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Differential geometry of generalized Lagrangian functions

By

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There are many generalizations of Finsler geometry. A Finsler metric function is defined on the tangent bundle of a differentialble manifold with some assumptions. Especially, it is assumed to be positively homogeneous. The importance of a generalized metric has been emphasized by many authors ([2], [5], [7]). Some of them studied the non-homogeneous "metric" space ([1], [3], [4]). In [1], they investigated generalized Lagrangian space (M, L) from the view point of Finsler spaces (M^*, L^*), where M^* is the $(n+1)$-dimensional manifold and L^* is positively homogeneous. The purpose of the present paper is to investigate the function without the assumption of homogeneity from another point of view.

§ 1. Generalized Lagrangian functions

Let M be an n-dimensional differentiable manifold and $T(M)$ its tangent bundle. A coordinate system $x=(x^i)$ in M induces the canonical coordinate system $(x, y)=(x^i, y^i)$ in $T(M)$. We put $\partial_i=\partial/\partial x^i$, $\dot{\partial}_i=\partial/\partial y^i$ and $T_0(M)=\{(x, y)\in T(M)\,|\,y\neq 0\}$. A function $L(x, y)$ on $T(M)$ is called a *generalized Lagrangian* if it is continuous on $T(M)$ and non-degenerate: $\det(g_{ij})\neq 0$ on $T_0(M)$, when $g_{ij}(x, y)$ is given by $g_{ij}=\dot{\partial}_i\dot{\partial}_j L$. A generalized Lagrangian function is called *postive definite*, if (g_{ij}) is positive definite on $T_0(M)$. A generalized Lagrangian tensor $g_{ij}(x, y)$ is a Finsler metric tensor if $L(x, y)$ is positively homogeneous of degree 2: $L(x, ty)=t^2L(x, y)$ for $t>0$.

We consider the cotangent bundle $T^c(M)$ of the manifold M. A coordinate system $x=(x^i)$ in M induces the canonical coordinate system $(x, p)=(x^i, p_i)$ in $T^c(M)$. The *Legendre transformation* Φ is a mapping of the tangent bundle $T_0(M)$ to the cotangent bundle $T^c(M)$:

$$\Phi: T_0(M)\longrightarrow T^c(M)\quad ((x^i, y^i)\longrightarrow (x^i, p_i)), \tag{1}$$

with a local expression $p_i=\dot{\partial}_i L$.

The restriction Φ_x to $T_x(M)$ of Φ is a local diffeomorphism, where $T_x(M)$ is the tangent space at a point x and if necessarily, 0 is excluded, we call the Lagrangian L *one-to-one* if Φ_x is one-to-one at any point x, i.e., $\Phi_x(x, y)=\Phi_x(x, y')\Leftrightarrow y=y'$.

We always assume that the dimension of M is more than two and the Lagrangian $L(x, y)$ is one-to-one. Consequently Φ_x is a one to one correspondence of $T_x(M)$ to

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some connected domain D_x of $T_x^c(M)$, where $T_x^c(M)$ is the cotangent space at the point x. Therefore we can write

(2) $$p^i=p^i(x, y), \qquad \text{inversely } y^i=y^i(x, p).$$

The *generalized Hamiltonian function* $H(x, p)$ on $D=\{(x, p)|x\in M, p\in D_x\}$ is defined by

(3) $$H(x, p)=p_i y^i(x, p)-L(x, y(x, p)).$$

From $g_{ij}(x, y)=\dot{\partial}_j p_i(x, y)$, we have the reciprocal g^{ij} of g_{ij} in the form:

(4) $$g^{ij}(x, y(x, p))=\partial y^i/\partial p_j,$$

and (3) gives

(5) $$y^i(x, p)=\partial H/\partial p_i.$$

From (4) and (5) we have

(6) $$h^{ij}(x, p)\equiv g^{ij}(x, y(x, p))=\partial^2 H/\partial p_i \partial p_j.$$

§ 2. Non-linear Connection associated with $L(x, y)$

Let $\pi: T_0(M)\to M$ be the projection of the tangent bundle. The vertical vector space $V_{(x,y)}(T_0(M))$ consists of the vertical vectors in $T_{(x,y)}(T_0(M))$, i.e.,

(7) $$V_{(x,y)}\equiv\{X\in T_{(x,y)}(T_0(M))|\pi_*(X)=0\}.$$

where π_* is the diferential of π.

