_{\alpha\in B}\frac{1}{d(\xi_\alpha)}\operatorname{Tr}(\xi_\alpha(h_\alpha h_\alpha'^{-1})). \tag{2.1}$$

The last expression depends only on a finite number of components ξ_α, $\alpha\in B$, of ξ. Therefore this function is measurable on the direct product measure space (Ξ, μ).

For (ii), fix an $\xi=(\xi_\alpha)\in\Xi$. Since each ξ_α is irreducible, the set of operators $\{\xi_\alpha(h_\alpha)\mid h_\alpha\in G_\alpha\}$ is total in $\boldsymbol{B}(\xi_\alpha)$. This fact together with properties of the tensor product shows that $\{\phi_h(\xi)\mid h\in G\}$ is total in the infinite tensor product Hilbert space $\mathcal{H}(\xi)$. //

Theorem 2.2. (i) *The unique irreducible decomposition of the representation* $L\cdot R$ *of* $G\times G$ *on* $L^2(G)$ *is given by the following direct integral:*

$$\{L\cdot R,\ L^2(G)\}\cong {}^{(\phi_h)}\!\!\int_{\Xi}^{\oplus}\{\rho_\xi\cdot\hat{\rho}_\xi,\ \mathcal{H}(\xi)\}\,d\mu(\xi).$$

A natural equivalence mapping is given by

$$L^2(G)\ni\delta_h \longleftrightarrow \phi_h\in {}^{(\phi_h)}\!\!\int_{\Xi}^{\oplus}\mathcal{H}(\xi)\,d\mu(\xi).$$

(ii) *The restriction to* $G\times\{e\}$ *(e is the unit of G) of this decomposition gives the central factor decomposition of the left regular representation L of G as*

$$\{L,\ L^2(G)\}\cong {}^{(\phi_h)}\!\!\int_{\Xi}^{\oplus}\{\rho_\xi,\ \mathcal{H}(\xi)\}\,d\mu(\xi).$$

The factor representations coming into this are of type II_1 *or type* I *finite.*

Here $\cong$ denotes unitary equivalence, and the notation ${}^{(\phi_h)}\!\!\int^{\oplus}$ follows Dixmier [3].

Proof. (i) We denote by P the representation in the right hand side in (i) given by the direct integral.

Firstly we compute the inner product of ϕ_h and $\phi_{h'}$. For h and h', we take a finite $B\subset A$ as in the proof of Lemma 2.1. By the formula (2.1) and the finiteness of B, we have

$$\begin{aligned}\langle\phi_h,\ \phi_{h'}\rangle&=\int_{\Xi}\langle\phi_h(\xi),\ \phi_{h'}(\xi)\rangle_{\mathcal{H}(\xi)}\,d\mu(\xi)\\&=\int_{\Xi}\prod_{\alpha\in B}\frac{1}{d(\xi_\alpha)}\operatorname{Tr}(\xi_\alpha(h_\alpha h_\alpha'^{-1}))\,d\mu(\xi)\\&=\prod_{\alpha\in B}\left\{\int_{\Xi_\alpha}\frac{1}{d(\xi_\alpha)}\operatorname{Tr}(\xi_\alpha(h_\alpha h_\alpha'^{-1}))\,d\mu_\alpha(\xi_\alpha)\right\}\\&=\delta_{hh'}\qquad\text{(Kronecker's }\delta).\end{aligned}$$

The last transfer comes from the orthogonality relation of irreducible characters $\operatorname{Tr}(\xi_\alpha(\cdot))$. Thus the set $\{\phi_h\mid h\in G\}$ gives an orthonormal basis of the Hilbert space ${}^{(\phi_h)}\!\!\int_{\Xi}^{\oplus}\mathcal{H}(\xi)\,d\mu(\xi)$.

