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Solutions of nonlinear kinetic equations on the level of the Navier-Stokes dynamics

By

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1. Introduction

This paper is a successive step in the mathematical description of relations between the kinetic theory of gases and the continuous fluid theory. In the previous paper [8] the limit of kinetic equations (the Boltzmann equation, the Enskog equation and the Povzner equation) corresponding with Euler fluid dynamics was studied. Solutions to the kinetic equations were requested in the form of a sum of a truncated Hilbert expansion series, of a truncated initial layer series and of a remainder. In this way the original kinetic equation was replaced by a system of Hilbert expansion equations, initial layer equations as well as a (weakly nonlinear) equation for the remainder. In this paper the approach initiated in [8] is continued. Instead of the Hilbert expansion, which results in the nonlinear and linearized Euler equations (see [7] and [1, 2, 4, 10]), the modified expansion proposed by Caflisch [1] is used. The equations resulting from this modified expansion are first the system of Navier-Stokes equations (N-SE) for compressible fluids and thereafter systems of linearized Navier-Stokes equations. The Caflisch modified expansion would also start with the system of Burnett equations and at higher order would yield linearized systems of Burnett equations.

The expansion can be modified in such a way that it results in different "hydrodynamical" systems and as in the linearized case (Ref. [3]), from the point of view of the kinetic equations, the Navier-Stokes system can be considered as an only one of many possible refinements of the Euler system. This "kinetic nonuniqueness of the Navier-Stokes equation" will be discussed in Section 8.

Our main results are the existence and asymptotic behaviour theorems for the kinetic equations (the Boltzmann, the Enskog and the Povzner equations) under suitable smallness assumptions on the nonhydrodynamical part of the initial datum. The analysis is carried out in C^0 setting with respect to the space variable. The solutions exist macroscopically as far as a smooth solution of the system of Navier-Stokes equations (N-SE) does and are approximated by the solution of the N-SE. In addition, the solutions of the kinetic equations are continuously differentiable and unique if the solution of the N-SE is unique. The advantage of using the N-SE, instead of the Euler system, as a starting point

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for constructing the existence theorem for kinetic equations is in that we can expect the existence of smooth solutions to the N-SE even when shocks appear (cf. [1]). Thus the time interval, on which our solutions exist can be longer than of the Euler system.

2. Kinetic equations

Kinetic equations are mathematical models which describe the time and space evolution of the one-particle distribution function $f = f(t, x, v)$, where t is the time, x - the space variable and v - the velocity. Similarly as in Ref. [8] all kinetic equations are assumed to be in the dimensionless form and to be singularly perturbed by a small parameter $\varepsilon > 0$ representing the scale of the mean free path. The dimensionless form of a kinetic equation is realized by referring the variables t, x, v as well as the distribution function f to the suitable characteristic quantities For such new variables we preserve the notations t, x, v and f. Throughout the paper we assume that all functions are periodic with respect to the space variable x with fundamental domain $\Omega \subset R^d$, where $d = 1, 2$ or 3 (for details see Ref. [7]). Consequently, the investigated problems can be written

$$Df_B = \frac{1}{\varepsilon} J_0(f_B, f_B), \tag{2.1}$$

$$Df_E = \frac{1}{\varepsilon} J_\delta(f_E, f_E) + \frac{\delta}{\varepsilon^2} E_{\delta,\varepsilon}(f_E; f_E, f_E), \tag{2.2}$$

$$Df_P = \frac{1}{\varepsilon} P_r(f_P, f_P), \tag{2.3}$$

with initial data

$$f_B|_{t=0} = f_E|_{t=0} = f_P|_{t=0} = F, \tag{2.4}$$

where $D = \dfrac{\partial}{\partial t} + v \cdot \mathrm{grad}_x$, δ is a dimensionless parameter representing the scale of the hard-sphere diameter in the Enskog model (2.2) and r is a dimensionless diameter of the sphere of interaction of particles in the Povzner equation. The reader is referred to [8] for all details as well as for the definitions of rather complicated collision operators: J_0 in the Boltzmann equation (2.1), J_δ and $E_{\delta,\varepsilon}$ in the Enskog equation (2.2) and P_r in the Povzner equation (2.3). Note that J_0 is a bilinear, symmetric operator acting only on the variable v and corresponding to Grad's cutoff hard potentials (however the notation for hard-spheres potential will be used for simplicity).

