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A parametrix at noninvolutively crossing characteristic points

By

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§ 0. Introduction

In this paper, we construct a parametrix for hyperbolic equations near a very plain double characteristics, namely, at a neighborhood of singular points of two non-involutively crossing hypersurfaces. From general discussion, we can reduce the problem to constructing a parametrix for the following simple equation of second order. Therefore, we shall consider only about it.

$$P=\partial_z(\partial_z-i\Lambda(z, x, D)+a(z, x, D))+c(z, x, D)+d(z, x, D),$$

where the principal part p_2 is $-(\zeta-\Lambda)\zeta$. We assume following conditions for the symbol of this operator.

1) a, b, c, Λ are classical pseudo-differential operators in (x, ξ) with the smooth parameter z belonging to $S^0_{1,0}$, $S^0_{1,0}$, $S^1_{1,0}$, and $S^{-1}_{1,0}$, respectively.
2) Λ is real and positively homogeneous order 1 in ξ.
3) $\partial_z\Lambda\neq 0$ at the points such that $\Lambda=0$.
   (Here, we assume $\partial_z\Lambda>0$ in proof.)
4) $\partial_z[c/\partial_z\Lambda]\equiv 0$ near $\Lambda=0$.

In this case the characteristics is the union of $\{\zeta=0\}$ and $\{\zeta-\Lambda=0\}$, which are crossing non-involutively on the common set, $\{\zeta, \zeta-\Lambda\}=-\partial_z\Lambda\neq 0$. So, we are interested in the construction of parametrix only near the set $\{\Lambda=0\}$.

V. Ya. Ivriĭ [2] studied this type of operator to show well-posedness. For this hyperbolic operator, the singularities of solutions may bifurcate when they pass through the singular points $\{\Lambda=0\}$. The parametrix, we construct, makes the order of singularity after bifurcating clear. The typical model case was considered by K. Taniguchi and Y. Tozaki [3]. This example shows us the prototype of the bifurcation of singularity. We can also show the same fenomena, as it, occars in our cases except for the special ones of the first order term c, where the singularity may not bifurcate. Related works exist in N. Hanges [1] and M. Taylor [4].

We devide characteristics of P to four parts. A_1, A_2, B_1 and B_2 are $\{\zeta=0, \Lambda>0\}$, $\{\zeta-\Lambda=0, \Lambda>0\}$, $\{\zeta=0, \Lambda<0\}$ and $\{\zeta-\Lambda=0, \Lambda<0\}$, respectively. We shall construt a parametrix of null solution of P whose singularity has a given asymptotic behavior on one of A_1, A_2 or B_1, however, is smooth on B_2 (has no singularity on B_2) The

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singularity of such null solution has same order on two points of A_1 and B_1 which are on the same bicharacteristic curve of $\zeta=0$. The difference between orders of singularity at two points P_2 and P_1 on A_2 and A_1, resp., is given by

$$\text{order at } P_2-\text{order at } P_1=\mathrm{Re}(c_1/i\Lambda_z)(P_3)-1/2,$$

where P_3 is a point on $\{\Lambda=0\}$, P_3 and P_2 are on a bicharacteristic curve of $\zeta-\Lambda=0$, and, P_3 and P_1 on one of $\zeta=0$. c_1 is the principal symbol of c. $c_1-i\Lambda_z/2$ is the subprincipal symbol of P. We here use the Wyle symbol of pseudo-differential operators. (See (1.12).)

