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On the singularity of the periods of abelian differentials with normal behavior under pinching deformation

Dedicated to Professor Tatsuo Fuji'i'e on his sixtieth birthday

By

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Introduction

In this note, we present a simple analytic method to determine the singularity of periods of the normalized abelian differentials, with a given normal behavior on an arbitrary Riemann surface, under the deformation by pinching a finite number of loops. For the sake of simplicity, we discuss only differentials of the first kind. The case of a compact Riemann surface has been deeply investigated, especially under the deformation by pinching a single loop (cf. [1] and [5]).

1. Notations and definitions

We first recall some definitions and known results (cf. [3]).

1) Let R_0 be an arbitrary Riemann surface, not necessarily of finite topological type, with a finite number of nodes. Denote by $N(R_0)$ the set $\{p_j\}_{j=1}^n$ of all nodes of R_0, and set $R'_0=R_0-N(R_0)$. Recall that R'_0 is a union of ordinary Riemann surfaces whose universal coverings are conformally equivalent to the unit disk, and that each p_j has a neighborhood homeomorphic to the subset $\{|z|<1, |w|<1, zw=0\}$ in $\boldsymbol{C}^2$.

For every j, we fix a neighborhood U_j of p_j on R_0 such that each component, say $U_{j,l}$ $(l=1, 2)$, of $U_j-\{p_j\}$ is mapped conformally onto $D_0=\{0<|z|<1\}$ by a mapping, say, $z=z_{j,l}(p)$. Also we assume that $\{\bar{U}_j\}_{j=1}^n$ are mutually disjoint. In the sequel, we consider $z_{j,l}$ also as a canonical local parameter on $U_{j,l}$ for every j and l.

Let m be a positive integer and $\{\mu(t): t\in\Delta^m\}$ with $\Delta=\{|\zeta|<1\}$ be a family of Beltrami coefficients on R_0 (i.e. bounded $(-1, 1)$ forms on R_0 with $\|\mu(t)\|_\infty<1$ for every t) such that $\mu(0)=0$ and that the support of every $\mu(t)$ is contained in $R_0-\bar{U}$, where U is the union of all U_j. Further suppose that $\mu(t)$ depends holomorphically on $t\in\Delta^m$ (with respect to the sup norm). Let f^t be a quasiconformal mapping of R'_0 onto another union R'_t of Riemann surfaces with the complex dilatation $\mu(t)$. Since f^t is conformal on U, we may identify $f^t(U_{j,l})$ with $U_{j,l}$, and hence may consider $z_{j,l}$ as a conformal mapping of $f^t(U_{j,l})$ onto D_0, or as a canonical local parameter on $f^t(U_{j,l})$, for every

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j and l.

Next let $t\in\Delta^m$ and $s=(s_1, \cdots, s_n)\in(\Delta')^n$ be given arbitrarily, where $\Delta'=\{|\zeta|<1/2\}$. Let $R_{t,s}$ be the Riemann surface possibly with nodes, obtained from R'_t by deleting two punctured disks $\{0<|z_{j,l}|<|s_j|^{1/2}\}$ ($l=1, 2$) in $U_j-\{0\}$ and by identifying the resulting borders into a loop $C_{j,t,s}$ under the mapping

$$z_{j,2}^{-1}(s_j/z_{j,1}(p)),$$

for every j. Here, for every j such that $s_j=0$, nothing should be removed from $U_j-\{0\}$, and $R_{t,s}$ has a node corresponding to p_j.

Thus we have a family $\{R_{t,s}:(t, s)\in\Omega=\Delta^m\times(\Delta')^n\}$ of Riemann surfaces possibly with nodes, which we call *the complex pinching deformation family with the center* R_0 *and with the deformation data* $(\{\mu(t)\}, U)$.

In the sequel, we consider $R''_{t,s}=R_{t,s}-\bigcup_{j=1}^n C_{j,t,s}$ as a subset of R'_t, and hence consider the mapping $(f^t)^{-1}$ as a quasiconformal mapping of $R''_{t,s}$ into R_0, for every $(t, s)\in\Omega$.