3. Navier-Stokes system

The macroscopic fluid-dynamic parameters are related to the distribution function f in the classical way: the mass density is defined by

$$\rho(t, x) = \int f(t, x, v)\,dv, \tag{3.1a}$$

the macroscopic velocity vector by

$$u(t, x) = \frac{1}{\rho(t, x)} \int v f(t, x, v)\,dv \tag{3.1b}$$

and the macroscopic temperature by

$$T(t, x) = \frac{1}{3\rho(t, x)} \int v^2 f(t, x, v)\,dv - \rho(t, x) u^2(t, x). \tag{3.1c}$$

An interesting problem in the analysis of relations between the kinetic theory and the continuous fluid theory is the relationship between the macroscopic parameters ρ, u, T and the corresponding solution $(\rho_{NS}, u_{NS}, T_{NS})$ of the system of Navier-Stokes equations (N-SE):

$$\frac{\partial}{\partial t}\rho_{NS} + \sum_{i=1}^{3} \frac{\partial}{\partial x^{(i)}}(\rho_{NS} u_{NS}^{(i)}) = 0, \tag{3.2a}$$

$$\frac{\partial}{\partial t}(\rho_{NS} u_{NS}^{(j)}) + \sum_{i=1}^{3} \frac{\partial}{\partial x^{(i)}}(\rho_{NS} u_{NS}^{(i)} u_{NS}^{(j)}) + \frac{\partial}{\partial x^{(j)}}(\rho_{NS} T_{NS})$$
$$= \varepsilon \left\{ \sum_{i=1}^{3} \frac{\partial}{\partial x^{(i)}} \left(\mu(T_{NS}) \left(\frac{\partial u_{NS}^{(j)}}{\partial x^{(i)}} + \frac{\partial u_{NS}^{(i)}}{\partial x^{(j)}} \right) \right) - \frac{2}{3} \frac{\partial}{\partial x^{(j)}} \left(\mu(T_{NS}) \sum_{i=1}^{3} \frac{\partial u_{NS}^{(i)}}{\partial x^{(i)}} \right) \right\}, \tag{3.2b}$$

$$\frac{\partial}{\partial t}\left(\rho_{NS} \left(\frac{3}{2} T_{NS} + \frac{1}{2} u_{NS}^2 \right) \right) + \sum_{i=1}^{3} \frac{\partial}{\partial x^{(i)}} \left(\rho\, u_{NS}^{(i)} \left(\frac{3}{2} T_{NS} + \frac{1}{2} u_{NS}^2 \right) + \rho_{NS} u_{NS}^{(i)} T_{NS} \right)$$
$$= \varepsilon \sum_{i=1}^{3} \frac{\partial}{\partial x^{(i)}} \left\{ \mu(T_{NS}) \sum_{j=1}^{3} \left(u_{NS}^{(j)} \left(\frac{\partial u_{NS}^{(j)}}{\partial x^{(i)}} + \frac{\partial u_{NS}^{(i)}}{\partial x^{(j)}} \right) - \frac{2}{3} u_{NS}^{(i)} \frac{\partial u_{NS}^{(j)}}{\partial x^{(j)}} \right) + \lambda(T_{NS}) \frac{\partial T_{NS}}{\partial x^{(i)}} \right\}, \tag{3.2c}$$

where $u^{(i)}$ is the i-th component of the vector u, μ and λ represent the coefficient of viscosity and that of heat conduction, respectively (cf. [2, 4, 6]), with initial data

$$(\rho_{NS}, u_{NS}, T_{NS})|_{t=0} = (\rho_0, u_0, T_0), \tag{3.3}$$

where ρ_0, u_0, T_0 are the fluid-dynamic parameters of the initial distribution function F (cf. (2.4)). The above problem can be formulated in terms of analysis of relationship between the solution f_B (or f_E or f_P) of the Boltzmann equation (the Enskog equation, the Povzner equation) and the local Maxwellian with fluid-dynamic parameters defined by ρ_{NS}, u_{NS} and T_{NS} i.e.

$$M_{NS}(t, x, v) = \rho_{NS}(t, x)(2\pi T_{NS}(t, x))^{-3/2} \exp\left(- \frac{|v - u_{NS}(t, x)|^2}{2T_{NS}(t, x)} \right). \tag{3.4}$$

The reader is referred to the paper [12] for a review of results on existence of solutions for the N-SE. In this paper the following assumption is the starting point.

Assumption 3.1. Let $t_0 \in (0, +\infty)$ and the initial datum (3.3) be such that

A.) a sufficiently smooth solution $(\rho_{NS}, u_{NS}, T_{NS})$ of the problem (3.2–3) exists on the time interval $[0, t_0]$

B.) the solution satisfies

$$\rho_{NS}(t, x) \geqslant c_\rho > 0, \qquad T_{NS}(t, x) \geqslant c_T > 0 \tag{3.5}$$
$$\forall (t, x) \in [0, t_0] \times \Omega,$$

where c_ρ and c_T are constants (independent of ε),

C.) the solution is such that the functions ρ_{NS}, u_{NS}, T_{NS} and their derivatives are bounded independently of $\varepsilon \in (0, \varepsilon_0]$ for some ε_0.