A non-linear connection N is a horizontal distribution in the tangent bundle ([6]), i.e., a subspace $H_{(x,y)}(T_0(M))$ of $T_{(x,y)}(T_0(M))$ is given at each point (x, y), satisfying $T_{(x,y)}(T_0(M))=H_{(x,y)}(T_0(M))+V_{(x,y)}(T_0(M))$ (direct sum). And we suppose that this distribution is differentiable. Therefore we write the sum totally in the form:

(8) $$T(T_0(M))=H(T_0(M))\oplus V(T_0(M)),$$

where $T(T_0(M))$ is the tangent bundle of $T_0(M)$.

The differential Φ_* of the Legendre transformation Φ is the mapping of $T(T_0(M))$ to $T(T^c(M))$. Precisely, since $\Phi(T_0(M))=D\subseteqq T^c(M)$, we have

(9) $$\Phi_*: T(T_0(M)) \longrightarrow T(D)\subseteqq T(T^c(M)).$$

At each point (x, y), we have

(9′) $$\Phi_*(x, y): T_{(x,y)}(T_0(M)) \longrightarrow T_{(x,p)}(D)\subseteqq T_{(x,p)}(T^c(M)).$$

Here, we put

$$V_{(x,p)}(D)\equiv\{X\in T_{(x,p)}(D)|\pi^c_{D*}(X)=0\},$$

where $\pi^c_D: D\to M$ is the restriction to D of the projection $\pi^c: T^c(M)\to M$. By the simple calculation, we get

(11) $$V_{(x,p)}(D)=\Phi_*(x, y)(V_{(x,y)}(T_0(M))).$$

Since $\Phi_*(x, y)$ is an isomorphism, if we put

(12) $$H_{(x,p)}(D)\equiv\Phi_*(x, y)(H_{(x,y)}(T_0(M))),$$

for a given non-linear connection N, then we have

(13) $$T(D)=H(D)\oplus V(D).$$

At each point (x, p), we have

(13′) $$T_{(x,p)}(D)=H_{(x,p)}(D)+V_{(x,p)}(D) \qquad \text{(direct sum)}.$$

For a local base of $H(T_0(M))$ and $V(T_0(M))$, we have $(\partial_i-N_i^j\dot{\partial}_j)$ and $(\dot{\partial}_i)$ respectively. Similarly, for $H(D)$ and $V(D)$, we have $(\partial_i-M_{ij}\partial^{*j})$ and (∂^{*i}) respectively, where we put $\partial^{*i}=\partial/\partial p_i$. We get easily the following formulas:

(14) $$\Phi_*(\dot{\partial}_i)=g_{ij}\partial^{*j},$$

(15) $$\Phi_*(\partial_i-N_i^j\dot{\partial}_j)=\partial_i-(N_{ij}-\partial_i\dot{\partial}_j L)\partial^{*j},$$

where we put $N_{ij}=g_{jk}N_i^k$. Thus from the definition (12), we have

(16) $$M_{ij}=N_{ij}-\partial_i\dot{\partial}_j L.$$

There is a natural 1-form θ on $T^c(M)$ with a local expression $\theta=p_i dx^i$. The exterior differential $d\theta$ has rank $2n$. We put $d\theta(X, Y)=\langle X, Y\rangle$ for $X, Y\in T_{(x,p)}(T^c(M))$. For a cotangent vector $X^c=(a_i, b^i)=a_i dx^i+b^i dp_i$ at $(x, p)\in T^c(M)$, we define $\Theta(X^c)\in T_{(x,p)}(T^c(M))$ by

(17) $$\langle Y, \Theta(X^c)\rangle=X^c(Y) \qquad \text{for any tangent vector } Y \text{ at } (x, p),$$

with a local expression:

(17′) $$X\equiv\Theta(x, p)(X^c)=(b^i, -a_i)=b^i\partial_i-a_i\partial^{*i}.$$

Consequently we have an isomorphism:

(18) $$\Theta: T^c(T^c(M))\longrightarrow T(T^c(M)).$$

At each point (x, p), we have

(18′) $$\Theta(x, p): T^c_{(x,p)}(T^c(M))\longrightarrow T_{(x,p)}(T^c(M)).$$

The Hamiltonian function $H(x, p)$ is defined on D. Consequently the exterior differential dH is a 1-form on D. By the restriction to $T^c(D)$ of Θ, $\Theta(dH)$ is a vector field on D.