2) Set $\Delta^*=\Delta'-\{0\}$ $(=\{0<|z|<1/2\})$. Then for every $(t, s)\in\Omega^*=\Delta^m\times(\Delta^*)^n$, $R_{t,s}$ is an ordinary Riemann surface. Fix $(t^*, s^*)\in\Omega^*$ once for all, and denote R_{t^*,s^*} simply by R^*. Let F be the embbeding $(f^{t^*})^{-1}$ of $(R^*)''$ $(=R''_{t^*,s^*})$ into R_0. Then it is easy to see that there is a canonical homology base $\Xi=\Xi(R^*)=\{A_k, B_k\}_{k=1}^g$ of R^* modulo the ideal boundary (, where g may be infinite,) which satisfies the following condition;

(#) for every $p_j\in N(R_0)$, $C_j=F^{-1}(\{|z_{j,1}|=1/2\})$ is either

(i) freely homotopic to some $A_k\in\Xi$ (with a suitable orientation), or

(ii) letting E_0 be the set of all A_k corresponding to nodes of R_0 as in (i), C_j is a loop on $R^*-\{A_k, B_k:A_k\in\Xi-E_0\}$ dividing $R^*-\{A_k:A_k\in E_0\}$.

Here note that all curves in Ξ except for $\{B_k:A_k\in E_0\}$ can be considered as ones on R'_0, and $\Xi_0=\Xi(R_0)=\{A_k, B_k:A_k\in\Xi-E_0\}$ as a homology base of R'_0 modulo the ideal boundary.

Next, fix a normal behavior space $\Gamma_0(R_0)$, i.e. a subspace of $\Gamma_h(R_0)$, the Hilbert space consisting of all square integrable complex harmonic differentials on R'_0, which satisfies the following conditions;

i) $\Gamma_0(R_0)\subset\Gamma_{\mathrm{hse}}(R_0)$,

ii) $\int_{A_k}\omega=0$ for every $\omega\in\Gamma_0(R_0)$ and $A_k\in\Xi_0$,

iii) $\Gamma_0(R_0)+{}^*\Gamma_0(R_0)=\Gamma_h(R_0)$ (a direct sum), and

iv) $\Gamma_0(R_0)=\overline{\Gamma_0(R_0)}$.

And we say that a complex harmonic or meromorphic differential ϕ on R'_0 has Γ_0-*behavior* if there exist α in $\Gamma_0(R_0)$ and df in $\Gamma_{e0}(R_0)$ such that $\phi=\alpha+df$ outside of a compact set on R_0. (This condition imposes nothing on ϕ when R_0 is compact.)

For every $A_k\in\Xi$, *the k-th normalized abelian differential* $\phi_k(R_0)$ *of the first kind with* Γ_0-*behavior* is, by definition, a holomorphic differential (on R'_0) with Γ_0-behavior on R_0 uniquely determined by the following conditions;

(i) $\int_{A_h} \phi_k(R_0)=\delta_{kh}$ for every $A_h \in \Xi$,

(ii) $\phi_k(R_0)$ has simple poles at two punctures of R_0' corresponding to every node p_j such that the algebraic intersection number $C_j \times B_k$ between C_j and B_k is non-zero, where C_j is as in (#) with positive orientation with respect to $\{0<|z_{j,1}|<1/2\}$.

(iii) $\phi_k(R_0)$ is holomorphic at two punctures of R_0' corresponding to every other node.

Remark. When R_0 is compact and without nodes, the above $\phi_k(R_0)$ is the classical k-th normal differential of the first kind with respect to Ξ_0.

2. Main theorem and proof

Fix $(t, s) \in \Omega$ arbitrarily. We can define a normal behavior space $\Gamma_0(R_{t,s})$ on $R_{t,s}$ corresponding to $\Gamma_0(R_0)$ in a natural manner (cf. the proof of [3, Theorem 1]). Also by the condition (#), we can regard every $A_k \in \Xi$ as a curve on $R_{t,s}$, which we denote by the same A_k. And we can define, similarly as above, the k-th normalized abelain differential $\phi_k(R_{t,s})$ of the first kind with Γ_0-behavior on $R_{t,s}$ for every k, which is again uniquely determined (cf. [3, § 2]).

On the other hand, B_k determines a curve on $R_{t,s}$ not uniquely, but only modulo $\{n \cdot A_k : n \in \boldsymbol{Z}\}$ for every $A_k \in E_0$. So the period

$$\pi_{kh}(t, s)=\int_{B_h} \phi_k(R_{t,s})$$

of $\phi_k(R_{t,s})$ along B_h should be considered only modulo $\boldsymbol{Z}$ when $h=k$. Any way, we know the following

Proposition (cf. [2, Theorem 6]). *Fix a relatively compact open ball W in Ω^* arbitrarily. Then any continuous branch of $\pi_{hk}(t, s)$ on W is holomorphic on B^*.*

Moreover,

$$d\pi_{hk}=\sum_{i=1}^{m} \frac{\partial \pi_{hk}}{\partial t_i} dt_i+\sum_{j=1}^{n} \frac{\partial \pi_{hk}}{\partial s_j} ds_j$$

is a well-defined holomorphic 1-form on Ω^*.

