J. Math. Kyoto Univ. (JMKYAZ)
32-1 (1992) 53–55

Openness of stability

By

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Let X be a smooth projective variety of dimension n. Let $N.S.(X)$ be the Neron-Severi group of divisors on X modulo numerical equivalence. Then $N.S.(X)$ is a finitely generated abelian group which embeds in $V = N.S.(X) \otimes_{\mathbf{Z}} \mathbf{R}$. By a result of Kleiman [1], there is an open cone C in V such that $C \cap N.S.(X)$ consists of all classes of ample divisors.

Let θ be an ample divisor on X. If $\mathscr{F}$ is a coherent sheaf on X, then $\deg_\theta \mathscr{F} \equiv$ the intersection number $c_1(\mathscr{F}) \cdot \theta^{n-1}$ where $c_1(\mathscr{F})$ is the first Chern class and the slope $\mu_\theta \mathscr{F} \equiv \deg_\theta \mathscr{F} / \operatorname{rank} \mathscr{F}$ if $\mathscr{F}$ is not torsion. A vector bundle $\mathscr{W}$ on X is θ-stable if $\mu_\theta(\mathscr{F}) < \mu_\theta(\mathscr{W})$ for all coherent $0 \subsetneq \mathscr{F} \subsetneq \mathscr{W}$.

In this paper we propose to prove

Theorem 1. *There is an open cone $D(\mathscr{W}) \subset C$ such that $N.S.(X) \cap D(\mathscr{W})$ consists of the classes of θ such that $\mathscr{W}$ is θ-stable.*

This result may be proven analytic over $\mathbf{C}$ replacing $N.S.(X)$ by the real $H^{1,1}$-classes and C by the classes of Kähler metrices. In this case the result follows from the openness of the differential operator in the equation for a Hermitian-Einstein metric on stable bundles using the Donaldson-Uhlenbeck-Yau theorem. Thus our result is mostly interesting in characteristic p unless one just wants an algebraic proof.

Also there are the openness theorems of Maruyama [2] where the polarization is essentially fixed but $\mathscr{W}$ and X vary algebraically. There should be a common generalization of our results but this would be too complicated in seeing the ideas clearly.

§1. Testing for stability

We first note

Lemma 2. *The θ-stability of $\mathscr{W}$ is equivalent to the condition*

*) *for all $0 \leq i < \operatorname{rank} \mathscr{W}$ and all invertible sheaves $\mathscr{L}$ on X such that $\deg_\theta \mathscr{L} \geq i\mu_\theta(\mathscr{W})$, there is no non-zero section of $\Lambda^i \mathscr{W} \otimes \mathscr{L}^{\otimes -1}$ that satisfies the Plücker relations at a generic point of X.*

Proof. If $0 \subset \mathscr{F} \subset \mathscr{W}$ is a destabilizing $\mathscr{F}$ then $\Lambda^i \mathscr{W} \otimes (\Lambda^i \mathscr{F})^{dual}$ has a non-zero section where $i = \operatorname{rank} \mathscr{F}$. Now $(\Lambda^i \mathscr{F})^{dual\ dual} = \mathscr{L}$ is invertible and

Communicated by Prof. Ueno, May 22, 1990

$\deg_\theta \mathscr{L} \geq \deg_\theta \mathscr{F}$. This gives one implication. The other implication is just as easy. Q.E.D.

We refer to the last part (there is $\cdots$) of * as the Plücker condition. It is independent of the polarization θ.

A set $S(\theta)$ of invertible sheaves $\mathscr{L}$ is a test set if we have the equivalence; each $\mathscr{L}$ in $S(\theta)$ satisfies the Plücker condition $\Leftrightarrow \mathscr{W}$ is θ-stable. For instance $R(\theta) \equiv \{\mathscr{L} \mid \deg_\theta \mathscr{L} \geq i\mu_\theta(\mathscr{W})$ for some such $i\}$ is a test set. We want to consider smaller test sets. Consider

Lemma 3. *Given $\mathscr{W}$, there are finitely many invertible sheaves $\mathscr{M}_1, \ldots, \mathscr{M}_r$ on X such that*

$$S(\theta) = \{\mathscr{L} \mid \Gamma(X, \mathscr{L}^{\otimes -1} \otimes \mathscr{M}_j) \neq 0 \text{ for some } j\} \cap R(\theta)$$

is a test set for all polarization θ.

Proof. For each i let $0 \subset R_1 \subset \cdots \subset R_d = \Lambda^i \mathscr{W}$ be coherent sheaves such that R_j/R_{j-1} is torsion free of rank 1. Let $\mathscr{M}_j = (R_j/R_{j-1})^{dual\ dual}$. Then

$$\Gamma(X, \Lambda^i \mathscr{W} \otimes \mathscr{L}^{\otimes -1}) \neq 0 \Longrightarrow \Gamma(X, \mathscr{M}_j \otimes \mathscr{L}^{\otimes -1}) \neq 0 \qquad \text{for some } j.$$

Thus this lemma follows from the last. Q.E.D.

§2. The proof of Theorem 1

If E be a divisor on X, $\deg_\theta E \equiv \deg_\theta \mathscr{O}_X(E)$. We may write our test set $S(\theta)$ in terms of effective divisors

$$S(\theta) = \{\mathscr{M}_j(-E) \mid \deg_\theta E \leq \deg_\theta \mathscr{M}_j - i\mu_\theta(\mathscr{W})\}.$$

To prove the theorem it will be enough to show that the images in V satisfy $S(\theta') \subseteq S(\theta)$ for θ' in C with direction close enough to θ.

By the finiteness of i and j it is enough to show that the image of $\{D \mid \deg_\theta D \leq x, D$ is effective$\}$ in V is finite for any constant x. Let us extend $\deg_\theta D$ to D in V. Consider the following well-known result of Chow and Van der Waerden,

Lemma 4. *If $D > 0$ then $\deg_\theta D > 0$ for D in the closed cone N spanned by effective divisors.*

If we prove Lemma 4 we will be done because the image of $\{D \mid \deg_\theta D \leq x, D$ is effective$\} = N.S.(X) \cap$ closed bounded subset of N because $\deg_\theta D$ is linear in D. Thus we need only prove Lemma 4.

Assume that $\deg_\theta D = 0$ with D in N. Then $\deg_{\theta'} D \geq 0$ for all θ' in C. So,

$$\deg_{\theta+\varepsilon} D = \deg_\theta D + (D \cdot \varepsilon \cdot \theta^{n-2}) + \text{higher order term} \geq 0$$

for all small ε in V. Thus $(D \cdot \varepsilon \cdot \theta^{n-2}) = 0$ for all ε in V. Let S be the surface $D_1 \cap \cdots \cap D_{n-2}$ where D_i is a generic divisor in $|m\theta|$ where $m \gg 0$. Then we have

$D|_S \cdot \varepsilon|_S = 0$ for all ε in V. Now Grothendieck has shown that $\mathrm{Pic}(X) \to \mathrm{Pic}(S)$ is an isomorphism. So $D|_S$ is numerical equivalent to zero. Hence by Matsusaka $D|_S$ is homological equivalent to zero on S with respect to the étale topology. By the Lefschetz theorem of Deligne this implies that D is homological equivalent to zero on X. Hence $D = 0$ in V. This proves the lemma.

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References

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