§ 1. A phase function and operators

Here we define a phase function, we use, and note some properties of it. $\zeta-\Lambda(s, X)$ is a part of product principal symbol, where $X=(x, \xi)$. We take a function φ which satisfies the equation

$$\begin{aligned} &\partial\varphi/\partial s-\Lambda(s, X+2^{-1}J\nabla\varphi)=0, \\ &\varphi|_{s=t}=0. \end{aligned} \tag{1.1}$$

The existence of φ is well known and φ is necessarily homogeneous of order 1 in ξ. For conveniently expressing the parametrix, we put it as

$$\Psi(z, t)=\varphi(z, z-t), \tag{1.2}$$

where it naturally depends on the variable X. Then $\Psi(z, t)$ is a solution of the equation

$$(\partial_z+\partial_t)\Psi-\Lambda(z, X+2^{-1}J\nabla\Psi)=0. \tag{1.3}$$

The initial condition is

$$\Psi(z, 0)=0, \tag{1.4}$$

therefore,

$$\begin{aligned} &\partial_z\Psi(z, 0)=0, \\ &\nabla\Psi(z, 0)=0, \\ &\partial_z\nabla\Psi(z, 0)=0. \end{aligned} \tag{1.5}$$

Using the equation,

$$\partial_t\Psi(z, 0)=\Lambda(z, X) \tag{1.6}$$

and

$$\partial_z\partial_t\Psi(z, 0)=-\partial_t^2\Psi(z, 0)=\Lambda_z(z, X)=((\partial_z\Lambda)(z, X)).$$

We shall construct the symbol of parametrix by the sum of the following Fourier integral operators' ones $\Phi(\alpha, k, f)$.

$$\Phi(\alpha, k, f)=\int_0^\infty e^{i\Psi(z,t)}t^{\alpha-1}(\log t)^k f(z, t)dt \tag{1.7}$$

$$(\Phi(\alpha, f)=\Phi(\alpha, 0, f)),$$

where α and f are smooth functions in (z, t, X), the support of f is bounded with respect to t, and k is a non-negative integer. We assume that α belongs to $S^0_{1,0}$ and f to $S_{1,0}$ in X. (We call f an amplitude.) If $\mathrm{Re}\,\alpha>0$, then Φ is well defined. For general α, it is defined by the analytic extension in α except for non-positive integers by using the relation that

(1.8) $$\Phi(\alpha, k, i\Psi_t f)+\Phi(\alpha-1, k, f(\alpha-1))+\Phi(\alpha-1, k-1, kf) +\Phi(\alpha, k, f_t)+\Phi(\alpha, k+1, f\alpha_t)=0,$$

where $\Phi(\alpha, k, f)=0$ if $k<0$, and where Ψ_t, α_t and f_t mean the derivatives in t of Ψ, α and f, respectively.

Lemma 1.1.

(1.9) $$\partial_z\Phi(\alpha, k, f)=\Phi(\alpha, k, i\Psi_z f)+\Phi(\alpha, k, f_z)+\Phi(\alpha, k+1, f\alpha_z).$$

(1.10) $$\partial_z\Phi(\alpha, k, f)-\Phi(\alpha, k, i\tilde{\Lambda} f) =\Phi(\alpha-1, k, f(\alpha-1))+\Phi(\alpha-1, k-1, fk) +\Phi(\alpha, k, (\partial_z+\partial_t)f)+\Phi(\alpha, k+1, f(\partial_z+\partial_t)\alpha).$$

Here

(1.11) $$\tilde{\Lambda}=\Lambda(z, X+2^{-1}J\nabla\Psi)$$

Proof. We consider it under the assumption $\mathrm{Re}\,\alpha-1>0$. The first one is the differentiation under integral. The second one is obtained by applying (1.3) to (1.8) and taking integral by parts. The analytic extension in α assures them for general α. q. e. d.

The operator $\Phi(\alpha, k, f, D)$ with the symbol $\Phi(\alpha, k, f)$ is defined as if a pseudo-differential operator $p(x, D)$ with a Wyle symbol $p(x, \xi)$ is. It is formally written as

(1.12) $$\Phi(\alpha, k, f, D)u =(2\pi)^{-n}\iint_{R^{2n}}\int_0^\infty e^{i[(x-y)\xi+\Psi(z,t,(x+y)/2,\xi)]}f(z, t, (x+y)/2, \xi) \times t^{\alpha(z,t,(x+y)/2,\xi)-1}(\log t)^k u(y)dtdyd\xi,$$

which is regorously verified for any $\alpha\in S^0_{1,0}$ and $f\in S_{1,0}$ by the oscillatory integral with the phase function

$$[(x-y)\xi+\Psi(z, t, (x+y)/2, \xi)]$$

in (ξ, y) and by the analytic extension in α. Therefore the integrals in t and in (ξ, y) are commutative in their order.