For the sake of convenience, we include in § 3 a standard proof of Proposition (which is strongly inspired by Ahlfors' argument).

Now the main purpose of this note is to give a simple analytic proof of the following

Theorem. *Fix h and k. Then*

$$d\pi_{hk}-\frac{1}{2\pi i} \cdot \sum_{j=1}^{n} (N_{j,k} \cdot N_{j,h}) \cdot \frac{ds_j}{s_j}$$

can be extended holomorphically to the whole Ω, where we set $N_{j,p}=C_j \times B_p$ for every j and p.

Remark. Compare with [4, Theorem 5]. The case that $n=1$ has been investigated in [1] and [5].

To prove Theorem, fix h and k once for all. Then in the proof of Proposition, we actually show the following

Lemma 1. *For every* $(t, s)\in\Omega^*$, $\frac{\partial\pi_{hk}}{\partial t_i}(t, s)$ *is equal to*

$$F_i(t, s)=\iint_{R_{t,s}}-\mu_{i,t}\phi_k(R_{t,s})\wedge\phi_h(R_{t,s}),$$

for every i, *where (considering* f^t *as a mapping of* $R_0-\bar{U}$ *into* $R_{t,s}$*) we set*

$$\mu_{i,t}=\left(\frac{\partial\mu}{\partial t_i}(t)(1-|\mu(t)|^2)^{-1}(f^t)_z/(\bar{f}^t)_{\bar{z}}\right)\circ(f^t)^{-1}$$

and $\frac{\partial\pi_{hk}}{\partial s_j}(t, s)$ *is equal to*

$$G_j(t, s)=\iint_{R_{t,s}}-\lambda_{j,s}\phi_k(R_{t,s})\wedge\phi_h(R_{t,s}),$$

for every j. *Here (considering* $z_{j,1}$ *as a mapping of* $f^t(\{1/2<|z_{j,1}|<1\})\subset R_{t,s}$ *into* D_0 *and) denoting by* $\chi(x)$ *the characteristic function of* $[3/5, 4/5]$ *on* $\boldsymbol{R}$, *we set*

$$\lambda_{j,s}=\left(\frac{-1}{2s_j\cdot\log(4/3)}\chi(|z|)\cdot(z/\bar{z})\frac{d\bar{z}}{dz}\right)\circ z_{j,1}.$$

Also we know the following

Lemma 2. *For every* $(t, s)\in\Omega^*$, *set* $X_{t,s}=R''_{t,s}-\bigcup_{j=1,l=1}^{n,2}f^t(\{0<|z_{j,l}|\leqq 1/2\})$. *Then for every* p, *the function* $E_p(t, s)=\|\phi_p(R_{t,s})\|_{X_{t,s}}$ *on* Ω^* *is locally bounded in* Ω. *i.e. for every relatively compact open ball* V *in* Ω, E_p *is bounded on* $V\cap\Omega^*$. *Here* $\|\phi\|_X^2$ *is the Dirichlet energy* $\iint_X\phi\wedge*\bar{\phi}$ *of* ϕ *on* X.

This lemma is an immediate corollary of [4, §3 Proposition]. But we will include a rather elementary proof in §4.

Now by Lemma 2, we can easily show the following

Lemma 3. $F_i(t, s)$ *and* $\tilde{G}_j(t, s)=s_j\cdot G_j(t, s)$ *are locally bounded on* Ω *for every* i *and* j, *respectively.*

Proof. By Lemma 1, $|F_i(t, s)|\leqq\|\mu_{i,t}\|_\infty\cdot E_h(t, s)\cdot E_k(t, s)$ and $|\tilde{G}_j(t, s)|\leqq\|s_j\cdot\lambda_{j,s}\|_\infty\cdot E_h(t, s)\cdot E_k(t, s)$. Hence the assertion follows by Lemma 2. q.e.d.

Proof of Theorem. By Proposition, Lemma 3 and Riemann's extension theorem, all $F_i(t, s)$ and $\tilde{G}_j(t, s)$ can be extended to holomorphic functions on the whole Ω.

So it remains only to show that, for every j, $\tilde{G}_j(t, s)$ tends to the constant

$\frac{1}{2\pi i}N_{j,k}\cdot N_{j,h}$ when $(t, s)\in\Omega^*$ tends to any $(T, S)\in\Omega-\Omega^*$ such that $S_j=0$.

