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Adams operation and γ-filtration in K-theory

By

Tsunekazu KAMBE

1. Introduction

In [3], M. F. Atiyah related operations in K-theory to the Steenrod power operations in cohomology for CW-complexes without torsion. We take the γ-filtration in stead of the ordinary filtration in Theorem 6.5 in [3] and show that the same relations hold for arbitrary CW-complexes. The methods proving Theorem 1 are simple and based on the splitting principle of complex vector bundles. We also consider the relation between the γ-filtration and the ordinary filtration. Theorem 2 states the best possible general result for coincidence of two filtrations in p-primary component. In applying theorem 2 for $(2p+1)$-skeleton of classifying spaces B_G of finite groups, we obtain Theorem 3 which states the relation of γ-filtration $R_{2n}^{\gamma}(G)$ and the topological filtration $R_{2n}(G)$ for $n \leqq p$.

2. Relation between Adams operation and γ-operation

Let $\mathrm{Vect}(X)$ be the semi-ring of isomorghism classes of complex vector bundle over a CW-complex X. $\lambda_t\colon \mathrm{Vect}(X) \to 1 + K(X)[t]$ is defined by $\lambda_t(E) = \sum_{i \geq 0} \lambda^i(E)t^i$. Here $1 + K(X)[t]$ denotes the group of power series of t with coefficients in $K(X)$ and the leading term 1, and $\lambda^i(E)$ denotes the i-th exterior of a vector bundle E. Since $\lambda_t(E_1 + E_2) = \lambda_t(E_1)\cdot\lambda_t(E_2)$, λ_t defines uniquely the group homomorphism $K(X) \to 1 + K(X)[t]$. This homomorphism is written by the same notation λ_t. For an element $x \in K(X)$, $\lambda_t(x) = \sum_{i \geqslant 0} \lambda^i(x)t^i$ defines the i-th exterior power operation λ^i over $K(X)$. The i-th γ-operations denoted $\gamma^i\colon K(X) \to K(X)$ defined by the requirement that $\gamma_t(x) = \lambda_{t/1-t}(x)$ where $\gamma_t(x) = \sum_{i \geqslant 0} \gamma^i(x)t^i$. γ_t satisfies also the relation $\gamma_t(x+y) = \gamma_t(x)\gamma_t(x)$ and so $\gamma^k(x+y) = \sum_{i+j=K} \gamma^i(x)\cdot\gamma^j(y)$. For a line bundle L, we see $\gamma_t(L-1) = 1 + (L-1)t$. For n line bundles $L_1, L_2, \ldots, L_n$, we put $x_i = L_i - 1$, then $\gamma_t(x_1 + x_2 + \cdots + x_n) = 1 + \sigma_1 t + \sigma_2 t^2 + \cdots + \sigma_n t^n$ where $\sigma_i = \sigma_i(x_1, x_2, \ldots, x_n)$ is the i-th elementary symmetric function of $x_1, x_2, \ldots, x_n$. This implies $\gamma^i(x_1 + x_2 + \cdots + x_n) = \sigma_i$ for $1 \leqq i \leqq n$, $= 0$ for $i > n$. Using the splitting principle and the naturality of γ^i-operation, we see $\gamma^i(E-n) = 0$ $i > n$ for a n-dimensional vector bundle E. It follows $\gamma^i(E-n) \in K_{2i}(X) = \mathrm{Ker}(K(X) \to K(X_{2i-1}))$. γ-filtration of $K(X)$ is

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defined by the subgroups $K_{2n}(X)$ generated by all monomials $\gamma^{i_1}(x_1)\ \gamma^{i_2}(x_2) \cdots \gamma^{i_k}(x_k)$ with $\sum_{i=1}^{k} i_j \geqq n$ and $x_i \in \tilde{K}(X) = K_2(X)$. The fact stated above implies $K^{\gamma}_{2n}(X) \subset K_{2n}(X)$ because $\{K_{2i}(X)\}$ makes $K(X)$ the filtered ring. Let $\psi_t(x) = \sum_{k \geqslant 0} \psi^k(x) t^k$ be given by the relation $\psi_{-t}(x) = -t((d/dt)\ \lambda_t(x))/\lambda_t(x))$. ψ^k is called the Adams operation. It is well-known that $\psi^k(x) = Q^k_n(\lambda^1(x), \lambda^2(x), \dots, \lambda^n(x))$ for $n \geqq k$ where $x_1^k + x_2^k + \cdots + x_n^k = Q^k_n(\sigma_1, \sigma_2, \dots, \sigma_n)$. We put $Q^k = Q^k_n$, $n \geqq k$. We wish to expless Adams operation in terms of γ^i-operations as in case of λ^i-operation. Let

$$L^k_n(\sigma_1, \sigma_2, \dots, \sigma_n) = \sum_{i=1}^{n} \{(x_i + 1)^k - 1\}.$$

Clearly we have $L^k_n = \sum_{i=1}^{k} \binom{k}{i} Q^k_n$.