Secondly, we study the natural mapping

$$\Phi : \delta_h \longmapsto \phi_h .$$

Since Φ gives a correspondence between orthonormal bases, it is unitary. It commutes with the actions of $G\times G$ as follows: for $h=(h_\alpha)\in G$, $\xi=(\xi_\alpha)\in \Xi$, $g_1=(g_{1\alpha})$, $g_2=(g_{2\alpha})\in G$,

$$\Phi((L\cdot R)(g_1, g_2)\delta_h)=\Phi(\delta_{g_1 h g_2^{-1}})=\phi_{g_1 h g_2^{-1}},$$

and

$$\begin{aligned}(P(g_1, g_2)\phi_h)(\xi)&=\rho_\xi(g_1)\hat{\rho}_\xi(g_2)\phi_h(\xi)\\ &=\rho_\xi(g_1)\hat{\rho}_\xi(g_2)(\otimes_{\alpha\in A}^{1(\xi)}\xi_\alpha(h))\\ &=\otimes_{\alpha\in A}^{1(\xi)}(\xi_\alpha(g_{1\alpha})\xi_\alpha(h_\alpha)\xi_\alpha(g_{2\alpha}^{-1}))\\ &=\otimes_{\alpha\in A}^{1(\xi)}\xi_\alpha(g_{1\alpha}h_\alpha g_{2\alpha}^{-1})=\phi_{g_1 h g_2^{-1}}(\xi).\end{aligned}$$

(ii) From (i), we see that the formula gives a decomposition of the left regular representation L. Fix a $\xi\in\Xi$. The von Neumann algebra generated by the family of operators $\{\rho_\xi(h)\mid h\in G\}$ is the infinite tensor product, with respect to the reference vector $1(\xi)=(1_{V(\xi_\alpha)})_{\alpha\in A}$, of the von Neumann algebras $\boldsymbol{B}(V(\xi_\alpha))$, which are generated by $\{\xi_\alpha(h_\alpha)\mid h_\alpha\in G_\alpha\}$ ([10, Chap. 7 §1]). In fact, since the infinite tensor product is inductive limit of tensor products of finite number of $\boldsymbol{B}(V(\xi_\alpha))$'s with respect to the reference vector $1(\xi)$ of the sequence of identity operators, the correspondence of generators of von Neumenn algebras is clear.

Since $\boldsymbol{B}(V(\xi_\alpha))$ is a factor of type I finite, the infinite tensor product is a factor of type II_1 in case $\boldsymbol{B}(V(\xi_\alpha))\not\cong \boldsymbol{C}$ for infinitely many $\alpha\in A$, or a factor of type I finite otherwise ([10, Chap. 7 §1]). Thus the direct integral gives the factor decomposition of L. //

§3. An irreducible decomposition of the left regular representation

We obtained in §2 the central decomposition of the left regular representation L by restricting to $G\times\{e\}$ the irreducible decomposition of the double regular representation:

$$\{L, L^2(G)\}\cong\{\rho, \mathcal{H}\}={}^{(\phi_h)}\int_{\Xi}^{\oplus}\{\rho_\xi, \mathcal{H}(\xi)\}d\mu(\xi).$$

In this section we derive many different irreducible decompositions of L from this factor decomposition by decomposing each factor representation ρ_ξ on $\mathcal{H}(\xi)=\otimes_{\alpha\in A}^{1(\xi)}\boldsymbol{B}(\xi_\alpha)$ to a direct integral of irreducible ones.

We fix an element $\xi=(\xi_\alpha)\in\Xi$. Let $N(\xi_\alpha)$ be the set $\{1, 2, \cdots, d(\xi_\alpha)\}$ of natural numbers, and define a measure ν_{ξ_α} on $N(\xi_\alpha)$ by

$$\nu_{\xi_\alpha}(\{m\})=\frac{1}{d(\xi_\alpha)} \qquad \text{for } m\in N(\xi_\alpha).$$

Let $(K(\xi), \nu_\xi)$ be the direct product of $(N(\xi_\alpha), \nu_{\xi_\alpha})$:

$$K(\xi)=\prod_{\alpha\in A}N(\xi_\alpha), \qquad \nu_\xi=\prod_{\alpha\in A}\nu_{\xi_\alpha}.$$

Fix an orthonormal basis $\{v_{\xi_\alpha, j} | 1 \leqq j \leqq d(\xi_\alpha)\}$ of $V(\xi_\alpha)$ for each $\xi_\alpha \in \Xi_\alpha$, denote by $\mathcal{O} = \mathcal{O}(\xi)$ this system of orthonormal bases of $V(\xi_\alpha)$, $\alpha \in A$. For $\kappa = (\kappa_\alpha) \in K(\xi)$, we take the infinite tensor product Hilbert space:

$$\boldsymbol{V}(\kappa) = \boldsymbol{V}(\xi, \mathcal{O}; \kappa) = \bigotimes_{\alpha \in A} \{V(\xi_\alpha), v_{\xi_\alpha, \kappa_\alpha}\}$$

with a reference vector $v(\kappa) = (v_{\xi_\alpha, \kappa_\alpha})_{\alpha \in A}$ consisting of $v_{\xi_\alpha, \kappa_\alpha} \in \mathcal{O}$.