Lemma 1.2.

$$(1.13)\qquad i\Lambda(z, x, D)\Phi(\alpha, k, f, D)$$
$$\sim\Phi(\alpha, k, i\tilde{\Lambda}f, D)+\Phi(\alpha, k, L_\Lambda f, D)+\Phi(\alpha, k+1, fL_\Lambda^0\alpha, D)$$
$$+\textit{terms with lower order amplitudes,}$$

where $\tilde{\Lambda}$ *is* (1.11) *and where* L_Λ^0 *and* L_Λ, *with a scalor term* $\nu(\Lambda)$ *of order* 0, *are the first order operators such that*

$$(1.14)\qquad L_\Lambda^0=\frac{1}{2}\sum_{|\alpha|=1}[\Lambda^{(\alpha)}(z, X+2^{-1}J\nabla\Psi)\partial_x^{(\alpha)}-\Lambda_{(\alpha)}(z, X+2^{-1}J\nabla\Psi)\partial_\xi^{(\alpha)}]$$

and

$$L_\Lambda=L_\Lambda^0+\nu(\Lambda).$$

Remark. The lower terms are denoted as sum of $\Phi(\alpha, k+l+1, g)$, $(l=-1, 0, 1, \cdots)$ because α does not change. The amplitude functions g have their order such that $\text{order}\, g\leqq \text{order}\, f-l$ if $l\geqq 1$, or $\text{order}\, g\leqq \text{order}\, f-1$ if $l=-1, 0$.

Proof. Expand asymptoticly the product of two Fourier integral operators as usually, and take the first two terms in the order of amplitudes. q. e. d.

We assume beforehand that

$$(1.17.1)\qquad \partial_z\alpha=0 \quad \text{at } t=0,$$

$$(1.17.2)\qquad (\partial_z+\partial_t-L_\Lambda^0)\alpha=0 \quad \text{on } t>0,$$

$$(1.17.3)\qquad i\Lambda_z(z, X)(\alpha-1)+c=0 \quad \text{at } t=0.$$

$$(1.17.4)\qquad i(\alpha-1)\alpha_{zt}+L_\Lambda^0\alpha=0 \quad \text{at } t=0.$$

Here c is a given function. α at $t=0$ is defined by (1.17.3). The condition (1.17.1) needs an assumption for c; $(\partial/\partial z)[c/\Lambda_z]=0$. We have here assumed it. If we get a solution of the equation (1.17.2) with the initial condition α at $t=0$, then (1.17.4) is automatically satisfied.

Let us consider an operator,

$$(1.18)\qquad P=\partial_z(\partial_z-i\Lambda(z, x, D)+a(z, x, D))+c(z, x, D)+d(z, x, D),$$

where Λ and c belong to $S_{1,0}^1$, a to $S_{1,0}^0$ and d to $S_{1,0}^{-1}$.

Then we obtain Lemma 1.3.