4. Some definitions

After Assumption 3.1 the local Maxwellian M_{NS}, as defined in (3.4), is related to the solution $(\rho_{NS}, u_{NS}, T_{NS})$ as in Assumption 3.1. Then a local and a global Maxwellians M_0 and M_+ such that

$$M_0 = M_{NS}|_{t=0} \tag{4.1}$$

and

$$(1 + v^2)^{\frac{\alpha}{2}} M_{NS}(t, x, v) \leqslant c_\alpha M_+(v) \tag{4.2}$$

for all $(t, x, v) \in [0, t_0] \times \Omega \times R^3$ and all $\alpha \in R^1$, where the constant c_α depends only on α. Analogously as in Ref. [8] the spaces $Y_0^{\alpha,s}$ and $Y_+^{\alpha,s}$ equipped with the norms

$$N_0^{\alpha,s}\{\cdot\} = \|(\|\cdot; C^s(\Omega; M_0^{-\frac{1}{2}})\|); B^\alpha\|$$

and

$$N_+^{\alpha,s}\{\cdot\} = \|(\|\cdot; C^s(\Omega)\|; B^\alpha(M_+^{-\frac{1}{2}})\|,$$

respectively, are introduced.

$C^s(\Omega)$ and $C^s(\Omega; M_0^{-\frac{1}{2}}(\cdot, v))$ are the spaces of the functions which are continuous together with all their derivatives of orders $|\gamma| \leqslant s$ and equipped with the norms

$$\|f; C^s(\Omega)\| = \sup_{\substack{0 \leqslant |\gamma| \leqslant s \\ x \in \Omega}} \left| \frac{\partial^{|\gamma|} f}{\partial x^\gamma} \right|$$

and

$$\|f; C^s(\Omega, M_0^{-\frac{1}{2}}(\cdot, v))\| = \sup_{\substack{0 \leqslant |\gamma| \leqslant s \\ x \in \Omega}} \left| M_0^{-\frac{1}{2}}(\cdot, v) \frac{\partial^{|\gamma|} f}{\partial x^\gamma} \right|.$$

B^α and $B^\alpha(W)$ are the spaces of continuous functions on R^3 with the norms

$$\|f;\, B^\alpha\| = \sup_{R^3} |w_\alpha f|$$

and

$$\|f;\, B^\alpha(W)\| = \sup_{R^3} |w_\alpha W f|$$

where $w_\alpha(v) = (1 + v^2)^{\frac{\alpha}{2}}$ and W is a positive continuous function. $L_2(R^3;\, W)$ is the space with the norm

$$\|f;\, L_2(R^3;\, W)\| = \left(\int_{R^3} (fW)^2 dv\right)^{\frac{1}{2}}$$

and with the inner product

$$(f, g)_{L_2(R^3;W)} = \int_{R^3} f \cdot g W^2 dv.$$

The norm and the inner product in the space $L_{2,NS} = L_2(R^3,\, M_{NS}^{-\frac{1}{2}})$ are defined by

$$\|f\|_{2,NS} = \left(\int_{R^3} f^2 M_{NS}^{-1}\, dv\right)^{\frac{1}{2}} \quad \text{and} \quad (f, g)_{NS} = \int f \cdot g\, M_{NS}^{-1}\, dv.$$

Finally we introduce the hydrodynamic and nonhydrodynamic subsets in $L_2(R^3;\, W)$:

$$\mathcal{N}(W) = \text{lin}\,\{W^{-2}\Psi_i\colon\ \Psi_0(v) = 1,\ \Psi_i(v) = v^{(i)}\ (i = 1, 2, 3),\ \Psi_4(v) = v^2\}$$

and

$$\begin{aligned} \mathcal{R}(W) &= (\mathcal{N}(W))^\perp \\ &= \left\{g \in L_2(R^3;\, W)\colon (g,\, W^{-2}\Psi_i)_{L_2(R^3;W)} = \int \Psi_i g\, dv = 0,\ i = 0, 1, \ldots, 4\right\}. \end{aligned}$$

5. Caflisch expansion

The bulk expansion proposed by Caflisch [1] is intermediate between the Hilbert and the Chapman-Enskog expansions. The equation resulting from this expansion are the N-SE although the nature of the expansion is rather more close to the Hilbert procedure than to that of Chapman-Enskog.

