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A property of an analytic semi-group

By

Takashi SADAMATSU

§ 1. Introduction

In this note we show a necessary condition for a system of partial differential operators to be a generator of an analytic semi-group and give an example that generates a C_0 semi-group, but does not an analytic semi-group.

Let $A(x, D)$ be a matrix of partial differential operators of order m in the form the form

(1.1) $$A(x, D)=H(x, D)+B(x, D) \qquad x\in R^d$$

where D stands for $\frac{1}{i}\frac{\partial}{\partial x}$, $H(x, \xi)$ is a matrix whose entries are homogeneous polynomials of degree m in ξ with smooth coefficients and $B(x, D)$ is a lower order term.

Let $\lambda_i(x, \xi)$ $(i=1, 2, \cdots, l)$ be the characteristic roots of $A(x, \xi)$: the roots of $\det(\lambda I-H(x, \xi))=0$, then we obtain

Proposition. *If there exist the constants c_0 (>0) and β_0 such that the estimate*

(1.2) $$\|(zI-A(x, D))U\|\geqq c_0|z-\beta_0|\,\|U\|$$

holds for $U\in H^m$ and $\operatorname{Re} z>\beta_0$, then

(1.3) $$\lambda_i(x, \xi)\in\{z\in \boldsymbol{C}\,;\operatorname{Re} z<0\}\cup\{z=0\} \qquad (i=1, \cdots, l)$$

where $\|\ \ \|$ denotes L^2-norm.

Remark. (1.3) shows that if $\operatorname{Re}\lambda_i(x_0, \xi^0)=0$ for some x_0, ξ^0, then $\operatorname{Im}\lambda_i(x_0, \xi^0)=0$, that is $\lambda_i(x_0, \xi^0)=0$.

From Proposition, we have, regarding $A(x, D)$ as an operator from $\{U\in L^2\,; AU\in L^2\}$ $(\subset L^2)$ to L^2,

Theorem. *If $A(x, D)$ generates an analytic semi-group, then the characteristic roots of $A(x, \xi)$ must be in $\{z\in \boldsymbol{C}\,;\operatorname{Re} z<0\}\cup\{z=0\}$.*

El Fiky, A. [1] showed that the conditions

$$\sup_{x\in R^d,\,\xi\in S^{d-1}}\operatorname{Re}\lambda_i(x, \xi)<0 \qquad (i=1, 2, \cdots l)$$

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are necessary and sufficient in order that there exist positive constants a, b and β_0 such that the estimate

$$\|(zI-A(x, D))U\| \geqq a(|z|-\beta_0)\|U\|+b\|U\|_m$$

holds for $U \in H^m$ and $\mathrm{Re}\, z \geqq \beta_0$, where $\| \quad \|_m$ denotes the H^m-norm.

On the other hand, Igari, K. [3] treated the degenerate parabolic differential equation:

$$(\partial_t+A)u=\partial_t u+\sum_{j,k} D_{x_j}((a_{jk}(x, t)D_{x_k}u)+\sum_j b_j(x, t)D_{x_j}u+d(x, t)u=f(x, t)$$

and proved that A generates a C_0 semi-group and others under some conditions.

§ 2. Proofs of Proposition and Theorem

To prove Proposition, we use the micro-local energy method devised by S. Mizohata [4, 5, 6]. Following S. Mizohata [6], we explain this method bliefly.

First of all, we introduce the micro-localizer. Let (x_0, ξ^0) be a point in $R^d \times R^d$ $(|\xi^0|=1)$. For any given positive number r_0, we take a C_0^∞-function $\beta(x)$ which satisfies

(i) $0 \leqq \beta(x) \leqq 1$,

(ii) $\beta(x)=\begin{cases} 1 & \text{for } |x-x_0| \leqq \frac{1}{2} r_0 \\ 0 & \text{for } |x-x_0| \geqq r_0 \end{cases}$