Lemma 1.3.

$$P\Phi(\alpha, k, f, D)$$
$$\sim\Phi(\alpha-1, k, (\alpha-1)f_z, D)$$
$$+\Phi(\alpha+1, k, it^{-1}\Psi_z(\partial_z+\partial_t-L_\Lambda+\tilde{a}+\tilde{b})f, D)$$
$$+\Phi(\alpha-1, k-1, kf_z, D)$$

$+\Phi(\alpha,\ k-1,\ it^{-1}\Psi_z kf,\ D)$

+lower terms,

where the lower terms mean one of $\Phi(\alpha,\ k+l+1,\ g,\ D)$ *such that*

$$(\text{order}\ g)\leqq\min[(\text{order}\ f)-l,\ (\text{order}\ f)-1],$$

or $\Phi(\alpha+1,\ k+l+1,\ g,\ D)$ *such that*

$$(\text{order}\ g)\leqq\min[(\text{order}\ f)-l+1,\ (\text{order}\ f)],$$

for all $l\geqq-(k+1)$, *and where the function b is defined by the relation that*

$$it^{-1}\Psi_z(\alpha-1)+\tilde{c}=i\Psi_z\tilde{b}.$$

§2. Construction of asymptotic solutions

Lemma 1.3 will be used for constructing an asymptotic solution,

$$F\sim\sum_{k=0}^{\infty}\Phi(\alpha,\ k,\ f_{(k)}). \tag{2.1}$$

We first write down the relation making inductively the order of remainders lower. We here note the total order of $\Phi(\alpha,\ k,\ f,\ D)$ as operators is at most the maximum of order f and $-\text{Re}\,\alpha+\text{order}\,f$ except for k-power of logalithmic order.

$$(\alpha-1)\partial_z f+g_1=0 \quad \text{at } t=0, \tag{2.2.1}$$

$$k(it^{-1}\Psi_z)f+g_2=0 \quad \text{at } t=0 \text{ and } \Psi_t(=\Lambda)=0, \tag{2.2.2}$$

$$it^{-1}\Psi_z(\partial_z+\partial_t-L_\Lambda+\tilde{a}+\tilde{b})f+g_3=0 \quad \text{on } t>0, \tag{2.2.3}$$

where $(\text{order}\ g_1)=(\text{order}\ g_2)-1=(\text{order}\ g_3)-1$.

At the first, we give $f_{(0)}$. We assume $k=0$, so (2.2.2) is free. We can choose any function $f_{(0)}$ of order 0 such that $\partial_z f_{(0)}=0$ at $t=0$, and solve

$$(\partial_z+\partial_t-L_\Lambda+\tilde{a}+\tilde{b})f_{(0)}=0$$

with the given initial condition $f_{(0)}$ at $t=0$. Then, the remainded terms are $\Phi(\alpha,\ l+1,\ g,\ D)$ with $(\text{order}\ g)\leqq\min[-l,\ -1]$, or $\Phi(\alpha+1,\ l+1,\ g,\ D)$ with $(\text{order}\ g)\leqq\min[-l+1,\ 0]$ for $l\geqq-1$.

Secondly, we choose $f_{(1)}$ of order -1 with $k=2$ as the term $\Phi(\alpha+1,\ 2,\ g_3,\ D)$ with order $g_3=0$ vanishes by the equation (2.2.3), where it may be possible that $f_{(1)}=0$ at $t=0$. Then the remainded terms are $\Phi(\alpha,\ l+1,\ g,\ D)$ with $(\text{order}\ g)\leqq\min[-l,\ -1]$, or $\Phi(\alpha+1,\ l+1,\ g,\ D)$ with $(\text{order}\ g)\leqq\min[-l+1,\ -1]$ for $l\geqq-1$, except for $\Phi(\alpha,\ 0,\ g_2,\ D)$ with order $g_2=0$, $\Phi(\alpha+1,\ 0,\ g_3^{(0)},\ D)$ and $\Phi(\alpha+1,\ 1,\ g_3^{(1)},\ D)$ with order $g_3^{(j)}=0$.