Now we consider the following cannonical conditions that a non-linear connection N associated with L should satisfy:

(C_1) $$\Theta(dH)\in H(D).$$

At each point (x, y), we have

(C_1') $$\Theta_{(x,p)}(dH)\in H_{(x,p)}(D).$$

Now we have the following expression of dH:

(19) $$dH=(\partial_i H)dx^i+(\partial^{*i}H)dp_i.$$

Consequently from (17′) we get

(20) $$\Theta(dH)=(\partial^{*i}H)\partial_i-(\partial_i H)\partial^{*i}.$$

Now the condition (C_1) is equivalent to

(21) $$\partial_j H-(\partial^{*i}H)M_{ij}=0.$$

Then (5), (21) and $\partial_j H(x, p)=-\partial_j L(x, y)$ from (3) yield

(22) $$\partial_j L-(\partial_i\dot{\partial}_j L)y^i+y^i N_{ij}=0.$$

Consequently we have

Theorem 1. *A non-linear connection N satisfies the condition* (C_1) *if and only if the coefficients of N are given by*

(23) $$y^i N_{ij}=(\partial_i\dot{\partial}_j L)y^i-\partial_j L.$$

Let us call $y^j N_j{}^i$ the *generalized spray defined by a non-linear connection* $N_j{}^i$ and denoted by N^i.

Remark 1. We must pay attention to the covariant derivative $H_{|j}=\partial_j H-(\partial^{*i}H)M_{ji}$ of H. The condition (C_1) is not the same as $H_{|j}=0$. But from (21), which is equivalent to the condition (C_1), we get $H_{|0}\equiv H_{|j}y^j(x, p)=0$, while $L_{|0}\equiv L_{|j}y^j\neq 0$ in general.

§ 3. Finsler type connections associated with L

According to [6], we write a Finsler type connection as $\Gamma(N, F, C)$, where $N_j{}^i(x, y)$ is a non-linear connection and plays an important role in our theory.

If we put

(24) $$G_j=(\partial_i\dot{\partial}_j L)y^i-\partial_j L, \qquad G^i\equiv g^{ij}G_j,$$

the condition (23) in the above theorem is the same as

(25) $$G^i=y^j N_j{}^i.$$

Moreover, let us put

(26) $$G_j{}^i\equiv\dot{\partial}_j\left(\frac{1}{2}G^i\right), \qquad G_j{}^i{}_k\equiv\dot{\partial}_k G_j{}^i,$$

Then $\Gamma(G_j{}^i, G_j{}^i{}_k, C_j{}^i{}_k)$ is a Finsler type connection but the non-linear connection $G_j{}^i$ does not satisfy the condition (25), unless G^i is positively homogeneous of degree 2. Here we consider the other conditions for a Finsler type connection:

(C_2) $(h)h$-torsion $T=0$, i.e., $T_j{}^i{}_k\equiv F_j{}^i{}_k-F_k{}^i{}_j=0$,

(C_3) the deflection tensor $D=0$, i.e., $D^i{}_j\equiv y^k F_k{}^i{}_j-N_j{}^i=0$,

(C_4) $(h)hv$-torsion $C=0$, i.e., $C_j{}^i{}_k=0$.

It is our problem whether there exist some connections satisfying the conditions $(C_1)\sim(C_4)$ or not. First, we define

$$U^i \equiv G^i - y^j G_j{}^i, \qquad U_j{}^i \equiv \dot{\partial}_j U^i, \tag{27}$$

$$V^i \equiv U^i - y^j U_j{}^i. \tag{28}$$

Since G^i of (24) have not the positively homogeneous property, it is necessary for us to define the above quantities. U^i and V^i are both considered as global vector fields on $T_0(M)$, even if they are defined by local expressions and $U_j{}^i$ is a globally defined tensor. By differentiating (27) with respect to y^j, we get the formula:

$$U_j{}^i = G_j{}^i - y^k G_k{}^i{}_j. \tag{29}$$

Here we suppose that there exists a covariant vetor field $\phi_j(x, y)$ on $T_0(M)$ satisfying

$$\phi_j(x, y) y^j = 1. \tag{30}$$

Let us call this $\phi_j(x, y)$ a *characteristic covariant vector field.*

Now, to define a Finsler type connection satisfying the condition $(C_1)\sim(C_4)$, we put

$$N_j{}^i \equiv G_j{}^i + \phi_j U^i, \tag{31}$$

$$F_j{}^i{}_k \equiv G_j{}^i{}_k + \phi_j U_k{}^i + \phi_k U_j{}^i + \phi_j \phi_k V^i, \tag{32}$$

$$C_j{}^i{}_k \equiv 0. \tag{33}$$

The above (N, F, C) gives a Finsler type connection. As for the conditions $(C_1)\sim(C_4)$, we must check as follows: From (31), (30) and (27), we have first

$$y^j N_j{}^i = y^j G_j{}^i + y^j \phi_j U^i = y^j G_j{}^i + U^i = G^i. \tag{C_1}$$