Proposition (2.1) *For an element $x \in \tilde{K}(X)$, we have a relation*

$$\psi^k(x) = L^k_n(\gamma^1(X), \gamma^2(x), \dots, \gamma^n(x)),\ n \geqq k.$$

Proof. We can put $\gamma_t(x) = 1 + \gamma^1(x)t + \cdots + \gamma^n(x)t^n = (1 + y_1 t)(1 + y_2 t) \cdots (1 + y_n t)$ for a sufficient large n. Since

$\gamma_{t/1+t} = \lambda_t$, we can compute as follows;

$$\begin{aligned}
\psi_{-t}(x) &= -t\frac{d}{dt}(\log(\lambda_t(x)) = -t\frac{d}{dt}\log((1 + y_1 t/1 + t)\cdots(1 + y_n t/1 + t)) \\
&= -t\sum_{i=1}^{n}((y_i + 1)/1 + (y_i + 1)t - 1/1 + t) \\
&= -t\sum_{i=1}^{n}((y_i + 1 - 1) - ((y_i^2 - 1)^2 - 1)t + ((y_i + 1)^3 - 1)t^2 + \cdots) \\
&= \sum_k (-1)^k t^k (\sum_{i=1}^{n}((y_i + 1)^k - 1)).
\end{aligned}$$

It follows $\psi^k(x) = \sum_{i=1}^{n}((y_i + 1)^k - 1))$ and so the proposition is proved.

Notation (2.2) *We put* $Q\gamma^k(x) = Q^k_n(\gamma^1(x), \gamma^2(x), \dots, \gamma^n(x))\ n \geqq k$ *for an element* $x \in \tilde{K}(X)$.

Let $x = x_1 + x_2 + \cdots + x_n$ be sum of stable classes of line bundles. Then we have $Q\gamma^k(x) = x_1^k + x_2^k + \cdots + x_n^k$. Since $Q\gamma^1(x) = x$, (2.1) implies $(\psi^k - k)(x) = \sum_{i \geqslant 2} \binom{k}{i} Q\gamma^i(x)$. We generalize this situation.

Lemma (2.3) *Let $x \in \tilde{K}(X)$. Then we have*

$$(\psi^k - k^m)Q\gamma^s(x) = \sum_{i\geqslant 1} a_i Q\gamma^{s+i}(x) \qquad \textit{for the case } m = s,$$
$$= \sum_{i\geqslant 0} b_i Q\gamma^{s+i}(x) \qquad \textit{for the cases } m \neq s.$$

where a_i and b_i are certain integers and $b_1 = k^s - k^m$.

proof. Put $x_i = L_i - 1$ for line bundles L_i, $i = 1, \ldots, n$. Then $Q\gamma^s(x_1 + x_2 + \cdots + x_n) = x_1^s + x_2^s + \cdots + x_n^s$ and so we see $(\psi^k - k^m)(Q\gamma^s(x)) = (((x_1+1)^k - 1)^s - k^m x_1^s) + \cdots + (((x_n+1)^k - 1)^s - k^m x_n^s)$.

Then (2.3) holds for $x = x_1 + \cdots + x_n$, and for general stable bundles it is seen from the splitting principle.

Proposition (2.4) *Let X be a finite CW-complex and $x \in \tilde{K}(X)$, then*

$$\prod_i (\psi^k - k^i)(x) = 0$$

$$\prod_{i \neq m} (\psi^k - k^i)(x) = \sum_{i\geqslant 0} a_i Q\gamma^{m+i}(x).$$

In the second equation, a_i is a certain integer which does not depend on x.

proof. This is an easy consequence of (2.3) and its proof.