Fix $U_{j,l}$ arbitrarily, and write $\phi_p(R_{t,s})\circ z_{j,l}^{-1}$ as $a_p(t, s, z)ds$ on $D=\{1/2<|z|<1\}$ for every p and $(t, s)\in\Omega$. Then it is known ([3, Corollary 4]) that $a_p(t, s, z)$ is holomorphic on $\Omega\times D$. In particular, when (t, s) tends to (T, S) in Ω, $a_p(t, s, z)$ converges to $a_p(T, S, z)$ locally uniformly on D. (Recall that the proof of [3, Corollary 4] uses a similar argument as that of Proposition does. So it is rather standard, and hence omitted.)

Hence for every point $(T, S)\in\Omega-\Omega^*$ with $S_j=0$, we can see that $\tilde{G}_j(t, s)$ converges to

$$I=-\iint_{\{3/5\leq|z|\leq4/5\}}\frac{-1}{2\cdot\log(4/3)}(z/\bar{z})a_k(T, S, z)a_h(T, S, z)d\bar{z}\wedge dz$$

$$=\int_0^{2\pi}\int_{3/5}^{4/5}a_k(T, S, re^{i\theta})\cdot a_h(T, S, re^{i\theta})\frac{(e^{i\theta})^2\cdot irdrd\theta}{\log(4/3)},$$

as $(t, s)\in\Omega^*$ tends to (T, S).

Since Laurent's expansion of $a_p(T, S, z)$ has such a form as $N_{jp}\cdot\frac{1}{2\pi i\cdot z}+\sum_{n=0}^{\infty}c_n(T, S)\cdot z^n$ for every p, we conclude that

$$I=\int_{3/5}^{4/5}\frac{N_{jk}\cdot N_{jh}}{\log(4/3)}\cdot\frac{1}{r}\cdot\frac{dr}{2\pi i}=N_{jk}\cdot N_{jh}/2\pi i.$$

Thus we have proved Theorem.

3. Proofs of Proposition and Lemma 1

Fix k, h and $(t_0, s_0)\in\Omega^*$ arbitrarily. First we will recall the proof of [2, Lemma 7], which shows that (any continuous branch of) π_{hk} is differentiable with respect to each t_i at (t_0, s_0). Let g^t $(=g^{t,s_0})$ be the quasiconformal mapping of $R=R_{t_0,s_0}$ to $R(t)=R_{t,s_0}$, coincident with $f^t\circ f^{-1}$ on R''_{t_0,s_0} (and hence conformal on $R-f(R_0-U_j)$), where and in the sequel, we set $f=f^{t_0}$. Then note that the complex dilatation $\nu(t)$ $(=\nu(t, s_0))$ of g^t is equal to

$$((\mu(t)-\mu(t_0))(1-\overline{\mu(t_0)}\mu(t))^{-1}(f_z)/(\bar{f}_{\bar{z}}))\circ f^{-1}$$

where we denote by z a generic local parameter on R. In particular, $\nu(t)$ depends holomorphically on t.

Next set

$$\omega_t=\phi_k(R(t))\circ g^t-\phi_k(R),$$

for every $t\in\Delta^m$, where and in the sequel, $\phi\circ g$ is the pull-back of ϕ by g.

Then as in the proof of [2, Lemma 7], we have

$$\pi_{kh}(t, s_0)-\pi_{kh}(t_0, s_0)=-\iint_R\omega_t\wedge\phi_h(R)$$

A standard argument originally due to Ahlfors shows that, for every i, $(\partial\pi_{hk}/\partial t_i)(t_0, s_0)$ exists and equals to

$$-\iint_R \frac{\partial\nu}{\partial t_i}(t_0)\phi_k(R)\wedge\phi_h(R).$$

Here, since

$$\frac{\partial\nu}{\partial t_i}(t_0)=\Big(\frac{\partial\mu}{\partial t_i}(t_0)\cdot(1-|\mu(t_0)|^2)^{-1}(f_z)/(\bar{f}_{\bar{z}})\Big)\circ f^{-1},$$

we also conclude the first assertion of Lemma 1.

Next to show the differentiablity with respect to each s_j at (t_0, s_0), recall that the deformation represented by the parameter s can be considered, locally, as a quasiconformal deformation depending holomorphically on s ([3, Lemma 5]). More precisely, fix j and set $s_0(\zeta)=((s_0)_1, \cdots, (s_0)_{j-1}, \zeta, (s_0)_{j+1}, \cdots, (s_0)_n)$. For every ζ with sufficiently small $|\zeta-(s_0)_j|$, define a quasiconformal mapping f^ζ of R to $R_\zeta=R_{t_0, s_0(\zeta)}$ by assuming that f^ζ is equal to the identity on $f(R_0-U_j)$, and by setting