Next we remove the terms $\Phi(\alpha,\ 0,\ g_2,\ D)$ and $\Phi(\alpha+1,\ 0,\ g_3^{(1)},\ D)$ by taking suitably $\Phi(\alpha,\ 1,\ f_{(1)2},\ D)$ with order $f_{(1)2}=-1$. The transport equations are that on $\{t=0\}$,

$$(\alpha-1)\partial_z f_{(1)2}=1$$

$$f_{(1)2}+g_2(it^{-1}\Psi_z)^{-1}=0 \qquad \text{at } \Psi_t=0,$$

and on $\{t>0\}$,

$$(\partial_z+\partial_t-L_A+\tilde{a}+\tilde{b})f_{(1)2}+g_3^{(1)}(it^{-1}\Psi_z)^{-1}=0.$$

If we take such a solution $f_{(1)2}$, then

$$\Phi(\alpha-1,\ 1,\ (\alpha-1)\partial_z f_{(1)2},\ D)=\Phi(\alpha,\ 1,\ g_{2(1)},\ D)$$

with order $g_{2(1)}=-1$, and

$$\Phi(\alpha,\ 1,\ (it^{-1}\Psi_z)f_{(1)2}+g_2,\ D)=\Phi(\alpha,\ 0,\ i\Psi_t\tilde{h},\ D)$$

with order $\tilde{h}=-1$. The last term $\Phi(\alpha, 0, i\Psi_t\tilde{h}, D)$ is passed to next steps by using the lemma.

Lemma 2.1.

$$\begin{aligned}&\Phi(\alpha,\ k-1,\ i\Psi_t\tilde{h},\ D)+\Phi(\alpha-1,\ k-1,\ (\alpha-1)\tilde{h},\ D)\\&\quad+\Phi(\alpha-1,\ k-2,\ (k-1)\tilde{h},\ D)+\Phi(\alpha,\ k-1,\ \tilde{h}_t,\ D)\\&\quad+\Phi(\alpha,\ k,\ \tilde{h}_t\alpha_t,\ D)=0.\end{aligned}$$

Therefore all remainded terms are lower ones except for $\Phi(\alpha-1, 0, (\alpha-1)\tilde{h}, D)$ and $\Phi(\alpha+1, 0, g^{(0)}, D)$. These terms are removed by $\Phi(\alpha, 0, f_{(1)3}, D)$ in which $f_{(1)3}$ is any solution of the equations

$$(\alpha-1)\partial_z f_{(1)3}+(\alpha-1)\tilde{h}=0 \qquad \text{on } t=0,$$

and

$$(\partial_z+\partial_t-L_A+\tilde{a}+\tilde{b})f_{(1)3}+g_3^{(0)}(it^{-1}\Psi_z)^{-1}=0 \qquad \text{on } t>0.$$

Therefore

$$\begin{aligned}P[&\Phi(\alpha,\ 0,\ f_{(0)},\ D)+\Phi(\alpha,\ 2,\ f_{(1)},\ D)\\&+\Phi(\alpha,\ 1,\ f_{(1)2},\ D)+\Phi(\alpha,\ 0,\ f_{(1)3},\ D)]\end{aligned}$$

is the sum of lower terms such that $\Phi(\alpha, l+1, g, D)$ with (order g)$\leqq\min[-l, -1]$, or $\Phi(\alpha+1, l+1, g, D)$ with (order g)$\leqq\min[-l+1, -1]$ for $l\geqq-1$.

Proceeding in this way inductively with respect to order of remainders and the exponent k of logalithm, we can decrease the order of remainders to any lower one. So we can construct the parametrix.

Theorem 1. *For any natural number* N, *there exist* $f_{(k)}\in S_{1,0}^{-[(k+1)/2]}$ $k=0, \cdots, 2N$, *such that*

$$P\left\{\sum_{k=0}^{2N}\Phi(\alpha,\ k,\ f_{(k)},\ D)\right\}$$

is equal modulo smoothing operators to the sum of lower terms, which are $\Phi(\alpha, l+1, g, D)$ *with* (order g)$\leqq\min[-l, -N]$, *or* $\Phi(\alpha+1, l+1, g, D)$ *with* (order g)$\leqq\min[-l+1, -N]$ *for* $l\geqq-1$, *where the principal part of* $f_{(0)}$ *is able to be any solution* $\in S_{1,0}^{0}$ *of* (2.2.1) *and* (2.2.3) *with* $g_j=0$.