The first equation in (2.4) implies that ψ^k as a linear transformation on $\tilde{K}(X)\otimes \mathbf{Q}$ has eigenvalues powers of k. Since an orthogonal decomposition of the identity is $\sum_{i\geqslant 1}\prod_{i\neq m}(\psi^k - k^m)/(k^i - k^m)$, the second equation implies that the eigenspace of ψ^k corresponding to k^m is as follows,

$$\{r(\sum_{i\geqslant 0} a_i Q\gamma^{m+i}(x) \mid r \in \mathbf{Q},\ x \in \tilde{K}(X)\}.$$

Notice that there is the next relation from the Newton formula.

(2.5) $Q\gamma^n(x) - Q\gamma^{n-1}(x)\cdot\gamma^1(x) + \cdots + (-1)^{n-1}Q\gamma^1(x)\cdot\gamma^{n-1}(x) + (-1)^n n\gamma^n(x) = 0.$

3. Adams operation and Steenrod operation

Throughout this section we suppose that p is a prime. First we compute $\psi^p(\gamma^n(x))$ for $x \in \tilde{K}(X)$.

Lemma (3.1) *Let $x \in \tilde{K}(X)$. Then there exist elements $a_i \in K_{2n+2i(p+1)}(X)$ $(i = 0, 1, \ldots, n)$ such that $\psi^p(\gamma^n(x)) = \sum_{i=0}^{n} p^{n-1} a_i$. Moreover we can choose a_0, a_n satisfying $[a_0] = [\gamma^n(x)] \in K_{2n}(x)/K_{2n+2}(X)$, $a_n = (\gamma^n(x))^p$.*

proof. Let's consider the case $x = x_1 + \cdots + x_n$ where $x_i = L_i - 1$ for line bundle L_i. Since $\gamma^n(x) = x_1 x_2 \cdots x_n$, we have

$$\psi^p(\gamma^n(x)) = ((x_1+1)^p - 1)\cdots((x_n+1)^p-1)$$
$$= (x_1^p + py_1)\cdots(x_n^p + py_n)$$

where $y_i = 1/p\left(\binom{p}{1}x_i + \binom{p}{2}x_i^2 + \cdots + \binom{p}{p-1}x_i^{p-1}\right)$. It follows

$$\psi^p(\gamma^n(x)) = \sum_{i=0}^{n} p^{n-i}\left(\sum y_1\cdots y_{n-1}x_{n-i+1}^p\cdots x_n^p\right)$$

where $\sum$ implies the symmetric sum over $y_1, y_2, \dots, y_n, x_1, x_2, \dots, x_n$. We put $a_i = \sum y_1\cdots y_{n-1}x_{n-i+1}^p\cdots x_n^p = \sum x_1\cdots x_{n-i}x_{n-i+1}^p\cdots x_n^p$ + higher terms. Then we can easily see $a_i \in K_{2n+2i(p-1)}$ and $[a_0] = [\gamma^n(x)]$, $a_n = (\gamma^n(x))^p$. Therefore we have proved (3.1) for elements which are the sum of stable classes of line bundles. In the above notation, let $a_i(x_1, \dots, x_n) = f_i(\sigma_1, \sigma_2, \dots, \sigma_n)$ where σ_i is the i-th elementary symmetric function of $x_1, \dots, x_n$. Then for an arbitrary $x \in \tilde{K}(X)$, using the splitting principle and the naturarity of γ^i-operation, we have $\psi^p(\gamma^n(x)) = \sum_{i=0}^{n} p^{n-i}f_i(\gamma^1(x), \gamma^2(x), \dots, \gamma^n(x))$ and $f_i \in K_{2n+i(p-1)}$ and $[f_0] = [\gamma^n(x)] \in K_{2n}(X)/K_{2n+2}(X)$, $f_n = (\gamma^n(x))^p$.

Let $H^{\mathrm{univ}}(X; Z) \subset H^*(X; Z)$ denote the subgroup of universal cycles in the Atiyah-Hirzebruch spectral sequence and $\varphi\colon H^{\mathrm{univ}}(X; Z) \to Gr(K^*(X))$ be the natural epimorphism.

