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Squaring operations in the Hermitian symmetric spaces

By

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§ 0. Introduction

In this paper we calculate the squaring operations in the mod 2 cohomology of the irreducible Hermitian symmetric spaces of compact type. Each of them is obtained as a quotient of an appropriate compact simple Lie group by the centralizer of an appropriate 1-dimensional torus, and they are devided into six classes:

AIII	$W(m, n) = U(m+n)/(U(m) \times U(n))$	$(m, n \geq 1)$
BDI	$Q_n = SO(n+2)/(SO(2) \times SO(n))$	$(n \geq 3)$
CI	$Sp(n)/U(n)$	$(n \geq 3)$
DIII	$SO(2n)/U(n)$	$(n \geq 4)$
EIII	$= E_6/(Spin(10) \cdot T^1)$	$(Spin(10) \cap T^1 \cong \mathbf{Z}_4)$
EVII	$= E_7/(E_6 \cdot T^1)$	$(E_6 \cap T^1 \cong \mathbf{Z}_3)$

Their cohomology rings have been obtained by several authers:

$$H^*(W(m, n); \mathbf{Z}) = \mathbf{Z}[a_1, \dots, a_m, b_1, \dots, b_n]/(\sum_{i+j=k} a_i b_j;\ k \geq 1);$$

$$H^*(Q_n; \mathbf{Z}) = \begin{cases} \mathbf{Z}[t, e]/(t^m - 2e, e^2) & (n = 2m-1), \\ \mathbf{Z}[t, s]/(t^{m+1} - 2st, s^2 - \delta_m st^m) & (n = 2m); \end{cases}$$

$$H^*(Sp(n)/U(n); \mathbf{Z}) = \mathbf{Z}[c_1, \dots, c_n]/(\sum_{i+j=2k} (-1)^i c_i c_j;\ k \geq 1);$$

$$H^*(SO(2n)/U(n); \mathbf{Z}) = \mathbf{Z}[e_2, e_4, \dots, e_{2n-2}]/(e_{4k} + \sum_{i=1}^{2k-1} (-1)^i e_{2i} e_{4k-2i})$$

(it should be understood that $e_{2j} = 0$ if $j \geq n$);

$$H^*(\mathrm{EIII}; \mathbf{Z}) = \mathbf{Z}[t, w]/(t^9 - 3w^2 t, w^3 + 15w^2 t^4 - 9wt^8);$$

$$H^*(\mathrm{EVII}; \mathbf{Z}) = \mathbf{Z}[u, v, w]/(v^2 - 2wu, u^{14} - 2A, w^2 - 2B)$$

where $\delta_m = \dfrac{1 + (-1)^m}{2}$, A and B are appropriate integral classes, and

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$$|a_i| = |b_i| = |c_i| = |e_{2i}| = 2i, \quad |t| = 2, \quad |s| = |e| = 2m,$$

$$|u| = 2, \quad |v| = 10 \quad \text{and} \quad |w| = 8 \text{ for EIII}; = 18 \text{ for EVII}.$$

For details see [1], [6], [7], [9] and § 1.4 in this paper.

For AIII and CI the cohomology rings are generated by Chern classes and for DIII by the suspension images of Stiefel-Whitney classes, whence the squaring operations are obtained by the Wu formula.

For BDI ($n = 2m$) we calculate in $H^*\left(\dfrac{SO(2m+2)}{U(1) \times U(m)}\right)$ through the homomorphism induced by the projection $\dfrac{SO(2m+2)}{U(1) \times U(m)} \to Q_{2m}$, and obtain

Theorem 1.4. $\mathrm{Sq}^{2i} s = \binom{m+1}{i} st^i \quad (i \geq 0).$

Corollary 1.5. $\mathrm{Sq}^{2i} e = \binom{m+1}{i} et^i \quad (i \geq 0).$

For the exceptional types EIII and EVII we calculate in G/T(T a maximal torus of G) using the fibration $G/T \xrightarrow{\tilde{i}} B\tilde{T} \to B\tilde{G}$, where $B\tilde{G}$ is the 4-connective cover of BG and $B\tilde{T}$ is defined in a similar way (for details see [3]). The results are

Theorem 2.5. (i) *In* EIII *we have*

$$\mathrm{Sq}^2 w' = w't, \quad \mathrm{Sq}^4 w' = t^6, \quad \mathrm{Sq}^8 w' = w'^2 \qquad (\textit{where } w' = w + t^4).$$