In the same way, we take $\alpha(\xi) \in C_0^\infty$ satisfying

(i) $0 \leqq \alpha(\xi) \leqq 1$

(ii) $\alpha(\xi)=\begin{cases} 1 & \text{for } |\xi-\xi^0| \leqq \frac{1}{2} r_0 \\ 0 & \text{for } |\xi-\xi^0| \geqq r_0. \end{cases}$

We put

$$\alpha_n(\xi)=\alpha\left(\frac{\xi}{n}\right)$$

where n is a large parameter. We define $\alpha_n(D)v(x)$ by

$$(\alpha_n v)^\wedge(\xi)=\alpha_n(\xi)\hat{v}(\xi)$$

where $\hat{v}(\xi)$ denotes the Fourier transform of $v(x)$. We call $\alpha_n(D)\beta(x)$ the micro-localizer and r_0 its size.

For $a(x, \xi) \in S_{1,0}^m$ $(m \geqq 0)$, we have

(2.1) $$\|a(x, D)\alpha_n(D)v\| \leqq Cn^m \|\alpha_n(D)v\| \quad \text{if } n \text{ is large}$$

and the asymptotic expression:

(2.2) $$a(x, D)\beta(x)=\sum_{|\nu| \leqq N} \frac{1}{\nu!}\beta_{(\nu)}(x)a(x, D)^{(\nu)}+r_N(x, D; n)$$

and the estimate of the remainder term:

(2.3) $$\|r_N(x, D; n)\|_{\mathcal{L}(L^2; L^2)} \leqq C_N n^{m-N-1}$$

holds, where N can be chosen large as we wish (see [6], p. 50~52, 55).

Next, let $\psi(\xi)$ be a C^∞-function which satisfies $\int|\psi(\xi)|d\xi=1$ with its support in $\{\xi;\ |\xi|\leqq 1\}$. We define

$$\tilde{\psi}_n(x)=e^{in\xi^0x}\tilde{\psi}(x)=(2\pi)^{-d}\int e^{ix\xi}\psi(\xi-n\xi^0)d\xi,$$

then the asymptotic expression (2.2) gives

$$\alpha_n(D)\beta(x)\tilde{\psi}_n=\beta(x)\alpha_n(D)\tilde{\psi}_n+r_1(x, D;\ n)\tilde{\psi}_n.$$

Since $\alpha_n(\xi)=1$ for $\xi\in\mathrm{supp}[\psi_n(\xi)]$, we have

$$\text{(2.4)}\qquad \begin{cases} \|\alpha_n(D)\beta(x)\tilde{\psi}_n\|\leqq\|\beta(x)\tilde{\psi}(x)\|+C_1n^{-1}\|\tilde{\psi}\| \\ \|\alpha_n(D)\beta(x)\tilde{\psi}_n\|\geqq\|\beta(x)\tilde{\psi}(x)\|-C_1n^{-1}\|\tilde{\psi}\|, \end{cases}$$

here we note $\|\beta(x)\tilde{\psi}(x)\|>0$.

Finally we define the micro-localized operator of $a(x, D)$. We take a C_0^∞-function $\phi(x)$ satisfying

(i) $0\leqq\phi(x)\leqq 1$

(ii) $\phi(x)=\begin{cases} 1 & \text{for } |x|\leqq r_0 \\ 0 & \text{for } |x|\geqq 2r_0, \end{cases}$

For an operator $a(x, D)$ whose symbol is $a(x, \xi)$, we put

$$\tilde{a}(x, \xi)=\phi(x-x_0)a(x, \xi)+(1-\phi(x-x_0))a(x_0, \xi)$$

$$\text{and } \Xi(\xi)=\phi(\xi-\xi^0)\xi+(1-\phi(\xi-\xi^0))\xi^0.$$

We define the micro-localized symbol of $a(x, D)$ by

$$a_{n,\mathrm{loc}}(x, \xi)=\tilde{a}\left(x,\ n\Xi\left(\frac{\xi}{n}\right)\right),$$