Lemma (3.2) *Let $x \in \tilde{K}(X)$ and $\psi^p(\gamma^n(x)) = \sum_{i=0}^{n} p^{n-i}a_i, a_i \in K_{2n+i(p-1)}$ Then there exist element $h_i \in H^{2n+2i(p-1)}(X; \mathbf{Z})$ such that $\varphi(h_i) = [a_i]$ and $P^i(\bar{h}_0) = \bar{h}_i$ where P^i is the Steenrod power operation ($P^i = Sq^{2i}$ for $p = 2$) and $\bar{h}_i \in H^{2n}(X; \mathbf{Z}/(p))$ is the mod p reduction of h_i.*

proof. We use the notations in the proof of (3.1). It is sufficient to prove (3.2) for the case $x = x_1 + \cdots + x_n$ where $x_i = L_i - 1$ for line bundles L_i. In this case we see that $[a_i] = [\sum x_1x_2\cdots x_{n-1}x_{n-i+1}^p\cdots x_n^p] \in Gr^{2n+2i(p-1)}(K(X)) = K_{2n+2i(p-1)}(X)/K_{2n+2i(p-1)+2}(X)$. Let $c_1(L_i)$ be the first Chern class of L_i. Then $\varphi(c_1(L_i)) = [x_i]$ as is seen in [4]. We put $h_i = \sum c_1(L_1)c_1(L_2)\cdots c_1(L_{n-i})c_1(L_{n-i+1})^p\cdots c_1(L_n)^p$. Then $\varphi(h_i) = [a_i]$ and from the Cartan formula of the Steenrod power operation, we can see easily $p^i(\bar{h}_0) = \bar{h}_i$.

We are ready to prove the following theorem.

Theorem (1) Let $x \in K_{2n}^{\gamma}(X)$. Then there exist elements $a_i \in K_{2n+2i(p-1)}(X)$ and $h_i \in H^{2n+2i(p-1)}(X; Z)$, $i = 0, 1, \dots, n$ such that $\psi^p(x) = \sum_{i=0}^{n} p^{n-i}a_i$, $[a_0] = [x]$, $\varphi(h_i) = [a_i]$ and $p^i(\bar{h}_0) = \bar{h}_i$.

proof. Since ψ^p and P^i are additive homomorphisms, we can suppose without loss of generality that $x = \gamma^{i_1}(x_1)\cdot\gamma^{i_2}(x_2)\cdot\cdots\cdot\gamma^{i_k}(x_k)$, $i_1 + i_2 + \cdots + i_k = m$, $m \geqq n$. Assume $n = m$, then from (3.1) and (3.2), Theorem (1) holds for elements $\gamma^{i_j}(x_j)$, $j = 1, 2, \ldots, k$. Now Theorem (1) holds for x is an easy consequence from the facts that ψ^p is a ring homomorphism and P^i satisfies the Cartan formula. For the case $m > n$, we see $[a_i] = 0$ and we can put $h_i = 0$.

We note that Theorem (1) derives the integrality theorem of Chern character from the augument in 7 in [3].

Corollary (3.3) *Let* $x \in K^{\gamma}_{2n}(x)$ *and* $m(q) = \prod_{p:\mathrm{prime}} p^{[q/(p-1)]}$. *Then* $m(q)$ $Ch_{n+q}(x)$ is integral where $Ch(x) = \sum_i Ch_i(x)$, $H^{2i}(x;\mathbf{Q}) \ni Ch_i(x)$.

4. Filtrations in *K*-theory and Atiyah's conjecture.

We introduce the new filtration Which is useful for the decomposition of K-theory localized at p into $(p-1)$-factors.

Definition (4.1) $K^{Q\gamma}_{2n}(X)$ is a subgroup generated by elements $Q\gamma^m(x)$, $m \geqq n$, $x \in \tilde{K}(X)$.

We have $K^{Q}_{2n}(X) \subset K^{\gamma}_{2n}(X) \subset K_{2n}(X)$.

J. F. Adams defined the additive operation $e_n\colon K(X) \to K(X)\otimes\mathbf{Q}$ as $e_n = Ch^{-1}\circ \Pi_n \circ Ch$ where $\Pi_n\colon H^*(X;\mathbf{Q}) \to H^{2n}(X;\mathbf{Q})$ is the natural projection. He proved that $E_\alpha = \sum_{\alpha \ni n} e_n$, $\alpha \in Z/(p-1)$ is the operation $K(X) \to K(X)\otimes \mathbf{Z}_{(p)}$ and obtained the decomposition of $K(X)\otimes\mathbf{Z}_{(p)}$ into $(p-1)$-factors. From the equation $Ch(\psi^k(x)) = \sum_{n \geqslant 0} k^n Ch_n(x)$, e_n is the projection operator corresponding to eigenspace of eigenvalue k^n. In the next proposition, the second equation is well-known.