Despite of fact that the Boltzmann equation (2.1) is singularly perturbed by a small parameter ε, its solution is searched in the form of a power series with respect to ε:

$$f_B(t) = f_0(t) + \varepsilon f_1(t) + \varepsilon^2 f_2(t) + \cdots, \tag{5.1a}$$

$$f_j = g_j + h_j, \tag{5.1b}$$

where g_j and h_j are the nonhydrodynamic and the hydrodynamic components of the j-th term of the expansion, i.e.

$$g_j \in \mathscr{R}(M_{NS}^{-\frac{1}{2}}), \qquad h_j \in \mathcal{N}(M_{NS}^{-\frac{1}{2}})$$

(cf. [7]).

Define in $L_{2,NS}$ projection operators P_{NS} and $P_{NS}^{\perp} = 1 - P_{NS}$ onto $\mathcal{N}(M_{NS}^{-\frac{1}{2}})$ and $\mathscr{R}(M_{NS}^{-\frac{1}{2}})$, respectively. Inserting (5.1) into (2.1) and decomposing the term g_1 into $g_1 = g_1' + g_1''$ we can obtain the following set of equations

$$J_0(f_0, f_0) = 0, \tag{5.2}$$

$$2J_0(f_0, g_1') = P^{\perp} Df_0, \tag{5.3a}$$

$$PDf_0 = -\varepsilon PDg_1', \tag{5.4a}$$

$$2J_0(f_0, g_1'') = \varepsilon P^{\perp} Dh_1, \tag{5.3b}$$

$$PDh_1 = -PDg_1'', \tag{5.4b}$$

$$2J_0(f_0, g_2) = \varepsilon P^{\perp} Dh_2 + P^{\perp} Dg_1 - J_0(g_1, g_1), \tag{5.3c}$$

$$PDh_2 = -PDg_2, \tag{5.4c}$$

and

$$2J_0(f_0, g_j) = \varepsilon P^{\perp} Dh_j + P^{\perp} Dg_{j-1} - \sum_{i=1}^{j-1} J_0(f_i, f_{j-1}), \tag{5.3d}$$

$$PDh_j = -PDg_j, \tag{5.4d}$$

for $j \geqslant 3$.

It is well known that the only solutions of the equation (5.2) are local Maxwellians. Thus f_0 must be a local Maxwellian with fluid-dynamic parameters $\tilde{\rho}$, $\tilde{u}$ and $\tilde{T}$. We can assume that $(\tilde{\rho}, \tilde{u}, \tilde{T})$ are given by Assumption 3.1:

$$(\tilde{\rho}, \tilde{u}, \tilde{T}) = (\rho_{NS}, u_{NS}, T_{NS}) \tag{5.5}$$

and then $f_0 = M_{NS}$. In that case, in virtue of the Fredholm theory (cf. [4, 7]) the integral equations (5.3) can be solved in $L_{2,NS}$:

$$g_1' = \mathscr{L}^{-1}(P^{\perp} Df_0), \tag{5.6a}$$

$$g_1'' = \mathscr{L}^{-1}(\varepsilon P^{\perp} Dh_1) \tag{5.6b}$$

and so on.

Now, (5.4a) leads to the equation

$$PDf_0 = -\varepsilon PD\mathscr{L}^{-1}(P^{\perp} Df_0) \tag{5.7}$$

which is precisely the N-SE (3.2) for $\tilde{\rho}$, $\tilde{u}$ and $\tilde{T}$ (cf. [1, 2, 4]). Thus by (5.5) the equation (5.7) is satisfied.
The equations (5.4b), (5.4c), (5.4d) lead to linearized Navier-Stokes systems for $h_1, h_2, \ldots$.
From the mathematical point of view the present procedure needs more caution than the Hilbert one ([7]) for the ε-dependence of the expansion terms. However, in contrast to the local Maxwellian $M_{NS}(t)$ for $t > 0$, the Maxwellians M_0 and M_+ are independent of ε. Moreover, the following lemma can be proposed.