then we have

$$\text{(2.5)}\qquad \begin{cases} a_{n,\mathrm{loc}}(x, \xi)=a(x, \xi) & \text{for } |x-x_0|\leqq r_0,\ |\xi-n\xi^0|\leqq nr_0 \\ a_{n,\mathrm{loc}}(x, \xi)=a(x_0, \xi^0) & \text{for } |x-x_0|\geqq 2r_0,\ |\xi-n\xi^0|\geqq 2nr_0 \\ |a_{n,\mathrm{loc}}(x, \xi)-a(x_0, n\xi^0)|\leqq Cr_0n^m & \text{for } |\xi-n\xi^0|\leqq 2nr_0 \end{cases}$$

and

$$\text{(2.6)}\qquad |a_{n,\mathrm{loc}}{}^{(\mu)}_{(\nu)}(x, \xi)|\leqq C_{\nu\mu}n^{m-|\mu|}$$

where the constants are independent of r_0 and n.

Concerning the micro-localized operator, the following facts hold.

$$\text{(2.7)}\qquad \|(a_{n,\mathrm{loc}}(x, D)-a(x, D))\alpha_n(D)\beta(x)\|_{\mathcal{L}(L^2;L^2)}\leqq Cn^{m-1}$$

and

$$\text{(2.8)}\qquad \|(a_{n,\mathrm{loc}}(x, D)-a(x_0, n\xi^0))\alpha_n(D)v\|\leqq(Cr_0n^m+C'n^{m-1/2})\|v\|$$

where the constants do not depend on r_0 and n.

In fact, we put $a'(x, D)=a_{n,\mathrm{loc}}(x, D)-a(x, D)$, then the asymptotic expression

(2.2) gives

$$a'(x, D)\alpha_n(D)\beta(x)=\beta(x)a'(x, D)\alpha_n(D)+r_1(x, D; n).$$

The 1-st term of the right hand side vanishes, therefore from the estimate of the remainder term (2.3), we have (2.7). Next, taking account of (2.5) and (2.6), we have (2.8) from the sharp form of the Gårding inequlity.

Proof of Proposition. We prove Proposition by a contradiction. Hence we assume that there exist x_0, ξ^0 ($|\xi^0|=1$) and one root, say $\lambda(x_0, \xi^0)$ such that $\mathrm{Re}\,\lambda(x_0, \xi^0)>0$ or $\mathrm{Re}\,\lambda(x_0, \xi^0)=0$, $\mathrm{Im}\,\lambda(x_0, \xi^0)\neq 0$ hold, then

$$\mathrm{Re}\,\lambda(x_0, \xi^0)+|\mathrm{Im}\,\lambda(x_0, \xi^0)|>0.$$

Let $\vec{h}$ ($|\vec{h}|=1$) be an eigenvector of $H(x_0, \xi^0)$ corresponding to $\lambda(x_0, \xi^0)$:

$$H(x_0, \xi^0)\vec{h}=\lambda(x_0, \xi^0)\vec{h}.$$

We take the sequences,

$$U_n=\alpha_n(D)\beta(x)\tilde{\varphi}_n\vec{h} \quad \text{and} \quad z_n=2\beta_0+\lambda(x_0, \xi^0)n^m,$$

then from the assumption of Proposition (1.2), we have

$$\|(z_nI-A(x, D))U_n\|\geqq c_0'(\mathrm{Re}\,\lambda(x_0, \xi^0)+|\mathrm{Im}\,\lambda(x_0, \xi^0)|)n^m\|U_n\|. \tag{2.9}$$

From now on we shall show that the estimate (2.9) fails to hold. We decompose $z_nI-A(x, D)$ as follows:

$$z_nI-A(x, D)=(z_nI-H(x_0, n\xi^0))+(H(x_0, n\xi^0)-H_{n,\mathrm{loc}}(x, D))$$
$$+(H_{n,\mathrm{loc}}(x, D)-H(x, D))-B(x, D).$$