Proposition (4.2) $K^{Q\gamma}_{2n}(X)\otimes\mathbf{Q} = K^{\gamma}_{2n}(X)\otimes\mathbf{Q} = K_{2n}(X)\otimes\mathbf{Q}$.

Proof. Since e_n is a scalar multiple of $\prod_{i \neq n}(\psi^k - k^i)$, from (2, 4) $e_n(x) = \sum_{i \geqslant 0} a_i Q\gamma^{n+i}(x)$ for some $a_i \in \mathbf{Q}$. Let $x \in K_{2n}(X)\otimes\mathbf{Q}$, then $e_0(x) = e_1(x) = \cdots = e_{n-1}(x) = 0$. Therefore $x = \sum_{i \geqslant 0} e_i(x) = \sum_{i \geqslant n} e_i(x) = \sum_{i \geqslant 0} b_i Q^{n+i}(x)$ for some $b_i \in \mathbf{Q}$ and we have $K_{2n}(X)\otimes \subset K^{Q\gamma}_{2n}(X)\otimes\mathbf{Q}$.

Lemma (4.3) *Let E be a complex vector bundle over X and* $x = E - \dim(E)$. *Then we have*

$$Ch(\gamma^i(x)) = c_i(E) + \text{higher terms}$$

$$Ch(Q\gamma^i(x)) = i!\, Ch_i(E) + \text{higher terms}.$$

Proof. It is suffisient to prove it for $E = L_1 + L_2 + \cdots + L_n$, L_i being line bundles. Put $x_i = L_i - 1$, then $x = x_1 + x_2 + \cdots + x_n$. We compute as follows; $Ch(\gamma^i(x)) = Ch(\sigma_i(x_1, x_2, \ldots, x_n)) = \sigma_i(Ch(x_1), \ldots, Ch(x_n)) = \sigma_i(c_1(L_1) +$ higher terms, $\ldots, c_1(L_n)$ + higher terms) $= \sigma_i(c_1(L_1), \ldots, c_1(L_n))$ + higher terms $= c_i(E)$ + higher terms. $Ch(Q\gamma^i(x)) = Ch(Q\gamma^i(x_1 + x_2 + \cdots + x_n)) = Ch(x_1^i + {}_2^i + \cdots + x_n^i) = ((\exp(c_1(L_1))^i - 1) + \cdots + ((\exp(c_1(L_n))^i - 1) = c_1(L_1)^i + \cdots + c_1(L_n)^i$ + higher terms $= i!\, Ch_i(x)$ + higher terms.

From the Newton formula (2.5) and (4.3), we obtain the following facts.

(4.4) *Let* $x \in K_{2i}(X)$, *then*

$$[Q\gamma^i(x)] = i![x] = (-1)^i \cdot i[\gamma^i(x)] \text{ in } K_{2i}(X)/K_{2i+2}(X).$$

(4.5) $e_i(x) = 1/i!\, Q\gamma^i(x)$ + higher terms.

Proposition (4.6) *Suppose that a CW-complex X has the dimension less than* $2p-1$. *Then we have* $K_{2i}^{Q\gamma}(X) \otimes \mathbf{Z}_{(p)} = K_{2i}^{\gamma}(X) \otimes \mathbf{Z}_{(p)} = K_{2i}(X) \otimes \mathbf{Z}_{(p)}$.

proof. In this case, e_i is the operation $K(X) \to K(X) \otimes \mathbf{Z}_{(p)}$. Hence the demonstration is done in a similar way to (4.2). We wish to generalize (4.6) to the space having the dimension as great as possible.