§ 3. Asymptotic expansions of parametrix

We may consider only two cases. The first one is that the phase function $\Psi(z, s)$ has no singular point with respect to s, namely, $\partial_s\Psi(z, s)\neq 0$ $(s\geqq 0)$. The other one is that $\Psi(z, s)$ has a discrete singular point with respect to s inside of $s>0$, because by the definition of $\Psi(z, s)$, $(\partial_s)^2\Psi(z, 0)=-\partial_s\partial_z\Psi(z, 0)=\partial_z\Lambda(z, X)\neq 0$, if $\partial_s\Psi(z, 0)=\Lambda(z, X)=0$, so that near $s=0$, it holds that $(\partial_s)^2\Psi(z, s)\neq 0$ if $\partial_s\Psi(z, s)=0$. The omited case ($\partial_s\Psi=0$ at $s=0$) is, however, important because it is the asymptotic expansion on the double characteristics of the original operator. Only the first case appears on $\{(z, X); \Lambda(z, X)<0\}$. The first case and the second one appear exactly on $\{(z, X); \Lambda(z, X)>0\}$. In fact, the solution $s=\eta(z, X)$ of $\partial_s\Psi=0$ is negative on $\{\Lambda(z, X)<0\}$ and positive on $\{\Lambda(z, X)>0\}$ because $\partial_\Lambda\eta=-\partial_s\partial_z\Psi/\partial_s^2\Psi\partial_z\Lambda>0$ on $\{\Lambda(z, X)=0\}=\{\partial_s\Psi=s=0\}$.

3.1. The first case.

Let us define $\Gamma_f(\alpha, k, \sigma)$ as

$$\Gamma_f(\alpha, k, \sigma)=\int_0^\infty e^{-s\sigma}s^{\alpha-1}(\log s)^k f(s)ds,$$

for $\mathrm{Re}\,\alpha>0$ and $\mathrm{Re}\,\sigma\geqq 0$, where f is a smooth function in $s\in[0, \infty)$ such that the support of $\partial_s f$ is compact in $(0, \infty)$, and $\lim_{s\to\infty}f=0$ if $\mathrm{Re}\,\sigma=0$, and where k is one of natural numbers including zero. We extend to meromorphic functions for the whole $\alpha\in\boldsymbol{C}$ as well as the Γ-function by the relation that

$$\sigma\Gamma_f(\alpha, k, \sigma)=(\alpha-1)\Gamma_f(\alpha-1, k, \sigma)+\Gamma_f(\alpha-1, k-1, \sigma)+\Gamma_{\partial_s f}(\alpha, k, \sigma).$$

We denote $\Gamma_f(\alpha, k, 1)$ with $f\equiv 1$ by $\Gamma(\alpha, k)$.

Lemma 3.1.

$$\Gamma_f(\alpha, k, 1)=f(0)\sigma^{-\alpha}\sum_{j=0}^k C_{jk}\log(1/\sigma)^j\Gamma(\alpha, k-j)+0(-\infty) \qquad \text{as } |\sigma|\to\infty,$$

except for $\alpha=0, -1, -2, \cdots$, *where the support of* f *is compact in* $[0, \infty)$.

For the purpose of this section it suffices to get the asymptotic behavior of

$$\Phi(\alpha, k, f)=\int_0^\infty e^{i\Psi(z,s)}f(z, s)s^{\alpha-1}(\log s)^k ds.$$

We consider the case that $\partial_s\Psi(s)<0$, (namely, on $\{\Lambda(z, X)<0\}$). Then, we put $t=-\Psi(s)|\xi|^{-1}$. We can write $s=\mu(t)=t\tilde{\mu}(t)$, $(\tilde{\mu}(t)>0)$. Then the integral $\Phi(\alpha, k, f)$ is

$$\Phi(\alpha, k, f)=\int_0^\infty e^{-it|\xi|}\tilde{f}(t)(t\tilde{\mu})^{\tilde{\alpha}-1}(\log\tilde{\mu})^k\mu' dt.$$