Lemma 5.1. *Let* $\mathscr{L} = 2J_0(M_{NS}, \cdot)$. *Then*

$$\| \mathscr{L}^{-1} g \|_{2,NS} \leqslant c_0 \| g \|_{2,NS} \tag{5.8}$$

for all $g \in \mathscr{R}(M_{NS}^{-\frac{1}{2}})$, *where* c_0 *is a constant independent of* t, x *and* ε. *Moreover*

$$\| \mathscr{L}^{-1} g ; B^{\alpha}(M_{NS}^{-\frac{1}{2}}) \| \leqslant c(\alpha) \| g ; B^{\alpha}(M_{NS}^{-\frac{1}{2}}) \| \tag{5.9}$$

for all $g \in \mathscr{R}(M_{NS}^{-\frac{1}{2}}) \cap B^{\alpha}(M_{NS}^{-\frac{1}{2}})$ *and* $\alpha \geqslant 2$, *where* $c(\alpha)$ *is a constant independent of* t, x *and* ε.

Proof. We first prove the following inequality

$$(g, \mathscr{L} g)_{NS} \leqslant -c_1 (g, g)_{NS} \tag{5.10}$$

for all $g \in \mathscr{R}(M_{NS}^{-\frac{1}{2}})$, where $c_1 > 0$ is a constant independent of t, x and ε. Denote

$$\omega(v) = (2\pi)^{-\frac{3}{2}} \exp\left(-\frac{v^2}{2} \right).$$

Then (cf. [8])

$$(g, \mathscr{L} g)_{NS} = \int_{R^3} \int_{R^3} \int_{S^2} \omega^{-1}\left(\frac{v - u_{NS}}{T_{NS}^{1/2}} \right) g(v) \left\{ \omega\left(\frac{v_1' - u_{NS}}{T_{NS}^{1/2}} \right) g(v') \right.$$
$$\left. + \, \omega\left(\frac{v' - u_{NS}}{T_{NS}^{1/2}} \right) g(v_1') - \omega\left(\frac{v - u_{NS}}{T_{NS}^{1/2}} \right) g(v_1) - \omega\left(\frac{v_1 - u_{NS}}{T_{NS}^{1/2}} \right) g(v) \right\}$$
$$\cdot \Phi(n \cdot (v_1 - v))\, dn\, dv_1\, dv$$

where

$$v_1' = v_1 - n \cdot (n \cdot (v_1 - v)),$$
$$v' = v + n \cdot (n \cdot (v_1 - v))$$

and

$$\Phi(y) = \max\{0, y\}.$$

Changing variables

$$v \longrightarrow \xi = \frac{v - u_{NS}}{T_{NS}^{1/2}} \quad \text{and} \quad v_1 \longrightarrow \xi_1 = \frac{v_1 - u_{NS}}{T_{NS}^{1/2}}$$

we have

$$(g, \mathscr{L} g)_{NS} = T^{\frac{7}{2}} \int_{R^3} \int_{R^3} \int_{S^2} \omega^{-1}(\xi) \hat{g}(\xi) \{\omega(\xi_1') \hat{g}(\xi') + \omega(\xi') \hat{g}(\xi_1')$$

$$- \omega(\xi) \hat{g}(\xi_1) - \omega(\xi_1) \hat{g}(\xi)\} \Phi(n \cdot (\xi_1 - \xi)) \, dn \, d\xi_1 \, d\xi, \tag{5.11}$$

where $\hat{g}(\xi) = g(T_{NS}^{\frac{1}{2}} \xi + u_{NS})$
and

$$\xi_1' = \xi_1 - n \cdot (n \cdot (\xi_1 - \xi))$$

$$\xi' = \xi + n \cdot (n \cdot (\xi_1 - \xi)).$$

It is very well known (see [5]: (80)) that the following inequality holds:

$$\int_{R^3} \int_{R^3} \int_{S^2} \omega^{-1}(v) g(v) \{\omega(v_1') g(v') + \omega(v') g(v_1') - \omega(v) g(v_1)$$

$$- \omega(v_1) g(v)\} \Phi(n \cdot (v_1 - v)) \, dn \, d\xi_1 \, d\xi \leqslant - c_2 (g, g)_{L_2(R^3; \omega^{-1/2})} \tag{5.12}$$

for $g \in \mathscr{R}(\omega^{-\frac{1}{2}})$, where $c_2 > 0$ is a constant (equal to the first negative eigenvalue of the operator $J(\omega, \cdot)$ in $L_2(R^3; \omega^{-\frac{1}{2}})$). Applying (5.12) to (5.11) we obtain

$$(g, \mathscr{L} g)_{NS} \leqslant - c_2 \rho_{NS} T_{NS}^{\frac{1}{2}} (g, g)_{NS} \tag{5.13}$$

for $g \in \mathscr{R}(M_{NS})$. Thus (5.10) with $c_1 = c_\rho c_T^{\frac{1}{2}} c_2$ (cf. (3.5)) follows. Then using (5.10), by the standard arguments, we obtain (5.8) with $c_0 = c_1^{-1}$. Next, the operator $\mathscr{L}$ can be split into regular and singular parts

$$\mathscr{L} f = \mathscr{K} f - \nu \cdot f \tag{5.14a}$$

where

$$\nu(t, x, v) = \int_{R^3} \int_{S^2} M_{NS}(t, x, v) \Phi(n \cdot (v_1 - v)) \, dn \, dv_1. \tag{5.14b}$$