Now we define a representation $\rho_{\xi, \kappa} = \rho^{\mathcal{O}}_{\xi, \kappa}$ of G on $\boldsymbol{V}(\kappa)$ as follows: for $g = (g_\alpha) \in G$ and a decomposable element $v = \bigotimes^{v(\kappa)}_{\alpha \in A} v_\alpha \in \boldsymbol{V}(\kappa)$,

$$\rho_{\xi, \kappa}(g) v = \bigotimes^{v(\kappa)}_{\alpha \in A} (\xi_\alpha(g_\alpha) v_\alpha).$$

Since g belongs to the restricted direct product, this action is well-defined and can be extended to the whole $V(\kappa)$.

Lemma 3.1. (i) *The representation* $\{\rho_{\xi, \kappa}, \boldsymbol{V}(\xi, \mathcal{O}; \kappa)\}$ *is irreducible for any* $\xi = \Xi$.

(ii) *If* $\xi \neq \xi'$, *then* $\rho_{\xi, \kappa}$ *is not equivalent to* $\rho_{\xi', \kappa'}$ *for any* $\kappa \in K(\xi)$ *and* $\kappa' \in K(\xi')$.

(iii) *The representation* $\rho_{\xi, \kappa}$ *is equivalent to* $\rho_{\xi, \kappa'}$, *if and only if* $\kappa_\alpha = \kappa'_\alpha$ *for almost all* $\alpha \in A$.

Proof. (i) Since each ξ_α is irreducible, the tensor product representation $\rho_{\xi, \kappa}$ is irreducible.

(ii) Obvious. (iii) This comes from Lemma 4.1 (i) in §4. //

For each $h = (h_\alpha) \in G$, $\xi \in \Xi$, we define a vector field $F_h(\xi)(\cdot)$ on the measure space $(K(\xi), \nu_\xi)$ as follows: for $\kappa = (\kappa_\alpha) \in K(\xi)$,

$$F_h(\xi)(\kappa) = \bigotimes^{v(\kappa)}_{\alpha \in A} (\xi_\alpha(h_\alpha) v_{\xi_\alpha, \kappa_\alpha}) \in \boldsymbol{V}(\kappa).$$

Since $\xi_\alpha(h_\alpha) v_{\xi_\alpha, \kappa_\alpha} = v_{\xi_\alpha, \kappa_\alpha}$ for almost all $\alpha \in A$, the value $F_h(\xi)(\kappa)$ actually belongs to $\boldsymbol{V}(\kappa)$.

Lemma 3.2. *The family* $\{\boldsymbol{V}(\kappa) | \kappa \in K(\xi)\}$ *is a measurable field of Hilbert spaces on the measure space,* $(K(\xi), \nu_\xi)$ *with respect to the fundamental sequence* $\{F_g(\xi)(\cdot) | h \in G\}$.

Proof. The proof is similar to that of Lemma 2.1. For the property (i), let $h = (h_\alpha)$, $h' = (h'_\alpha) \in G$, and B be a finite subset of A such that $h_\alpha = h'_\alpha = e_\alpha$ for $\alpha \in A \setminus B$. Then the function $\langle F_h(\xi)(\kappa), F_{h'}(\xi)(\kappa) \rangle_{V(\kappa)}$ in κ is expressed as

$$\begin{aligned} \langle F_h(\xi)(\kappa), F_{h'}(\xi)(\kappa) \rangle_{V(\kappa)} &= \prod_{\alpha \in A} \langle \xi_\alpha(h_\alpha) v_{\xi_\alpha, \kappa_\alpha}, \xi_\alpha(h'_\alpha) v_{\xi_\alpha, \kappa_\alpha} \rangle_{V(\xi_\alpha)} \\ &= \prod_{\alpha \in A} \langle \xi_\alpha(h'^{-1}_\alpha h_\alpha) v_{\xi_\alpha, \kappa_\alpha}, v_{\xi_\alpha, \kappa_\alpha} \rangle_{V(\xi_\alpha)} \\ &= \prod_{\alpha \in B} \langle \xi_\alpha(h'^{-1}_\alpha h_\alpha) v_{\xi_\alpha, \kappa_\alpha}, v_{\xi_\alpha, \kappa_\alpha} \rangle_{V(\xi_\alpha)}. \qquad (3.1) \end{aligned}$$