We evaluate the each term of the right hand side. We have easily from (2.4)

$$\|(z_nI-H(x_0, n\xi^0))U_n\|\leqq C\|\beta(x)\tilde{\varphi}(x)\|+C_1n^{-1}\|\tilde{\varphi}\|,$$

from (2.8) and (2.7) we have

$$\|(H(x_0, n\xi^0)-H_{n,\mathrm{loc}}(x, D))U_n\|\leqq(Cr_0n^m+C'n^{m-1/2})\|\beta(x)\tilde{\varphi}\|$$

and

$$\|(H_{n,\mathrm{loc}}(x, D)-H(x, D))U_n\|\leqq C_1n^{m-1}\|\tilde{\varphi}\|$$

and finally we get from (2.1) and (2.4)

$$\|B(x, D)U_n\|\leqq Cn^{m'}(\|\beta(x)\tilde{\varphi}\|+C_1n^{-1}\|\tilde{\varphi}\|) \qquad (m>m')$$

where the constants are independent of r_0 and n.

These estimates and (2.9) imply

$$Cr_0n^m\|\beta(x)\tilde{\varphi}\|\geqq c_0'(\mathrm{Re}\,\lambda(x_0, \xi^0)+|\mathrm{Im}\,\lambda(x_0, \xi^0)|)n^m\|\beta(x)\tilde{\varphi}\| \tag{2.10}$$

for large n. Since $\|\beta(x)\tilde{\varphi}\|>0$ and the size of the micro-localizer r_0 can be chosen small as we wish, the formula (2.9) fails to hold.

Proof of Theorem. If A is the generator of an analytic semi-group, then there exist $\gamma\left(0<\gamma\leqq\frac{\pi}{2}\right)$ and β_0 such that

$$\rho(A)\supset\{z\in \boldsymbol{C}\,;\ |\arg(z-\beta^0)|<\frac{\pi}{2}+\gamma,\ z-\beta_0\neq 0\}$$

where $\rho(A)$ denotes the resolvent set of A, and for any ε $(0<\varepsilon<\gamma)$, there exists M_ε such that

$$\|(zI-A)^{-1}\|\leqq\frac{M_\varepsilon}{|z-\beta_0|} \qquad \text{for} \quad z\in\{z\in \boldsymbol{C}\,;\ |\arg(z-\beta_0)|\leqq\frac{\pi}{2}+\gamma-\varepsilon,\ z-\beta_0\neq 0\}$$

(e.g. [2] p. 246).

Therefore there exists c_0 such that

$$\|(zI-A)U\|\geqq c_0|z-\beta_0|\,\|U\| \qquad \text{for} \quad \operatorname{Re} z>\beta_0,\ U\in H^m\subset\mathcal{D}(A)$$

holds. Then from Proposition the characteristic roots of $A(x, \xi)$ must be in $\{z\in \boldsymbol{C}\,;\ \operatorname{Re} z<0\}\cup\{z=0\}$.

§ 3. example

Let us consider

$$A_c=-D((a(x)+ic)D)+b(x)D+d(x)$$

where $D=\frac{1}{i}\frac{d}{dx}$, $a(x)$, $b(x)$ and $d(x)$ are real valued smooth functions. We assume that $a(x)\geqq 0$ ($a(x_0)=0$ for some x_0), $b(x)^2\leqq Ka(x)$ and c is a real constant.

1) There exists a C_0 semi-group whose generator is A_c. In fact, Igari, K. ([3] p. 497) showed that for large λ, $(\lambda-A_c)$ defines a one to one surjective mapping of $\mathcal{D}(A_c)=\{u\in L^2\,;\ A_c u\in L^2\}$ onto L^2 and that there exists a constant β such that

$$\|(\lambda-A_c)^{-1}\|_{\mathcal{L}(L^2;\,L^2)}\leqq\frac{1}{\lambda-\beta} \qquad \text{for any } \lambda>\beta.$$

Noting that $[\rho_{\varepsilon *}, A_c]=[\rho_{\varepsilon *}, A_0]$ ([,] denotes the commutator and $\rho_{\varepsilon *}$ is Friedrichs' mollifier in [3]) and Remark ([3], p. 501), by Hille-Yosida's theorem, there exists a C_0 semi-group whose generator is A_c.