The well-known Grad's inequalities lead to

$$c_3^- w_1 \leqslant \nu \leqslant c_3^+ w_1, \tag{5.15}$$

$$\| \mathscr{K} f; B^0(M_{NS}^{-\frac{1}{2}}) \| \leqslant c_4 \| f \|_{2,NS}, \tag{5.16}$$

$$\| \mathscr{K} f; B^\alpha(M_{NS}^{-\frac{1}{2}}) \| \leqslant c^\#(\alpha) \| f; B^{\alpha-1}(M_{NS}^{-\frac{1}{2}}) \| \tag{5.17}$$

for $\alpha \geqslant 1$, where c_3^-, c_3^+, c_4 and $c^\#(\alpha)$ are positive constants independent of t, x and ε. Thus, by the Grad's arguments [5-Section V] we obtain (5.8).

By Lemma 5.1 we conclude that none of the operations described in this Section does not cause any singularity with respect to ε.

6. The initial layer expansion

It is very well known that the bulk approximation, like that presented in Section 5, has to be completed by the initial layer approximation as in Section 4 of the paper [7]:

$$f_B(t) = \tilde{f}_0(\tau) + \varepsilon \tilde{f}_1(\tau) + \varepsilon^2 \tilde{f}_2(\tau) + \cdots + f_0(\varepsilon\tau) + \varepsilon f_1(\varepsilon\tau) + \varepsilon^2 f_2(\varepsilon\tau) + \cdots, \quad (6.1a)$$

where $\tau = \frac{t}{\varepsilon}$ is the "stretched" time variable, f_j are the bulk approximation terms defined in Section 5 and $\tilde{f}_j$ are the initial layer terms. The latter are decomposed

$$\tilde{f}_j = \tilde{g}_j + \tilde{h}_j, \quad (6.1b)$$

where $\tilde{g}_j \in \mathscr{R}(M_0^{-\frac{1}{2}})$ for $j = 0, 1, \ldots$; $\tilde{h}_j \in \mathscr{N}(M_0^{-\frac{1}{2}})$ for $j = 1, 2, \ldots$ and $\tilde{h}_0 = 0$. The bulk approximation terms are expanded in the power series in τ:

$$f_j(\varepsilon\tau) = \sum_{i=0}^{k-1} \varepsilon^i f_{j,i}(\tau) + \varepsilon^k f^{\#}_{j,k}(\tau, \varepsilon). \quad (6.1c)$$

Then the representation (6.1) leads to the initial layer equations (cf. [7]):

$$\frac{\partial \tilde{f}_0}{\partial \tau} = J_0(\tilde{f}_0, \tilde{f}_0) + 2J_0(M_0, \tilde{f}_0), \quad (6.2)$$

$$\tilde{f}_0|_{\tau=0} = G, \quad (6.3)$$

where $G = F - M_0$,

$$\frac{\partial \tilde{h}_j}{\partial \tau} = -P_0(v \cdot \text{grad}_x \tilde{f}_{j-1}), \quad (6.4)$$

$$\tilde{h}_j|_{\tau=0} = \tilde{H}_j, \quad (6.5)$$

$$\frac{\partial \tilde{g}_j}{\partial \tau} = 2J_0(M_0 + \tilde{f}_0, \tilde{g}_j) + 2J_0(\tilde{f}_0, \tilde{h}_j) + \sum_{\substack{i,k \geq 1 \\ i+k=j}} J_0(\tilde{f}_i, \tilde{f}_k)$$

$$+ \sum_{\substack{i,k,k' \geq 0 \\ i<j \\ i+k+k'=j}} 2J_0(\tilde{f}_i, f_{k',k}) - P_0^{\perp}(v \cdot \text{grad}_x \tilde{f}_{j-i}), \quad (6.6)$$

$$\tilde{g}_j|_{\tau=0} = -g_{j,0}, \quad (6.7)$$

for $j = 1, 2, \ldots$, where P_0 and $P_0^{\perp}$ are the projection operators onto $\mathscr{N}(M_0^{-\frac{1}{2}})$ and $\mathscr{R}(M_0^{-\frac{1}{2}})$, respectively, and $\tilde{H}_j$ has to be specified. The initial value of $\tilde{g}_j$ is determined by the solutions of Eqs (5.3b), (5.3c) and (5.3d) at $t = 0$ as in