Secondly, from (32) and $G_j{}^i{}_k - G_k{}^i{}_j = 0$, we have

$$T_j{}^i{}_k = 0. \tag{C_2}$$

Thirdly, from (32), (30), (29), (28) and (31), we have

$$\begin{aligned} y^j F_j{}^i{}_k &= y^j G_j{}^i{}_k + U_k{}^i + \phi_k y^j U_j{}^i + \phi_k V^i \\ &= G_k{}^i + \phi_k U^i \\ &= N_k{}^i. \end{aligned} \tag{C_3}$$

Finally, from the definition (33), we have

$$C_j{}^i{}_k = 0. \tag{C_4}$$

Therefore we have our first conclusion:

Theorem 2. *There exists a Finsler type connection $\Gamma(N, F, C)$ satisfying the conditions $(C_1)\sim(C_4)$, if the generalized Lagrange space (M, L) admits a characteristic covariant vector field ϕ_i on $T_0(M)$ and the Lagrangian L is one-to-one.*

If a generalized Lagrangian $L(x, y)$ is positive definite there exists a characteristic field ϕ_j:

(34) $$\phi_j \equiv g_{ji} y^i / g_{ab} y^a y^b .$$

Consequently, from Theorem 2 we get

Theorem 3. *If a generalized Lagrangian $L(x, y)$ is positive definite and one-to-one, there exists a Finsler type connection $\Gamma(N, F, C)$ satisfying the conditions* $(C_1)\sim(C_4)$.

Remark 2. We have proved the existence of a connection satisfying the above four conditions, but it is an unsolved problem to find additional conditions for determining a Finsler type connection uniquely. The condition (C_1) will be most interesting, because it determines the spray N^i uniquely from $L(x, y)$.

Example 1. $L(x, y) \equiv \frac{1}{6} a_{ijk}(x) y^i y^j y^k + \frac{1}{2} a_{ij}(x) y^i y^j - b_i(x) y^i - c(x).$

For this generalized Lagrangian, the straightforward calculation leads us to

(35) $$G_i = \frac{2}{3} \{jkl, i\} y^j y^k y^l + \{jk, i\} y^j y^k + E_{ji} y^j + C_i ,$$

(36) $$\{jkl, i\} \equiv \frac{1}{4} (\partial_j a_{ikl} + \partial_k a_{jil} + \partial_l a_{jki} - \partial_i a_{jkl}),$$

(37) $$\{jk, i\} \equiv \frac{1}{2} (\partial_j a_{ik} + \partial_k a_{ji} - \partial_i a_{jk}),$$

(38) $$E_{ji} \equiv \partial_i b_j - \partial_j b_i ,$$

(39) $$C_i \equiv \partial_i c .$$

The above $\{jk, l\}$ are the Christoffel symbols, characterized by the following:

(40) $$\partial_k a_{ij} - \{jk, i\} - \{ik, j\} = 0 \quad \text{and} \quad \{jk, i\} = \{kj, i\} .$$

Similarly, we find that $\{jkl, i\}$ are determined by the following properties:

(41) $$\partial_i a_{jkl} - \{kli, j\} - \{jli, k\} - \{jki, l\} + \{jkl, i\} = 0$$

and $\{jkl, i\}$ are symmetric with respect to j, k and l. Therefore, we may call these $\{jkl, i\}$ the *higher order Christoffel symbols*, more precisely the *third order Christoffel symbols*. The Christoffel symbols are the coefficients of a connection which is a geometrical object of class 2. We can prove that the third order Christoffel symbols are the coefficients of certain geometrical object of class 2 ([10]).