From Theorem we have

2) if A_c generates an analytic semi-group, then $c=0$.

3) If $c=0$, then, A_0 generates an analytic semi-group.

We prove this fact. Let $A=A_0$ and $\mathcal{D}(A)=\{u\in L^2\,;\ Au\in L^2\}$. If $u\in\mathcal{D}(A)$, then $\sqrt{a(x)}Du\in L^2$. In fact, we take a fixed large real number t, by virtue of Lemma ([3], p. 495), then

$$\|\rho_{\varepsilon *}(t-A)u\|^2\geqq \text{const.}(\|\sqrt{a(x)}Du_\varepsilon\|^2+\|u_\varepsilon\|^2)$$

and $\rho_{\varepsilon *}(t-A)u\to(t-A)u$ in L^2, $u_\varepsilon=\rho_{\varepsilon *}u\to u$ in L^2. Hence $\sqrt{a(x)}Du_\varepsilon\to v$ in L^2 and $\sqrt{a(x)}Du_\varepsilon\to\sqrt{a(x)}Du$ in $\mathcal{D}'^{1}_{L^2}$ imply $v=\sqrt{a(x)}Du\in L^2$

From Oleinik's lemma ([7] p. 972), we get

$$(Da(x))^2 \leqq Ka(x) \quad \text{and} \quad |D\sqrt{a(x)}| \leqq K \quad \text{for some } K.$$

Using these facts and the assumption for $b(x)$, we have, any $z \in \boldsymbol{C}$ ($\mathrm{Re}\, z > 0$)

$$\|\rho_{\varepsilon *}(z-A)u\|^2 \geqq |z|^2 \|u_\varepsilon\|^2 + \mathrm{Re}\, z(\|\rho_{\varepsilon *}(\sqrt{a(x)}Du)\|^2 - C\|u_\varepsilon\|^2)$$

$$-C|z|\|u\|^2 - \|C_\varepsilon(\sqrt{a(x)}Du,\ u)\|^2$$

where $\|C_\varepsilon(\sqrt{a(x)}Du,\ u)\|$ tends to 0 when ε tends to 0. By passing to the limit, we have, for some $\beta(>0)$

$$\|(z-A)u\|^2 \geqq (|z|-\beta)^2 \|u\|^2 \qquad \mathrm{Re}\, z > \beta,\ u \in \mathcal{D}(A)$$

$R(z,\ A) = (z-A)^{-1}$ satisfies

$$\|R(z,\ A)\| = \|(z-A)^{-1}\| \leqq \frac{1}{|z|-\beta} \leqq \frac{M}{|z-\beta_0|} \qquad \text{for} \quad \mathrm{Re}\, z > \beta_0$$

In general, if $z_0 \in \rho(A)$, then

$$\{z \in \boldsymbol{C}\ ;\ |z-z_0| < \|R(z_0,\ A)\|^{-1}\} \subset \rho(A).$$

Taking account of this fact, we can show that $R(z,\ A)$ is holomorphic in $\{z \in \boldsymbol{C}\ ;\ |\arg(z-\beta_0)| < \frac{\pi}{2} + \gamma,\ z-\beta_0 \neq 0\}$, here $\gamma = \sin^{-1}\frac{1}{M}$ and that for any $\varepsilon > 0$ $(0 < \varepsilon < \gamma)$,

$\|R(z,\ A)\| \leqq \frac{M_\varepsilon}{|z-\beta_0|}$ for $z \in \{z \in \boldsymbol{C}\ ;\ |\arg(z-\beta_0)| \leqq \frac{\pi}{2} + \gamma - \varepsilon,\ z-\beta_0 \neq 0\}$, here $M_\varepsilon = \frac{M}{(1-r_\varepsilon)}$ and $r_\varepsilon = 1 - \cot\gamma \tan\varepsilon$. Since A is a closed operator and $\mathcal{D}(A)$ is dense in L^2, there exists an analytic semi-group whose generator is A.

DEPARTMENT OF APPLIED MATHEMATICS,
EHIME UNIVERSITY

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