(6.7). Note, that in order to reduce singularities with respect to ε the initial layer terms must decay rapidly (exponentially) with $\tau \to \infty$. The null order equation (6.2) is the spatially uniform nonlinear Boltzmann equation (the space variable x is only a parameter). Note, that all its terms as well as the initial datum (6.3) are independent of ε. An exponentially decaying solution is constructed for the initial datum G satisfying the smallness condition

$$N_0^{4,0}\{G\} \leqslant \theta, \tag{6.8}$$

where θ is a critical constant independent of ε (cf. [7, 8]). As the initial layer terms should vanish at infinity (with respect to τ), the initial data (6.5) has to be

$$\tilde{H}_j = \int_0^{+\infty} P_0(v \cdot \operatorname{grad}_x \tilde{f}_j)\, d\tau, \tag{6.9}$$

$j = 1, 2, \ldots$ (cf. [7]).
This specifies the initial conditions for Eqs (5.4b), (5.4c) and (5.4d):

$$h_j|_{t=0} = -\tilde{h}|_{\tau=0} = -\tilde{H}_j, \qquad j = 1, 2, \ldots. \tag{6.10}$$

7. A weakly nonlinear equation

The considerations of Sections 5 and 6 as well as the methods from [7] lead to the following theorem.

Theorem 7.1. *Let Assumption* 3.1 *be satisfied. Let the initial datum* (2.4) *be decomposed into hydrodynamic and nonhydrodynamic parts as follows*

$$F = M_0 + G, \tag{7.1}$$

where M_0 *is a local Maxwellian whose fluid-dynamic parameters are* $\rho_{NS}|_{t=0}$, $u_{NS}|_{t=0}$, $T_{NS}|_{t=0}$ *and* G *is a function with null fluid-dynamic parameters, i.e.* $G \in \mathscr{R}(M_0^{-\frac{1}{2}})$ *such that* $G \in Y_0^{\alpha,s}$ *with* α *and* s *being large enough and such that the smallness condition* (6.8) *is satisfied. Then*

A.) *there exist solutions* $f_1, \ldots, f_a$ *of the bulk expansion equations* (5.3–5.6) *sufficiently smooth with respect to* $t \in [0, t_0]$ *and* $x \in \Omega$, *and such that*

$$\left| w_\alpha \frac{\partial^{k+|\gamma|}}{\partial t^k \partial x^\gamma} f_j \right| \leqslant \text{const} \cdot M_{NS}^{\frac{1}{2}} \tag{7.2}$$

for all $\alpha \geqslant 0$, $k \geqslant 0$, $|\gamma| \geqslant 0$, $j \in \{0, 1, \ldots, a\}$, $t \in [0, t_0]$, $x \in \Omega$, $v \in R^3$ *and* $\varepsilon > 0$, *where the constant denoted by* "const" depends only on α, k, $|\gamma|$ *and* j;

B.) *there exist solutions* $\tilde{f}_0, \ldots, \tilde{f}_a$ *of the initial layer equations* (6.3)–(6.8) *and numbers* α_j, s_j, $\delta_j > 0$ *such that*

$$\tilde{f}_j \in C^1([0, +\infty);\ Y_0^{\alpha_j, s_j}) \tag{7.3a}$$

and

$$\sup_{\tau \geqslant 0} N_0^{\alpha_j, s_j}\{\exp(\delta_j \tau)\tilde{f}_j(\tau)\} \leqslant \text{const} \tag{7.3b}$$

for $j = 0, \ldots, a$;

C.) *the Boltzmann equation* (2.1) *with the initial datum* F *is equivalent to the following nonlinear equation*

$$Dz = \frac{2}{\varepsilon} J_0(M_{NS}, z) + \frac{2}{\varepsilon} J_0(\tilde{f}_0, z) + 2\sum_{j=1}^{a} \varepsilon^{j-1} J_0(f_j + \tilde{f}_j, z)$$
$$+ \varepsilon^{b-1} J_0(z, z) + \varepsilon^{a-b}\mathfrak{A} \tag{7.4}$$

with initial data

$$z|_{t=0} = 0, \tag{7.5}$$

where $\mathfrak{A}$ *is a complicated term depending on* $f_0, \ldots, f_a$ *and* $\tilde{f}_0, \ldots, \tilde{f}_a$, *but with a regular behaviour with respect to* ε *as* $\varepsilon \to 0$.