By using the Taylor expansion in t and $t\log t$, we get, asymptoticly,

$$\Phi(\alpha,\, k,\, f)$$

$$\sim\int_0^\infty e^{-it|\xi|}\tilde{f}(0)t^{\tilde{\alpha}(0)-1}(\log t)^k\tilde{\mu}(0)^{\tilde{\alpha}(0)-1}\mu'(0)X(t)dt$$

$$+\sum_{(a,b,l)}\int_0^\infty e^{-it|\xi|}t^{\tilde{\alpha}(0)-1}(\log t)^l t^a(t\log t)^b C_{abl}X(t)dt\,,$$

where the parameters (a, b, l) of $\sum$ vary on

$$\{0\leqq l\leqq k,\ 0\leqq a,\ 0\leqq b \text{ and } a+b+k-l\neq 0\}\,,$$

$X(t)$ is a cut off function near 0 and C_{abl} are symbols of pseudo-differential operators with the same order as f has.

Lemma 3.2.

$$\Phi(\alpha,\, k,\, f)=\int_0^\infty e^{i\Psi(z,s)}f(z,\, s)s^{\alpha-1}(\log s)^k ds\,.$$

$$\sim b_0|\xi|^{-\alpha(0)}(\log 1/|\xi|)^k$$

$$+\sum_{(a,b,l)} b_{abl}|\xi|^{-\alpha(0)-a}(\log 1/|\xi|)^l((\log 1/|\xi|)/|\xi|)^b,$$

where the variables (a, b, l) of $\sum$ are on

$$\{0\leqq l\leqq k,\ 0\leqq a,\ 0\leqq b \text{ and } a+b+k-l\neq 0\},$$

$$b_0=(|\xi|/i\Lambda)^{\alpha(0)}f(0)$$

and b_{abl} has the same order as b_0 has.

Remark. In case of $\partial_s\Psi>0$, namely, on $\{\Lambda>0\}$, the same type of expansion holds with the same principal symbol b_0.

3.2. The second case. This case has a singular point $\partial_s\Psi=0$ inside of the integral domain $s\in[0, \infty)$. The singularity is the simplest case that $\partial_s^2\Psi<0$ at $\partial_s\Psi=0$, and also the integrands are smooth. Therefore it has the asymptotic expansion near the singular points $\partial_s\Psi=0$ as

$$\Phi(\alpha,\, k,\, f)\sim e^{i\Psi(\sigma)}\Gamma(1/2)e^{-i\pi/4}(-\Psi_{ss}(\sigma))^{-1/2}$$

$$\times[b_0+b_1|\xi|^{-1}+\cdots b_j|\xi|^{-j}+\cdots]\,,$$

where

$$b_0=(\sigma)^{\alpha(\sigma)}(\log\sigma)^k f(\sigma)\,,$$

and b_j are symbols of pseudo-differential operators with the same order as b_0, namely, f, and where $s=\sigma(z, X)$ is a solution of

$$\partial_s\Psi(s)=0\,,$$

so that it is a symbol of pseudo-differential operator of order 0 and $\varphi=\Psi(\sigma)$ satisfies

$$\partial_z\varphi-\Lambda(z,\, X+2^{-1}J\nabla\varphi)=0\,,$$

$$\varphi=0 \qquad \text{on } \Lambda=0.$$

This fact is important. The phase function φ is one for $\zeta-\Lambda=0$ starting at $\Lambda=0$.