The last expression contains a finite number of components κ_α, $\alpha \in B$, of κ. Hence the function is measurable on $K(\xi)$.

For the property (ii), fix a $\kappa = (\kappa_\alpha) \in K(\xi)$. Since each ξ_α is irreducible, the set of vectors $\{\xi_\alpha(h_\alpha) v_{\xi_\alpha, \kappa_\alpha} | h_\alpha \in G_\alpha\}$ is total in $V(\xi_\alpha)$. This fact shows that $\{F_h(\xi)(\kappa) | h \in G\}$ is total in $\boldsymbol{V}(\kappa)$. //

Theorem 3.3. *Fix $\xi=(\xi_\alpha)\in\Xi$. Take a system $\mathcal{O}=\{v_{\xi_\alpha,j}|1\leqq j\leqq d(\xi_\alpha),\ \alpha\in A\}$ of orthonormal bases of $V(\xi_\alpha)$'s. Then an irreducible decomposition of the factor representation $\{\rho_\xi, \mathcal{H}(\xi)\}$ is given by*

$$\{\rho_\xi, \mathcal{H}(\xi)\}\cong {}^{(F_h(\xi))}\int_{K(\xi)}^{\oplus}\{\rho^{\mathcal{O}}_{\xi,\kappa}, V(\xi, \mathcal{O}\,;\kappa)\}d\nu_\xi(\kappa)$$

A natural equivalence mapping is given by

$$\mathcal{H}(\xi)\ni\phi_h(\xi)\longleftrightarrow F_h(\xi)(\cdot)\in {}^{(F_h(\xi))}\int_{K(\xi)}^{\oplus} V(\xi, \mathcal{O}\,;\kappa)d\nu_\xi(\kappa).$$

Proof. Denote by ρ'_ξ the direct integral of $\rho^{\mathcal{O}}_{\xi,\kappa}$. We consider the natural mapping

$$\Phi:\phi_h(\xi)\longmapsto F_h(\xi)(\cdot).$$

Let h, $h'\in G$ and B be the same as in the proof of Lemma 3.2. By the formula (3.1) and the finiteness of B, we have

$$\begin{aligned}\langle F_h(\xi)(\cdot), F_{h'}(\xi)(\cdot)\rangle&=\int_{K(\xi)}\prod_{\alpha\in B}\langle\xi_\alpha(h'^{-1}_\alpha h_\alpha)v_{\xi_\alpha,\kappa_\alpha}, v_{\xi_\alpha,\kappa_\alpha}\rangle_{V(\xi_\alpha)}d\nu_\xi(\kappa)\\&=\prod_{\alpha\in B}\int_{N(\xi_\alpha)}\langle\xi_\alpha(h'^{-1}_\alpha h_\alpha)v_{\xi_\alpha,\kappa_\alpha}, v_{\xi_\alpha,\kappa_\alpha}\rangle_{V(\xi_\alpha)}d\nu_{\xi_\alpha}(\kappa_\alpha)\\&=\prod_{\alpha\in B}\frac{1}{d(\xi_\alpha)}\mathrm{Tr}(\xi_\alpha(h'^{-1}_\alpha h_\alpha)).\end{aligned}$$

The last transfer comes from the definition of the trace and the measure ν_{ξ_α}. On the other hand, the inner product of $\mathcal{H}(\xi)$ is given by (2.1) as

$$\langle\phi_h(\xi), \phi_{h'}(\xi)\rangle_{\mathcal{H}(\xi)}=\prod_{\alpha\in B}\frac{1}{d(\xi_\alpha)}\mathrm{Tr}(\xi_\alpha(h_\alpha h'^{-1}_\alpha)).$$

Therefore the mapping Φ keeps the inner prodct of $\mathcal{H}(\xi)$. Moreover the totality of the families $\{\phi_h(\xi)\}$ and $\{F_h(\xi)(\cdot)\}$ shows that Φ is unitary.