(ii) *In* EVII

$$\mathrm{Sq}^2 v = 0, \qquad \mathrm{Sq}^4 v = vu^2 + u^7, \qquad \mathrm{Sq}^8 v = w + vu^4 + u^9;$$

$$\mathrm{Sq}^2 w = u^{10}, \quad \mathrm{Sq}^4 w = vu^6 + u^{11}, \quad \mathrm{Sq}^8 w = vu^8 + u^{13}, \quad \mathrm{Sq}^{16} w = vu^{12}.$$

As an application we give in the final section the Stiefel-Whitney classes of EIII and EVII by use of the Wu classes.

Throughout the paper $H^*(\ )$ denotes exclusively the mod 2 cohomology (integral cohomology is always denoted by $H^*(\ ;\mathbf{Z})$). $\mathbf{F}_2$ denotes the prime field of characteristic 2. For an integral element x its mod 2 reduction is denoted by $\rho(x)$, or simply by x unless there is danger of confusion. $\sigma_i(x_1,\dots,x_n)$ denotes the i-th symmetric polynomial in $x_1,\dots,x_n$ ($i \geq 0$). $\Delta(a_1,\dots,a_n)$ denotes an algebra with simple system of generators $a_1,\dots,a_n$.

§ 1. Classical types

First recall the Wu foromula. Let $x_1,\dots,x_n$ be elements of degree d with $\mathrm{Sq}^i x_j = 0$ $(0 < i < d)$, and put $c_j = \sigma_j(x_1,\dots,x_n)$ $(j \geq 0)$. Then we have

$$\mathrm{Sq}^{di}c_j = \sum_{0\le r\le i}\binom{j-i+r-1}{r}c_{j+r}c_{i-r}. \tag{1.1}$$

1.1. The Grassmannian $W(m, n)$. We have the fibration

$$W(m, n) \xrightarrow{\iota} BU(m) \times BU(n) \longrightarrow BU(m+n).$$

Put $a_i = \iota^*(c_i \times 1)$ and $b_i = \iota^*(1 \times c_i)$, where c_i is the i-th universal Chern class in $BU(m)$ or $BU(n)$ $(i \ge 0)$. Then

$$H^*(W(m, n); \mathbf{Z}) = \mathbf{Z}[a_1,\dots,a_m, b_1,\dots,b_n]/(\sum_{i+j=k} a_i b_j;\ k \ge 1),$$

whence by the naturality the operation of Sq^i is obtained from the Wu formula (1.1).

1.2. The space $Sp(n)/U(n)$. The operation of Sq^i is again obtained from the Wu formula since

$$H^*(Sp(n)/U(n); \mathbf{Z}) = \mathbf{Z}[c_1,\dots,c_n]/(\sum_{i+j=2k}(-1)^i c_i c_j;\ k \ge 1),$$

where c_i is the i-th Chern class $(i \ge 0)$.

1.3. The space $SO(2n)/U(n)$. We extract from [6], Chap. 3, §6. Using the fibration

$$SO(2n)/U(n) \xrightarrow{\iota} BU(n) \longrightarrow BSO(2n),$$

we have unique elements $e_{2i} = \frac{1}{2}\iota^* c_i \in H^{2i}(SO(2n)/U(n); \mathbf{Z})$ $(1 \le i \le n-1)$. Let $p\colon SO(2n) \to SO(2n)/U(n)$ be the projection and $\sigma\colon H^*(BSO(2n)) \to H^*(SO(2n))$ the suspension. Then

$$p^*(e_{2i}) = \sigma(w_{2i+1}) \qquad (w_j \text{ the Stiefel-Whitney classes}); \tag{1.2}$$

$$H^*(SO(2n)/U(n)) = \Delta(e_2, e_4,\dots,e_{2n-2}),\quad e_{2i}^2 = e_{4i} \tag{1.3}$$

(it should be understood that $e_{2j} = 0$ if $j \ge n$).