Example 2. $L(x, y) = \frac{1}{2} a_{ij}(x) y^i y^j - b_i(x) y^i - c(x).$

This is a special case of Example 1. As to this Lagrangian, we get

(42) $$g_{ij}(x, y) = a_{ij}(x),$$

(43) $$G_i = \{jk, i\} y^j y^k + E_{ji} y^j + C_i ,$$

where E_{ji} and C_i are given by (38) and (39) respectively,

(44) $$G^i=\{_j{}^i{}_k\}y^jy^k+E_j{}^iy^j+C^i,$$

(44¹) $$\{_j{}^i{}_k\}\equiv g^{il}\{jk,l\}=a^{il}\{jk,l\},$$

(44²) $$E_j{}^i\equiv g^{ik}E_{jk}=a^{ik}E_{jk},$$

(44³) $$C^i\equiv g^{ij}C_j=a^{ij}C_j,$$

(45) $$G_j{}^i=\{_j{}^i{}_k\}y^k+\frac{1}{2}E_j{}^i,$$

(46) $$U^i=\frac{1}{2}E_j{}^iy^j+C^i,$$

(47) $$U_j{}^i=\frac{1}{2}E_j{}^i,$$

(48) $$V^i=C^i$$

(49) $$G_j{}^i{}_k=\{_j{}^i{}_k\}$$

§4. Variation calculus

We shall deal with the variation calculus for a generalized Lagrangian function $L(x, y)$. We put $\lambda(x, y)=\sqrt{2L(x, y)}$. Along a curve $C: x=x(t)$ $(a\leqq t\leqq b)$, we consider the definite integral:

(50) $$J(C)=\int_a^b\lambda(x(t), y(t))dt, \qquad y^i(t)=d_tx^i\equiv dx^i/dt.$$

For a family of curves C_u: $x=x(t, u)$ $(a\leqq t\leqq b, -\varepsilon\leqq u\leqq\varepsilon)$ with fixed end points, i.e., $x(a, u)=x(a, 0)$ and $x(b, u)=x(b, 0)$, we have

(51) $$d_u(J(C_u))=\int_a^b(\partial_i\lambda-\partial_t\dot{\partial}_i\lambda)Y^idt, \qquad Y^i\equiv\partial_ux^i,$$

where we put $d_u=d/du$, $\partial_u=\partial/\partial u$ and $\partial_t=\partial/\partial t$. Therefore, we get the equation of an extremal that is called the generalized Euler equation:

(52) $$\partial_i\lambda-d_t\dot{\partial}_i\lambda=0.$$

The equation (52) is transformed by the simple calculation into the following form:

(52′) $$g_{ij}d_ty^j+(\partial_j\dot{\partial}_iL)d_tx^j-\partial_iL=0,$$

where the parameter t satisfies the equation $L(x(t), y(t))=k$ (constant). The above equation is equivalent to

(53) $$d_td_tx^i+G^i=0,$$

where G^i is given by (24).

By Theorem 2, there exists a Finsler type connection $\Gamma(N, F, C)$ satisfying the four conditions $(C_1)\sim(C_4)$ in the generalized Lagrangian space (M, L) with a characteristic covariant vector field ϕ_i. We denote the Finsler type connection Γ satisfying the four conditions $(C_1)\sim(C_4)$ by $\Gamma^4(L)$ or $\Gamma^4(L, \phi)$ if there exists a characteristic covariant vector field ϕ_i and $\Gamma(N, F, C)$ is determined by (31), (32) and (33).

The equation of a path in $\Gamma^4(L)$ is written in the form

$$d_t d_t x^i + F_j{}^i{}_k (d_t x^j)(d_t x^k) = 0. \tag{54}$$

From the conditions (C_3) and (C_1), we get $F_j{}^i{}_k y^j y^k = G^i$. Consequently, we have

Theorem 4. *In the generalized Lagrangian space (M, L) with a Finsler type connection $\Gamma^4(L)$, the path is coincident with the extremal.*

A Randers space is a Finsler space (M, L') with a special metric function

$$L'(x, y) = \frac{1}{2}(\sqrt{a_{ij} y^i y^j} - b_i y^i)^2. \tag{55}$$

The equation of the extremal of this space is written in the following form:

$$d_t d_t x^i + \{{}_j{}^i{}_k\} d_t x^j d_t x^k + E_j{}^i d_t x^j = 0, \tag{56}$$

where $E_j{}^i$ is defined by the same form as the definitions (38) and (44^2) and the parameter t is the arc-length of the extremal. It follows from (44) that the equation (56) is the same as that of the path in a generalized Lagrangian space (M, L), where L is given by

$$L \equiv \frac{1}{2}(a_{ij} y^i y^j) - b_i y^i. \tag{57}$$

Consequently we have

Theorem 5. *The paths in a generalized Lagrangian space (M, L) are coincident with the ones of a Randers space (M, L') if L and L' are defined by* (57) *and* (55) *respectively.*

Remark 3. The equation (56) is related with the unified field theory ([8]).

Remark 4. In a generalized Lagrangian space, the parameter t has a significant meaning. If the Lagrangian function $L(x, y)$ is positively homogeneous of degree 2 with respect to the variables y^i, the integral (50) does not change the value even if the parameter of the curve C changes into another parameter. But the general transformation of the parameters changes the value of the integral if $L(x, y)$ is not homogeneous of degree 2.

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