The commutativity of the mapping Φ with G-actions is seen as follows: for $g=(g_\alpha)$, $h=(h_\alpha)\in G$ and $\xi=(\xi_\alpha)\in\Xi$,

$$\begin{aligned}\Phi(\rho_\xi(g)\phi_h(\xi))&=\Phi(\otimes^{1(\xi)}_{\alpha\in A}(\xi_\alpha(g_\alpha)\cdot\xi_\alpha(h_\alpha)))\\&=\Phi(\phi_{gh}(\xi))=F_{gh}(\xi)(\cdot),\end{aligned}$$

and

$$\begin{aligned}(\rho'_\xi(g)F_h(\xi)(\cdot))(\kappa)&=\rho_{\xi,\kappa}(g)F_h(\xi)(\kappa)\\&=\rho_{\xi,\kappa}(g)(\otimes^{v(\kappa)}_{\alpha\in A}(\xi_\alpha(h_\alpha)v_{\xi_\alpha,\kappa_\alpha}))\\&=\otimes^{v(\kappa)}_{\alpha\in A}(\xi_\alpha(g_\alpha)\xi_\alpha(h_\alpha)v_{\xi_\alpha,\kappa_\alpha})=F_{gh}(\xi)(\kappa).\end{aligned}$$

Therefore the unitary mapping Φ gives an irreducible decomposition of the factor representation ρ_ξ, thanks to Lemma 3.1 (i). //

Remark. This decomposition essentially depends on the choice of the system $\mathcal{O}=\mathcal{O}(\xi)$ of orthonormal bases.

By Theorems 2.2, 3.3, and Lemma 3.1, we obtain an irreducible decomposition of the left regular representation of G.

Theorem 3.4. *Fix systems $\mathcal{O}(\xi)=\{v_{\xi_\alpha,j}\mid 1\leqq j\leqq d(\xi_\alpha),\ \alpha\in A\}$ of orthonormal bases of $V(\xi_\alpha)$'s for all $\xi=(\xi_\alpha)\in\Xi$. An irreducible decomposition of the left regular representation L is given by the following direct integral:*

$$\{L,\ L^2(G)\}\cong^{(\phi_h)}\int_{\Xi}^{\oplus}\left\{{}^{(F_h(\xi))}\int_{K(\xi)}^{\oplus}\{\rho_{\xi,\kappa}^{\mathcal{O}(\xi)},\ V(\xi,\ \mathcal{O}(\xi);\ \kappa)\}d\nu_\xi(\kappa)\right\}d\mu(\xi).$$

Two irreducible components $\rho_{\xi,\kappa}^{\mathcal{O}(\xi)}$ and $\rho_{\xi',\kappa'}^{\mathcal{O}(\xi')}$ are mutually equivalent if and only if $\xi=\xi'$ and $\kappa_\alpha=\kappa'_\alpha$ for almost all α.

Remark. i) This double integral can be rewritten in a single integral easily.
ii) The multiplicities of irreducible components are infinite in general.

§4. Continuously many, mutually disjoint irreducible decompositions

Let $G=\prod'_{\alpha\in A}G_\alpha$ be the restricted direct product group, with discrete topology, of finite groups G_α. We obtained in Theorem 3.4 an irreducible decomposition of the left regular representation L of G. Here we study how the decomposition depends on the choice of the systems $\mathcal{O}(\xi)$, $\xi\in\Xi$, of orthonormal bases.

We recall a theorem of C. C. Moore for unitary equivalence of infinite tensor product of representations, restricting it to the present case.