$$\begin{aligned} z_{j,1}\circ f^\zeta\circ z_{j,1}^{-1}(z)&=z \qquad \text{on} \quad \Big\{\frac{4}{5}\leqq|z|<1\Big\}, \\ &=z\cdot\Big(\frac{4/5}{|z|}\Big)^{\log(\zeta/(s_0)_j)/\log(4/3)} \qquad \text{on} \quad \Big\{\frac{3}{5}<|z|<\frac{4}{5}\Big\}, \\ &=(\zeta/(s_0)_j)\cdot z \qquad \text{on} \quad \Big\{|(s_0)_j|<|z|\leqq\frac{3}{5}\Big\}, \end{aligned}$$

where we consider $z_{j,1}$ as a conformal mapping of $R-f(R_0-U_j)$ or $R_\zeta-f(R_0-U_j)$ onto $\{|(s_0)_j|<|z|<1\}$ or $\{|\zeta|<|z|<1\}$, respectively, and take the branch of log so that $\log 1=0$. Then f^ζ is well-defined for every ζ sufficiently near to $(s_0)_j$, and a simple computation shows that the complex dilatation $\mu(\zeta)$ of f^ζ depends holomorphically on ζ. Actually $(d\mu/d\zeta)((s_0)_j)$ has the support in $R-f(R_0-U_j)$ and is equal to

$$\lambda_{j,s_0}=\Big(\frac{-1}{2(s_0)_j\cdot\log(4/3)}\chi(|z|)(z/\bar{z})\frac{d\bar{z}}{dz}\Big)\circ z_{j,1}.$$

Now the same argument as before shows that $(\partial\pi_{hk}/\partial s_j)(t_0, s_0)$ exists and equal to

$$-\iint_{R_{T,S}}\lambda_{j,s_0}\phi_k(R)\wedge\phi_h(R),$$

which implies the second assertion of Lemma 1.

Since (t_0, s_0) is arbitrary, the assertion of Proposition follows by Hartogs' theorem.

4. Proof of Lemma 2

To show Lemma 2, fix p and a point $(T, S)\in\Omega$ arbitrary. Then for every $s\in(\Delta^*)^n$, $(T, s)\in\Omega^*$. Fix such an s. Then by a standard argument due to Ahlfors, we have (cf. [2, Theorems 2-5])

$$\begin{aligned} E_p(t, s)=\|\phi_p(R_{t,s})\|_{X_{t,s}}&\leqq\|\phi_p(R_{t,s})\circ g^t\|_{X_{T,s}} \\ &\leqq\|\phi_p(R_{t,s})\circ g^t-\phi_p(R_{T,s})\|_{X_{t,s}}+\|\phi_p(R_{T,s})\|_{X_{T,s}} \end{aligned}$$

$$\leqq K_t \|\phi_p(R_{T,s})\|_{X_{T,s}},$$

for every t, where g^t is as in §3 with $(t_0, s_0)=(T, s)$ and K_t is the maximal dilatation of g^t, which is independent of s.

Since $\lim_{t\to T} K_t=1$, the following lemma implies that there is an open ball V with the center (T, S) such that $E_p(t, s)$ is bounded on $V\cap\Omega^*$. Since (T, S) is arbitrary, we can conclude the assertion of Lemma 2.

Lemma 3. *Set* $\phi_s=\phi_p(R_{T,s})$ *and consider* ϕ_s *as a holomorphic differential on* $X_{T,S}$ *for every* s. *Then we have*

$$\lim_{s\to S} \|\phi_s-\phi_S\|_{X_{T,S}}=0,$$

Proof. Set $\psi_s=\phi_s-\phi_S$. Then ψ_s is holomorphic (hence in particular $*\psi_s=-i\cdot\psi_s$) on $X_{T,S}$ and $\int_{A_p}\psi_s=0$ for every A_p (considered as a curve on $X_{T,S}$). So by the same argument as in the proof of the bilinear relation (cf. the proof of [2, Lemma 1]), we have

$$\|\psi_s\|^2_{X_{T,S}}=\int_{\partial X_{T,S}} \Psi_s\cdot{}^*\bar{\psi}_s,$$

where $\partial X_{T,S}$ is the relative boundary of $X_{T,S}$ in $R_{T,S}$ and Ψ_s is a single-valued branch of the abelian integral of ψ_s on $\partial X_{T,S}$. (Note that, by the condition (#), $\int_C \psi_s=0$ for every s and every component C of $\partial X_{T,S}$, which also implies that the choice of integral constants of Ψ_s does not affect the value of the above integral.)

Now since ϕ_s converges to ϕ_S uniformly on $\partial X_{T,S}$ as s tends to S by [3, Corollary 4] (cf. Proof of Theorem), we have the assertion. q.e.d.

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