It follows that p^* is injective. So we calculate in $SO(2n)$:

$$p^*(\mathrm{Sq}^{2i}e_{2k}) = \sigma(\mathrm{Sq}^{2i}w_{2k+1}) = \sigma\Big(\sum_{0\le r\le 2i}\binom{2k-2i+r}{r}w_{2k+1+r}w_{2i-r}\Big)$$

by the Wu formula. Since σ annihilates decomposables, we have

Proposition 1.1. $\mathrm{Sq}^{2i}e_{2k} = \binom{k}{i}e_{2k+2i} \qquad (i, k \ge 0).$

1.4. The complex quadric $Q_n = SO(n+2)/(SO(2) \times SO(n))$. We have the fibration

$$Q_n \xrightarrow{\iota} BSO(2) \times BSO(n) \longrightarrow BSO(n+2).$$

Let $t \in H^2(BSO(2); \mathbf{Z})$ be the canonical generator, and put $t = \iota^*(t \times 1)$. Here we distinguish the two cases (a) n is even, and (b) n is odd.

(a) $n=2m$. Let $\chi \in H^{2m}(BSO(2n); \mathbf{Z})$ be the Euler class and $p_i \in H^{2i}(BSO(n); \mathbf{Z})$ the i-th Pontrjagin class. Using the fibration above we see that

$$\iota^*(t \times \chi) = 0, \quad \iota^*(1 \times p_m) = (-1)^m t^{2m} \quad \text{and}$$
$$\iota^*(1 \times \chi + t^m \times 1) \equiv 0 \mod (2).$$

Since $H^*(Q_{2m}; \mathbf{Z})$ has no torsion we have a unique element $s \in H^{2m}(Q_{2m}; \mathbf{Z})$ with $2s = \iota^*(1 \times \chi + t^m \times 1)$. Then the relations above yield

$$2st = t^{m+1} \quad \text{and} \quad 4s^2 = 2(1 + (-1)^m) st^m.$$

Considering the Serre spectral sequence for the fibration $SO(2m+2)/SO(2m) \to Q_{2m} \to BSO(2)$, we obtain

Theorem 1.2. $H^*(Q_{2m}; \mathbf{Z}) = \mathbf{Z}[t, s]/(t^{m+1} - 2st, s^2 - \delta_m st^m)$.

Now consider the diagram:

$$(1.4)\qquad \begin{tikzcd}
 & SO(2m)/U(m) \arrow[d] & & \\
P_m(\mathbf{C}) = \dfrac{U(m+1)}{U(1) \times U(m)} \arrow[r, "j"] & \dfrac{SO(2m+2)}{U(1) \times U(m)} \arrow[r, "q"] \arrow[dd, "p"] \arrow[dr, "\iota'"] & SO(2m+2)/U(m+1) \arrow[dr] & \\
 & & BU(1) \times BU(m) \arrow[r, "q'"] \arrow[d] & BU(m+1) \\
 & Q_{2m} = \dfrac{SO(2m+2)}{SO(2) \times SO(2m)} \arrow[r, "\iota"] & BSO(2) \times BSO(2m) &
\end{tikzcd}$$

Define t, $e_{2i} (i \geq 1)$ and $e \in H^*\left(\dfrac{SO(2m+2)}{U(1) \times U(m)}; \mathbf{Z}\right)$ by

$$t = p^*(t) = \iota'^*(c_1 \times 1), \qquad e_{2i} = q^*(e_{2i}) \quad \text{and} \quad e = \sum_{i=1}^{m} (-1)^i e_{2i} t^{m-i}.$$

Then

Lemma 1.3. (i) q^* induces an isomorphisms of algebras

$$H^*\left(\frac{SO(2m+2)}{U(1) \times U(m)}; \mathbf{Z}\right) \cong H^*\left(\frac{SO(2m+2)}{U(m+1)}; \mathbf{Z}\right)[t]/(t^{m+1} + 2te).$$

(ii) $$p^*(s) = \delta_m t^m + (-1)^m e.$$

Proof. By the definition of t we see that j^*t generates the ring $H^*(P_m(\mathbf{C}); \mathbf{Z})$. Since the spectral sequence for the row in (1.4) collapses, (i) holds as an isomorphism of modules. Let c, c' and c'' be the total Chern classes of $BU(m)$, $BU(1)$ and $BU(m+1)$, respectively. Then $q'^*(c'') = c' \times c$. Applying ι'^* we have

$$1 + 2e_2 + 2e_4 + \cdots + 2e_{2m} = (\iota'^*(1 \times c))\cdot(1 + t),$$

whence

$$\iota'^*(1 \times c_m) = (-1)^m(t^m + 2e).$$

Then $(t^m + 2e)t = (-1)^m(\iota'^*(1 \times c_m))\cdot t = 0$, which completes the proof of (i).