Lemma ([7]). *Assume that we are given irreducible representations $\{\pi_\alpha,\ V_\alpha\}$ of G_α, $\alpha\in A$, and two reference vectors $a=(a_\alpha)$ and $b=(b_\alpha)$ with $a_\alpha,\ b_\alpha\in V_\alpha$, $\|a_\alpha\|=\|b_\alpha\|=1$. Let π^a and π^b be the infinite tensor product representations of π_α with repect to reference vectors a and b respectively:*

$$\pi^a:=\bigotimes_{\alpha\in A}^{a}\{\pi_\alpha,\ V_\alpha\},\qquad \pi^b:=\bigotimes_{\alpha\in A}^{b}\{\pi_\alpha,\ V_\alpha\}.$$

Then $\pi^a\cong\pi^b$ if and only if

$$\frac{1}{2}\sum_{\alpha\in A}(\inf_{|c|=1}\|ca_\alpha-b_\alpha\|)^2=\sum_{\alpha\in A}(1-|\langle a_\alpha,\ b_\alpha\rangle|)<\infty.$$

As a direct consequence of this lemma, we get

Lemma 4.1. (i) *Let $\mathcal{O}(\xi)$ be a system of orthonormal bases $\{v_{\xi_\alpha,j}\mid 1\leqq j\leqq d(\xi_\alpha),\ \alpha\in A\}$ of $V(\xi_\alpha)$'s, and choose two sequences of indices $\kappa=(\kappa_\alpha)$ and $\kappa'=(\kappa'_\alpha)$ such that $1\leqq\kappa_\alpha$, $\kappa'_\alpha\leqq d(\xi_\alpha)$, $\alpha\in A$. Let two reference vectors $v(\kappa)$ and $v(\kappa')$ be $(v_{\xi_\alpha,\kappa_\alpha})_{\alpha\in A}$ and $(v_{\xi_\alpha,\kappa'_\alpha})_{\alpha\in A}$. Then $\pi^{v(\kappa)}$ and $\pi^{v(\kappa')}$ are unitary equivalent if and only if $\kappa_\alpha=\kappa'_\alpha$ for almost all $\alpha\in A$.*

(ii) *Let $\mathcal{O}^1(\xi)$, $\mathcal{O}^2(\xi)$ be two systems of orthonormal bases given as*

$$\mathcal{O}^i:=\{v_{\xi_\alpha,j}^i\mid 1\leqq j\leqq d(\xi_\alpha),\ \alpha\in A\}.$$

Assume that there exists some $\varepsilon>0$ such that $|\langle v_{\xi_\alpha,j}^1,\ v_{\xi_\alpha,i}^2\rangle|\leqq 1-\varepsilon$ for infinitely many $\alpha\in A$, $1\leqq\forall i,\ j\leqq d(\xi_\alpha)$. Then $\rho_{\xi,\kappa}^{\mathcal{O}1}\not\cong\rho_{\xi,\kappa'}^{\mathcal{O}2}$.

In the case where the condition

$$\text{(Dim)} \quad d(\xi_\alpha) \geqq 2 \qquad \text{for infinitely many } \alpha \in A,$$

holds, we have many mutually inequivalent irreducible tensor product representations by Lemma 4.1.

Theorem 4.2. *Fix an element $\xi=(\xi_\alpha)\in\Xi=\prod_{\alpha\in A}\Xi_\alpha$. Assume that $V(\xi_\alpha)$'s satisfy the above condition (Dim). Then there exist continuously many mutually disjoint irreducible decompositions of the factor representation $\{\rho_\xi, \mathcal{H}(\xi)\}$.*

Proof. Fix a system $\mathcal{O}=\mathcal{O}(\xi)$ of orthonormal bases $\{v_{\xi_\alpha,j} \mid 1\leqq j\leqq d(\xi_\alpha), \alpha\in A\}$. For each θ, $0\leqq\theta\leqq\pi/4$, we define a system of orthonormal bases $\mathcal{O}^\theta=\{v^\theta_{\xi_\alpha,j}=U_{\xi_\alpha}(\theta)v_{\xi_\alpha,j} \mid 1\leqq j\leqq d(\xi_\alpha), \alpha\in A\}$, when $d(\xi_\alpha)\geqq 2$. Here $U_{\xi_\alpha}(\theta)$ denotes a rotation on $V(\xi_\alpha)$ described with respect to the fixed basis $v_{\xi_\alpha,j}$ by a $d(\xi_\alpha)$-dimensional blockwise diagonal matrix $\mathrm{Diag}(U_2(\theta), U_2(\theta), \cdots, U_2(\theta), U_k(\theta))$ where $k=2$ or 3 and $U_2(\theta)=\exp\theta X_2$, $U_3(\theta)=\exp\theta X_3$, with

$$X_2=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}, \qquad X_3=\frac{1}{\sqrt{2}}\begin{bmatrix}0 & -1 & 0\\ 1 & 0 & -1\\ 0 & 1 & 0\end{bmatrix}.$$