Remark 7.1. Points A and B of Theorem 7.1 were proved in the paper [7] (see also [8]). Next, inserting

$$f(t) = M_{NS}(t) + \tilde{g}_0\left(\frac{t}{\varepsilon}\right) + \sum_{j=1}^{a} \varepsilon^j \left(f_j(t) + \tilde{f}_j\left(\frac{t}{\varepsilon}\right)\right) + \varepsilon^b z(t) \tag{7.6}$$

into the Boltzmann equation (2.1), the equation (7.5) is obtained. The nonlinear and nonhomogeneous terms of Eq. (7.4) are multiplied by numbers ε^{b-1} and ε^{a-b}, respectively. Therefore, for a and b chosen properly, Eq. (7.4) is weakly nonlinear. The analysis of this equation has been included in the paper [8] where the Euler limit was studied.

Now the following theorem on the Navier-Stokes hydrodynamic limit can be proposed.

Theorem 7.2. *Let* $k \geqslant 0$ *and Assumption* 3.1 *as well as the conditions of Theorem* 7.1 *be satisfied with* α *and* s *sufficiently large depending on* k. *If* $0 < \varepsilon \leqslant \varepsilon_c$, *where* $\varepsilon_c = \varepsilon_c(t_0)$ *is a critical value, then a solution* f_B *of the Boltzmann equation* (2.1)*-with the initial datum* F*-exists in* $L_\infty([0, t_0]; Y_+^{\gamma,k})$ *and*

$$\sup_{t \in [0, t_0]} N_+^{\gamma,k}\left\{f_B(t) - M_{NS}(t) - \tilde{f}_0\left(\frac{t}{\varepsilon}\right) - \varepsilon\left(f_1(t) + \tilde{f}_1\left(\frac{t}{\varepsilon}\right)\right)\right\} \leqslant c_{t_0}\varepsilon^2 \tag{7.7}$$

for all $\beta \geqslant 0$, *where* c_{t_0} *is a constant depending on* t_0. *Moreover, if* $k \geqslant 2$ *then*

$$f_B \in C^0([0, t_0]; Y_+^{\gamma,k-1}) \cap C^1([0, t_0]; Y_+^{\gamma,k-2}). \tag{7.8}$$

Remark 7.2. Theorem 7.2 follows from Theorem 7.1 and by an analysis of the weakly nonlinear equation (7.4) in Ref. [8].

Remark 7.3. The asymptotic relationships between the Enskog equation (2.2) or the Povzner equation (2.3) and the Navier-Stokes equations (3.2) can be formulated analogously as it has been done for the Euler system case in Ref. [8]-Sections 10 and 11.

8. Discussion

Theorem 7.2 refers to the solution to the Boltzmann equation under the suitable smallness assumption on the nonhydrodynamic part G of the initial datum. The solution exists as far as a smooth solution of the Navier-Stokes equations exists and the asymptotic relationship (7.7) is satisfied.
In addition, the solution is unique if the solution of the N-SE is unique and is continuously differentiable in $Y_+^{\gamma,k-2}$.
By Theorem 7.2 the corresponding theorems for the Enskog equation and the Povzner equation can be formulated (Remark 7.3).

The paper [9] suggests that the following theorem can be proved: the local Maxwellian M_{NS} is well approximated by a solution of an equation describing the dynamics of a system of spheres interacting through elastic collisions with a stochastic distance of interaction.

The expansion used in this paper is obtained from the Hilbert expansion by a rearrangement of terms and thus its nature is different from the Chapman-Enskog procedure ([1, 2, 4, 6, 11]). The detailed discussion can be found in Ref. [1]. The idea of rearrangement of terms leads to a conclusion that a system different from that of Navier-Stokes can be used to establish hydrodynamic approximation of the kinetic equations. Using the expansion equations

$$2J_0(f_0, g_1') = P^{\perp}Df_0 + P^{\perp}\mathscr{F}[f_0] \tag{8.1a}$$

and

$$2J_0(f_0, g_1'') = \varepsilon P^{\perp}Dh_1 - P^{\perp}\mathscr{F}[f_0] \tag{8.1b}$$

instead of Eqs (5.3a) and (5.3b) the asymptotic theory can be formulated starting from the following "hydrodynamic" equation rather than (5.6):

$$PDf_0 = -\varepsilon PD\mathscr{L}^{-1}P^{\perp}(Df_0 + \mathscr{F}[f_0]). \tag{8.2}$$

For example, the case of $\mathscr{F}[f_0] = -Df_0$ leads to the Euler system approximation. The similar effect of "nonuniqueness" of the Navier-Stokes hydrodynamic approximation has been described by Ellis and Pinsky [3] in the linearized case.

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