Theorem (2) *Suppose that a CW-complex X has the dimension less than* $2p+1$. *Then we have* $K_{2i}^{\gamma}(X) \otimes \mathbf{Z}_{(p)} = K_{2i}(X) \otimes \mathbf{Z}_{(p)}$.

proof. Let X_i be the i-skelton of X. Consider the exact sequence $K_{2p}(X) \to K(X) \to K(X_{2p-1})$. Let $x \in K_{2i}(X)$, then $\mathrm{res}(x) \in K_{2i}(X_{2p-1})$ where res is the restriction homomorphism $K(X) \to K(X_{2p-1})$. From (4, 6) and its proof, there exist $a_j \in \mathbf{Z}_{(p)}$ such that $\mathrm{res}(x) = a_i Q\gamma^i(\mathrm{res}(x)) + a_{i+1} Q\gamma^{i+1}(\mathrm{res}(x)) + \cdots + a_{p+1} Q\gamma^{p-1}(\mathrm{res}(x))$. Therefore we have $x - a_i Q\gamma^i(x) - a_{i+1} Q\gamma^{i+1}(x) - \cdots - a_{p-1} Q\gamma^{p-1}(x) \in K_{2p}^{\gamma}(X) \otimes \mathbf{Z}_{(p)}$. Hence if we show $K_{2p}^{\gamma}(X) \otimes \mathbf{Z}_{(p)} = K_{2p}(X) \otimes \mathbf{Z}_{(p)}$, we complete the demonstration. Let $y \in K_{2p}(X)$, then from (4, 4) we see $y = 1/p!\, Q\gamma^p(y) = (-1)^p/(p-1)!\, \gamma^p(y)\, K_{2p}^{\gamma}(X) \otimes \mathbf{Z}_{(p)}$.

We notice that for every prime p, the dimension $2p+1$ in Theorem (2) is best possible. That is, there exist CW-complexes of dimension $2p+2$ Which does not satisfy the equation in the theorem. We can take as such CW-complexes $(2p+2)$-skeleton of the classifying space of some finite groups. These are the sources of the counter-examples for Atiyah's conjecture. [4], [5], [6]

Let's apply Theorem (2) to $(2p+1)$-skeleton $B_{G,2p+1}$ of a classifying space B_G of a finite group G. Let $\alpha_i\colon R(G) \to K(B_{G,2i-1})$ be the natural homomorphism and $R_{2i}(G)$ its kernel.

Lemma (4.7)

(1) $\alpha_{m+1}^{-1}(K_{2i}(B_{G,2m+1})) = R_{2i}(G)$ *for* $i \leqq m+1$.

(2) $\alpha_{p+1}^{-1}(K_{2i}^{\gamma}(B_{G,2p+1}) \otimes \mathbf{Z}_{(p)}) = (R_{2i}^{\gamma}(G) + R_{2p+2}(G)) \otimes \mathbf{Z}_{(p)}$ *for* $i \leqq p$.

proof. (1) is trivial from definitions. Clealy, we have

$$K_{2i}^{\gamma}(B_{G,2p+1})) \otimes \mathbf{Z}_{(p)} \supset \alpha_{p+1}(R_{2i}^{\gamma}(G) + R_{2p+2}(G)) \otimes \mathbf{Z}_{(p)}.$$

Therefore it is sufficient to show that

$$\alpha_{p+1}^{-1}(K_{2i}^{\gamma}(B_{G,2p+1}))\otimes \mathbf{Z}_{(p)} \subset (R_{2i}^{\gamma}(G)+R_{2p+2}(G))\otimes \mathbf{Z}_{(p)}.$$

At first, consider the case $i=p$. Let $x\in\alpha_{p+1}^{-1}(K_{2p}(B_{G,2p+1})\otimes \mathbf{Z}_{(p)}$, then $\alpha_{p+1}(x)\in K_{2p}^{\gamma}(B_{G,2p+1})\otimes \mathbf{Z}_{(p)}=K_{2p}(B_{G,2p+1})\otimes \mathbf{Z}_{(p)}$ and so $\alpha_{p+1}(x)=1/(p-1)!\ \gamma^p(\alpha_{p+1}(x)) = \alpha_{p+1}(1/(p-1)!\ \gamma^p(x)$ as in the proof of Theorem (2). Therefore $x-1/(p-1)!\ \gamma^p(x)\in R_{2p+2}(G)\otimes \mathbf{Z}_{(p)}$ and $x\in(R_{2p}^{\gamma}(G)+R_{2p+2}(G))\otimes \mathbf{Z}_{(p)}$. For a general i, let $x\in\alpha_{p+1}^{-1}(K_{2i}^{\gamma}(B_{G,2p+1}))\otimes \mathbf{Z}_{(p)}$, then there exist $a_j\in \mathbf{Z}_{(p)}$ such that $\alpha_{p+1}(x-a_iQ\gamma^i(x) - a_{i+1}Q\gamma^{i+1}(x)-\cdots-a_{p-1}Q\gamma^{p-1}(x))\in K_{2p}(B_{G,2p+1})\otimes \mathbf{Z}_{(p)}$. It follows easily that $x\in(R_{2i}^{\gamma}(G)+R_{2p+2}(G))\otimes \mathbf{Z}_{(p)}$.