Next from the diagram (1.4) we have

$$2p^*(s) = \iota'^*(c_m \times 1) + t^m = 2\delta_m t^m + (-1)^m 2e,$$

which proves (ii) since $H^*\left(\dfrac{SO(2m+2)}{U(1) \times U(m)}; \mathbf{Z}\right)$ is torsion free. q.e.d.

Theorem 1.4. $\mathrm{Sq}^{2i}s = \dbinom{m+1}{i} st^i \quad (i \geq 0)$.

Proof. As is well known $\mathrm{Sq} = 1 + \mathrm{Sq}^1 + \mathrm{Sq}^2 + \cdots$ is an algebra homomorphism, and by 1.2 we can put $\mathrm{Sq}(s) = s(1 + \varepsilon_1 t + \varepsilon_2 t^2 + \cdots)$ $(\varepsilon_i \in \mathbf{F}_2)$. Applying p^* we have

(1.5) $$\mathrm{Sq}(\delta_m t^m + e) = (\delta_m t^m + e)(1 + \varepsilon_1 t + \varepsilon_2 t^2 + \cdots).$$

Now put $A = H^*(P_m(\mathbf{C}) \times P_m(\mathbf{C})) = \mathbf{F}_2[a, b]/(a^{m+1}, b^{m+1})$. Then the correspondence $t \mapsto a$, $e_{2i} \mapsto b^i (i \geq 1)$ extends to an algebra homomorphism φ: $H^*\left(\dfrac{SO(2m+2)}{U(1) \times U(m)}\right) \to A$, which commutes with the squaring operations. Apply φ to (1.5):

(1.6) $$\mathrm{Sq}((\delta_m + 1)a^m + c) = ((\delta_m + 1)a^m + c)(1 + \varepsilon_1 a + \varepsilon_2 a^2 + \cdots),$$

where $c = a^m + a^{m-1}b + \cdots + b^m$. First calculating in the quotient field of $\mathbf{F}_2[[a, b]]$, we obtain

$$\mathrm{Sq}(c) = \sum_{i=0}^{m} \binom{m+1}{i}(a^{m+i} + a^{m+i-1}b + \cdots + b^{m+i})\cdot \sum_{j \geq 0} (a+b)^j,$$

and then in A using the equalities $a^{m+i} + a^{m+i-1}b + \cdots + b^{m+i} = ca^i$ and $c(a+b) = 0$,

$$\mathrm{Sq}(c) = c(1 + a)^{m+1}.$$

Comparing the coefficients of $a^m b^i$ in both sides of (1.6), we obtain

$\varepsilon_i = \binom{m+1}{i}$, which proves the theorem. q.e.d.

(b) $n = 2m - 1$. According to [6]

$$H^*(Q_{2m-1}; \mathbf{Z}) = \mathbf{Z}[t, e]/(t^m - 2e, e^2),$$

where t is the same as ours. The inclusion $SO(2m+1) \subset SO(2m+2)$ $(X \mapsto X \oplus 1)$ yields a comuutative diagram

$$\begin{array}{ccc} Q_{2m-1} & \xrightarrow{\quad f \quad} & Q_{2m} \\ \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \iota} \\ BSO(2) \times BSO(2m-1) & \xrightarrow{f'} & BSO(2) \times BSO(2m). \end{array}$$

From $f'^*(1 \times \chi) = 0$ and $f'^*(t \times 1) = t \times 1$ it follows $f^*(t) = t$ and $f^*(s) = e$, and we have

Corollary 1.5. $\mathrm{Sq}^{2i} e = \binom{m+1}{i} et^i \qquad (i \geq 0).$

§ 2. Exceptional types

In this section $\ell = 6$ or 7. Let G be the simply connected exceptional Lie group of type E_ℓ and T a maximal torus of G. Take the root system $\{\alpha_1, \ldots, \alpha_\ell\}$ as in [2], and define K to be the centralizer of the 1-dimensional torus defined by the equations $\alpha_i = 0$ $(i \neq \ell)$. Then the quotient space G/K is the irreducible Hermitian symmetric space EIII ($\ell = 6$) or EVII ($\ell = 7$).