3.3. The conclusion. In constructing the parametrix $E=\sum_k \Phi(\alpha, k, f_{(k)}, D)$ at Theorem 1, we take the principal amplitude function f_0 of $f_{(0)}$ as f_0 is an elliptic symbol being a constant in z $(\partial_z f^0=0)$ at $s=0$ and extend it on $s>0$ by

$$(\partial_z+\partial_s-L_\Lambda+\tilde{a}+\tilde{b})f_0=0$$

by following the process of construction.

$$2\times(\text{the order of } f_{(k)}-\text{the order of } f_{(0)})+k$$

never exceed zero. Then the parametrix has the asymptotic behavior as follows.

Theorem 2. 1) *On* $\{(z, X); \Lambda(z, X)<0\}$, *$E$ is asymptoticly expanded as* I_1.

2) *On* $\{(z, X); \Lambda(z, X)>0\}$, *$E$ is asymptoticly expanded as the sum of* I_1 *and* I_2.

$$I_1\sim a_0|\xi|^{-\alpha}+\sum_{(l,m,n)\in K} a_{lmn}|\xi|^{-\alpha-m}(\log 1/|\xi|)^l((\log 1/|\xi|)/|\xi|)^n,$$

$$(K\equiv\{0\leq l\leq 2m,\ 0\leq m,\ 0\leq n \text{ and } 3m+n-l\neq 0\}),$$

$$I_2\sim e^{i\varphi}\Gamma(1/2)e^{-i\pi/4}\Big(\sum_{j=0}^{\infty} b_j|\xi|^{-j-1/2}\Big),$$

where

$$a_0=(|\xi|/i\Lambda)^\alpha f_0\neq 0$$

and

$$b_0=(\sigma_z|\xi|\tilde{\Lambda}_z(\sigma))^{1/2}\sigma^{\tilde{\alpha}(\sigma)}\tilde{f}_0(\sigma)\neq 0.$$

a_j and b_j have the same order as f_0 has.

Remarks. 1) $\alpha=1-c(i\Lambda_z)^{-1}$.

2) φ is a solution of the equation

$$\partial_z\varphi-\Lambda(z, X+2^{-1}J\nabla\varphi)=0,$$

$$\varphi=0 \qquad \text{on } \Lambda(z, X)=0.$$

Then $(\partial_z\varphi, \nabla\varphi)=0$ on $\Lambda(z, X)=0$.

3) σ is a solution of the equation

$$\Lambda(z-\sigma, X+2^{-1}J\nabla\varphi)=0,$$

so that

$$(\partial_0-L_\Lambda^0)\sigma=1,$$

and

$$\sigma=0 \qquad \text{on } \Lambda=0,$$

where

$$L^0_{\Lambda(\sigma)}=\frac{1}{2}\sum_{|\alpha|=1}[\Lambda^{(\alpha)}(z, X+2^{-1}J\nabla\varphi)\partial_x^{(\alpha)}+\Lambda_{(\alpha)}(z, X+2^{-1}J\nabla\varphi)\partial_\xi^{(\alpha)}].$$

4) $\tilde{\Lambda}_z(\sigma)$ and $\tilde{\alpha}(\sigma)$ are solutions of the equation

$$(\partial_z - L^0_{\Lambda(\sigma)})g = 0,$$

with the initial conditions

$$g|_{\Lambda=0} = \Lambda_z \quad \text{and } \alpha\ (=g_0),$$

respectively. Namely,

$$g = g_0(z-\sigma,\ X + 2^{-1}J\nabla\varphi).$$

5) $\tilde{f}_0(\sigma)$ are solutions of

$$(\partial_z - L_{\Lambda(\sigma)} + \tilde{a}(\sigma) + \tilde{b}(\sigma))g = 0,$$

with the initial condition

$$g|_{\Lambda=0} = f_0,$$

where

$$L_{\Lambda(\sigma)} = L^0_{\Lambda(\sigma)} + \nu(\Lambda(\sigma)),$$

$\nu(\Lambda(\sigma))$ is a scalor term depending on the phase function φ, and $\tilde{a}(\sigma)$, $\tilde{b}(\sigma)$ are scalor terms appearing in the construction of parametrix.

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