Theorem (3)

$$R_{2i}(G)\otimes \mathbf{Z}_{(p)}=(R_{2i}^{\gamma}(G)+R_{2p+2}(G))\otimes \mathbf{Z}_{(p)}, \qquad \textit{for } i\leqq p.$$

proof. This is an easy consequence from Theorem (2) and (4, 7). Let $CH^*(G)$ denote the subring of $H^*(G)$ generated by all Chern classes and $CH^{2i}(G)$ its $2i$-th component. Let $H^{\text{univ}}(G)$ denote the subring of universal cycles in the Atiyah's spectral sequence $H^*(G)\Rightarrow\widehat{R(G)}$ and: $H^{\text{univ}}(G)\to Gr(R(G))=\sum_n R_{2n}(G)/R_{2n+2}(G)$ the natural epimorphism. Then $CH^*(G)\subset H^{\text{univ}}(G)$ and the Atiyah's conjecture (i.e. $R_{2n}(G)=R_{2n}^{\gamma}(G)$ for all n and G) is equivalent to the following conjecture; $\varphi(CH^*(G))=Gr(R(G))$.

Corollary (4.8)

$$\varphi\otimes \mathbf{Z}_{(p)}\colon CH^{2i}(G)\mathbf{Z}_{(p)}\longrightarrow R_{2i}(G)/R_{2i+2}(G)\otimes \mathbf{Z}_{(p)}$$

is surjective for $i\leqq p$.

proof. From Theorem (3) and the relations $R_{2i}^{\gamma}(G)\subset R_{2i}(G)$, $R_{2p+2}(G)\subset R_{2i+2}(G)\subset R_{2i}(G)$, we have $R_{2i}(G)\otimes \mathbf{Z}_{(p)}=(R_{2i}^{\gamma}(G)+R_{2i+2}(G))\otimes \mathbf{Z}_{(p)}$, and so $(R_{2i}(G)/R_{2i+2}(G))\otimes \mathbf{Z}_{(p)}=((R_{2i}^{\gamma}(G)+R_{2i+2}(G))/R_{2i+2}(G)\otimes \mathbf{Z}_{(p)}\triangleq(R_{2i}^{\gamma}(G)/R_{2i}^{\gamma}(G)\cap R_{2i+2}(G))\otimes \mathbf{Z}_{(p)}$. Since $\varphi(CH^{2i}(G))=R_{2i}^{\gamma}(G)/R_{2i}^{\gamma}(G)\cap R_{2i+2}(G)$, (4.8) is proved.

We notice that for every prime p there exists a group G such that $\varphi\colon CH^{2p+2}(G)\otimes \mathbf{Z}_{(p)}\to(R_{2p+2}(G)/R_{2p+4}(G))\otimes \mathbf{Z}_{(p)}$ is not surjective. The next corollary was proved algebraicly in [4]. We can give a new proof.

Corollary (4.9) *Let G be a Artin-Tate group* (*with periodic cohomology*), *then* $R_{2n}^{\gamma}(G)=R_{2n}(G)$ *for all* n.

proof. In this case, it is known that $H^*(G)=H^{\text{even}}(G)=H^{\text{univ}}(G)$. We shall show that $CH^*(G)\otimes \mathbf{Z}_{(p)}=H^{\text{univ}}(G)\otimes \mathbf{Z}_{(p)}$ for all prime p and hence $CH^*(G)=H^{\text{univ}}(G)$. Since $H^{2i}(G)=R_{2i}(G)/R_{2i+2}(G)$, we have from (4.8) $CH^{2i}(G)\otimes \mathbf{Z}_{(p)}=H^{2i}(G)\otimes \mathbf{Z}_{(p)}$ for $i\leqq p$. It is proved in [4] that $H^*(G)\otimes \mathbf{Z}_{(p)}$ has a period $2q$ where q is a divisor of $(p-1)$ for an odd prime and 2 or 4 for $p=2$. It follows $CH^*(G)=H^{\text{univ}}(G)$ and (4.9) is proved.

DEPARTMENT OF MATHEMATICS,
NARA WOMEN'S UNIVERSITY

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