Consider the fibration $K/T \to G/T \xrightarrow{p} G/K$. By the classical theorem of Bott the odd dimensional parts of the cohomology of both the fibre and the base vanish. Hence the spectral sequence for the fibration collapses, and p^*: $H^*(G/K: A) \to H^*(G/T; A)$ is injective for any coefficient ring A. Therefore the action of Sq^i in G/K is derived from that in G/T.

First we fix a system of generators of $H^*(BT; \mathbf{Z})$ after [7] and [9]. Let $\{w_1, \ldots, w_\ell\}$ be the fundamental weights of G. Being regarded as elements of $H^2(BT; \mathbf{Z})$, they form a basis of it. Let R_j be the reflection in the plane $\alpha_j = 0$, and put

$$t_\ell = w_\ell,\ t_i = R_{i+1}(t_{i+1})(\ell > i > 1),\ t_1 = R_1(t_2) \quad \text{and} \quad c_i = \sigma_i(t_1, \ldots, t_\ell)\ (i \geq 0).$$

Then

$$H^*(BT; \mathbf{Z}) = \mathbf{Z}[t_1, \ldots, t_\ell, x]/(c_1 - 3x).$$

As the canonical mapping $i: G/T \to BT$ does not induce a surjection in $H^*(\ )$, we introduce $B\tilde{G}$ the 4-connective fibre space over BG to have the commutative diagram with two fibrations

$$\begin{array}{ccccc} G/T & \xrightarrow{\tilde{i}} & B\tilde{T} & \longrightarrow & B\tilde{G} \\ \| & & \downarrow{\scriptstyle g} & & \downarrow \\ G/T & \xrightarrow{i} & BT & \longrightarrow & BG \end{array}$$

In $H^*(B\tilde{T}; \mathbf{Z}_{(2)})$ we have new generators g_i ($i = 3, 5, 9$) with

$$2g_3 = c_3, \quad 2g_5 = c_5' = c_5 + c_4c_1 \quad \text{and} \quad 2g_9 = c_9' = c_7c_1^2 + c_6c_1^3$$

(Note that the symbol g^* is omitted here). We put

$$\gamma_3 = \rho(g_3), \quad \gamma_5 = \mathrm{Sq}^4\gamma_3 \quad \text{and} \quad \gamma_9 = \mathrm{Sq}^8\gamma_5 \qquad \in H^*(B\tilde{T}).$$

Then

(2.1) $$\gamma_5 = \rho(g_5 + g_3c_1^2 + c_4c_1) \quad \text{and}$$
$$\gamma_9 = \rho(g_9 + g_5(c_4 + c_1^4) + g_3(c_6 + c_4c_1^2 + c_1^6) + c_7c_1^2 + c_4^2c_1 + c_4c_1^5).$$

For details see [3]. Note that our $\gamma_i (i = 5, 9)$ are slightly different from those in [9].

Recall that the generator of maximum degree is w in each case (see §0). So it is sufficient for us to consider in the range of degree $\leq d_\ell$, where $d_6 = 14$ and $d_7 = 34$. Define polynomials

$$I_6 = \gamma_3^2 + c_4c_1^2 + c_1^6, \qquad I_8 = c_6c_1^2 + c_4^2 + c_4c_1^4 + c_1^8,$$
$$I_{10} = \gamma_5^2 + c_6c_1^4 + c_1^{10}, \qquad I_{12} = c_6^2 + c_6c_4c_1^2 + c_4^2c_1^4 + c_4c_1^8,$$
$$I_{14} = c_7^2 + c_6c_4c_1^4 + c_6c_1^8$$

and sets

$$R_6 = \{c_2, c_3, c_5', I_6\}, \qquad R_7 = \{c_2, c_3, c_5', I_6, I_8, c_9', I_{10}, I_{12}, I_{14}\}.$$

Then from §3 in [4] we have

Lemma 2.1. (1) *Up to degree d_ℓ*

$$H^*(G/T) = \begin{cases} \mathbf{F}_2[t_1, \ldots, t_6, \gamma_3]/(R_6) & (\ell = 6), \\ \mathbf{F}_2[t_1, \ldots, t_7, \gamma_3, \gamma_5, \gamma_9]/(R_7) & (\ell = 7). \end{cases}$$

(2) $$\mathrm{Sq}^2\gamma_3 = c_4, \qquad \mathrm{Sq}^4\gamma_3 = \gamma_5 (= c_4c_1 + c_1^5 \text{ if } \ell = 6);$$
$$\mathrm{Sq}^2\gamma_5 = c_4c_1^2 + c_1^6, \quad \mathrm{Sq}^4\gamma_5 = c_7', \quad \mathrm{Sq}^8\gamma_5 = \gamma_9;$$
$$\mathrm{Sq}^2\gamma_9 = c_4c_1^6, \qquad \mathrm{Sq}^4\gamma_9 = 0, \quad \mathrm{Sq}^8\gamma_9 = c_7'c_6, \quad \mathrm{Sq}^{16}\gamma_9 = c_7'c_6c_4$$

where $c_7' = c_7 + c_6c_1$.