If $\theta\neq\theta'$, then for $\varepsilon=1/4(\theta-\theta')^2>0$ we have

$$|\langle v_{\xi_\alpha(\theta),i}, v_{\xi_\alpha(\theta'),j}\rangle|\leqq 1-\varepsilon \qquad \text{for } \forall\alpha \text{ and } \forall i, j \ (d(\xi_\alpha)\geqq 2).$$

By Lemma 4.1 (ii) it follows that if $\theta\neq\theta'$, then $\rho^{\mathcal{O}^\theta}_{\xi,\kappa}\not\simeq\rho^{\mathcal{O}^{\theta'}}_{\xi,\kappa'}$ for any $\kappa, \kappa'\in K(\xi)$. Consequently we have constructed mutually disjoint irreducible decompositions parametrized by θ, $0\leqq\theta\leqq\pi/4$:

$$^{(F_h(\xi))}\int^{\oplus}_{K(\xi)}\{\rho^{\mathcal{O}^\theta}_{\xi,\kappa}, V(\xi, \mathcal{O}^\theta; \kappa)\}d\nu_\xi(\kappa). \quad //$$

Apply this theorem to every $\xi\in\Xi$, then we get one of our main results.

Theorem 4.3. *Let $G=\prod'_{\alpha\in A}G_\alpha$ be the restricted direct product group of a countably infinite number of finite groups G_α with discrete topology. Assume that G_α is not commutative for infinitely many $\alpha\in A$. Then there exist continuously many, mutually disjoint irreducible decompositions of the left regular representation L of G of the following type:*

$$\{L, L^2(G)\}\cong {}^{(\phi_h)}\int^{\oplus}_{\Xi}\left\{{}^{(F_h(\xi))}\int^{\oplus}_{K(\xi)}\{\rho^{\mathcal{O}(\xi)}_{\xi,\kappa}, V(\xi, \mathcal{O}(\xi); \kappa)\}d\nu_\xi(\kappa)\right\}d\mu(\xi).$$

Proof. In Theorem 4.2 we constructed continuously many, mutually disjoint decompositions of ρ_ξ for such $\xi\in\Xi$ that satisfies the condition (Dim). Put $X=\{\xi\in\Xi \mid \xi$ does not satisfy (Dim)$\}$. Then it is sufficient to prove that X is measurable and $\mu(X)=0$.

Note that $\xi\in X$ is characterized by the property "$d(\xi_\alpha)=1$ for almost all $\alpha\in A$". Since A is countable, we have an increasing sequence of finite subsets A_i of A such that $\bigcup_{i=1}^\infty A_i=A$. Let X_i be a subset of Ξ such that $\{\xi\in\Xi \mid d(\xi_\alpha)=1$ for $\forall\alpha\notin A_i\}$. Then $X_i\subset X_{i+1}$ and $\bigcup_{i=1}^\infty X_i=X$, whence X is measurable.

We denote by B the set of $\alpha\in A$ for which G_α is not commutative. For $\alpha\in B$, the order of the commutator group $[G_\alpha, G_\alpha]$ is more than or equal to 2. Let χ_α be the set of all (unitary) charactors of G_α. Since χ_α is isomorphic to $G_\alpha/[G_\alpha, G_\alpha]$, the order of χ_α is equal to the index $|G_\alpha:[G_\alpha, G_\alpha]|\leqq|G_\alpha|/2$. Therefore, for $\alpha\in B$, we have

$$\mu_\alpha(\chi_\alpha)=|G_\alpha|^{-1}|\chi_\alpha|\leqq\frac{1}{2}.$$

From the assumption of the theorem we have $|B|=\infty$ and so

$$\mu(X_i)\leqq\prod_{\alpha\in B\setminus A_i}\mu_\alpha(\chi_\alpha)=0.$$

Therefore the measure of $X=\bigcup_{i=1}^\infty X_i$ should be zero.

The proof of the theorem is now completed. //

Remark. Contrary to the situation in Theorem 4.3, if G_α is commutative except a finite number of $\alpha\in A$, then the irreducible decomposition of L is unique.

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