Now we interpret the results in [7] and [9] to our situation.

(2.2) $$H^*(\mathrm{EIII}) = \mathbf{F}_2[t, w']/(w'^2t, w'^3 + t^{12}),$$

where t, $w' = w + t^4 \in H^*(E_6/T;\, \mathbf{Z})$ *satisfy*

$$t \equiv c_1 + t_1,\ w' \equiv c_4 + (\gamma_3 + c_1^2 t + c_1 t^2)t \qquad \bmod(2).$$

(2.3) $$H^*(\mathrm{EVII}) = \mathbf{F}_2[u, v, w]/(v^2, u^{14}, w^2),$$

where $u = t_7$, v, $w \in H^*(E_7/T;\, \mathbf{Z})$ *satisfy*

$$\begin{aligned} 2v &\equiv \bar{c}_5 - \bar{c}_4\chi + \bar{c}_3\chi^2 - \bar{c}_2\chi^3 + 2\chi^5 + 2u^5 \qquad \bmod(4),\\ 2w &\equiv \bar{c}_6\bar{c}_3 + \bar{c}_5\bar{c}_4 + 2\bar{c}_5\bar{c}_2^2 + (2\bar{c}_6\bar{c}_2 - \bar{c}_4^2 + 2\bar{c}_4\bar{c}_2^2)\chi \\ &\quad - (\bar{c}_5\bar{c}_2 - \bar{c}_4\bar{c}_3 + 2\bar{c}_3\bar{c}_2^2)\chi^2 + (\bar{c}_6 + 2\bar{c}_2^3)\chi^3 - (\bar{c}_5 + \bar{c}_3\bar{c}_2)\chi^4 \\ &\quad - (\bar{c}_4 - \bar{c}_2^2)\chi^5 - \bar{c}_3\chi^6 - \bar{c}_2\chi^7 + 2\chi^9 + 2vu^4 \qquad \bmod(4) \end{aligned}$$

with $\chi = \frac{1}{3}c_1 - u$ *and* $\bar{c}_i = \sigma_i\left(t_1 - \frac{1}{3}u, \ldots, t_6 - \frac{1}{3}u\right)$ $(i \geq 0)$.

We must describe v and w modulo 2 in terms of the t_i and the γ_j. For EVII the results are a little complicated. So we calculate modulo (c_1). Note that $c_2 \equiv 0 \bmod(4)$ (see [8]) and recall the relation (2.1). Then after some calculations modulo $(4, 2c_1)$ we obtain the following:

Lemma 2.2. *Modulo* (c_1)

$$t \equiv t_1,\quad w' \equiv \gamma_3 t_1 + c_4 \qquad (\ell = 6);$$

$$u \equiv t_7,\quad v \equiv \gamma_5 + \gamma_3 t_7^2 + c_4 t_7,\quad w \equiv \gamma_9 + \gamma_3 t_7^6 + c_6 t_7^3 \qquad (\ell = 7).$$

Fortunately we have

Lemma 2.3. (i) *In* $H^*(G/T)$ *the ideal* (c_1) *is closed under the operation of the* Sq^i.

(ii) *Up to degree* d_ℓ *the composition of* $p^*\colon H^*(G/K) \to H^*(G/T)$ *and the projection* $\pi\colon H^*(G/T) \to H^*(G/T)/(c_1)$ *is injective.*

Proof. (i) This follows from the Cartan formula.

(ii) For $\ell = 6$ put $b_i = \sigma_i(t_2, \ldots, t_6)$ $(i \geq 0)$, and let

$$A = \mathbf{F}_2[t_1, \ldots, t_6, \gamma_3] \quad \text{and} \quad B = \mathbf{F}_2[t_1, b_1, \ldots, b_5, \gamma_3].$$

Then A is free as a B-module, and hence for any $R \subset B$ the canonical map $B/BR \to A/AR$ is injective, where BR denotes the ideal generated by R in B, and similarly for AR.

Note that B coincides with $\mathbf{F}_2[t_1, c_1, \ldots, c_5, \gamma_3]$. So it contains the set $R = \{c_1\} \cup R_6 = \{c_1, c_2, c_3, c_5', I_6\}$, and B/BR can be regarded as a subset of $A/AR = H^*(G/T)/(c_1)$. Then by 2.2 the image of $\pi \circ p^*$ is contained in $B/BR = \mathbf{F}_2[t_1, c_4, \gamma_3]/(\gamma_3^2)$, and it is easily seen that up to degree d_ℓ the map $H^*(G/T) \to B/BR$ is injective.

Similarly for $\ell = 7$. q.e.d.

Now the operation of the Sq^i are obtained from 2.2 and 2.1, (2) by calculating modulo (c_1). The results are as follows:

Theorem 2.4. (i) *In* $H^*(\mathrm{EIII})$ *we have*

$$\mathrm{Sq}^2 w' = w't, \quad \mathrm{Sq}^4 w' = t^6, \quad \mathrm{Sq}^8 w' = w'^2.$$

(ii) *In* $H^*(\mathrm{EVII})$

$$\mathrm{Sq}^2 v = 0, \quad \mathrm{Sq}^4 v = vu^2 + u^7, \quad \mathrm{Sq}^8 v = w + vu^4 + u^9;$$

$$\mathrm{Sq}^2 w = u^{10}, \quad \mathrm{Sq}^4 w = vu^6 + u^{11}, \quad \mathrm{Sq}^8 w = vu^8 + u^{13}, \quad \mathrm{Sq}^{16} w = vu^{12}.$$

§ 3. Stiefel-Whitney classes

For the Hermitian symmetric spaces of classical type the Chern classes have been obtained in [1], whence so have the Stiefel-Whitney classes by mod 2 reduction. In this section as an application of the previous section we give the Stiefel-Whitney classes of EIII and EVII by using the Wu classes.

For a compact n-manifold M the i-th Wu class $u_i \in H^i(M)$ is characterised by

$$u_i \cdot x = \mathrm{Sq}^i x \qquad \text{for any } x \in H^{n-i}(M).$$

Using the Wu classes the Stiefel-Whitney classes w_i are given by $w_i = \sum_{j \geq 0} \mathrm{Sq}^{i-j} u_j$, or equivalently

$$W = \mathrm{Sq}\, U, \tag{3.1}$$

where $W = \sum w_i$ and $U = \sum u_i$ (see [5], for example).

From the previous section it follows that

(3.2) $H^*(\mathrm{EIII})$ *has* $\{w'^n t^i, w'^2 \mid n = 0, 1;\ 0 \leq i \leq 12\}$ *as a basis, and*

$$\mathrm{Sq}^{2r} w' t^{12-r} = w' t^{12} \ (r = 0, 2, 6), \quad \mathrm{Sq}^{16} w^2 = wt^{12},$$

$\mathrm{Sq}^{2r} b = 0$ *for the other* b *of degree* 32–$2r$ *in the basis*;

(3.3) $H^*(\mathrm{EVII})$ *has* $\{w^n v^m u^i \mid n, m = 0, 1;\ 0 \leq i \leq 13\}$ *as a basis, and*

$$\mathrm{Sq}^{2r} wvu^{13-r} = wvu^{13} \ (r = 0, 4, 12),$$

$\mathrm{Sq}^{2r} b = 0$ *for the other* b *of degree* 54–$2r$ *in the basis.*

Therefore

Theorem 3.1. *The non-zero Wu classes are given as follows*:

for EIII, $\quad u_0 = 1, \ u_4 = t^2, \ u_{12} = t^6, \ u_{16} = w'^2;$

for EVII, $\quad u_0 = 1, \ u_8 = u^4, \ u_{24} = u^{12}.$

Then by use of the formula (3.1) we obtain

Corollary 3.2. *The total Stiefel-Whitney class W is given as follows*:

for EIII, $W = 1 + t^2 + t^4 + t^6 + (w'^2 + t^8) + t^{10} + w't^{12}$;

for EVII, $W = 1 + u^4 + u^8 + u^{12}$.

Remark 3.3. For EIII the result coincides with the mod 2 reduction of that in [8].

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