J. Math. Kyoto Univ. (JMKYAZ)
32-2 (1992) 373-386

Hypoellipticity for some infinitely degenerate elliptic operators of second order

By

Tatsushi MORIOKA

§ 1. Introduction and results

We are mainly concerned with the hypoellipticity of degenerate elliptic operators in $\boldsymbol{R}^3$ of the form

$$L=D_t^2+f(t)D_x^2+g(x)D_y^2 \tag{1.1}$$

satisfying

$$f(0)=g(0)=0, \quad f(t)>0, \quad g(x)>0 \qquad \text{for} \quad t\neq 0, \ x\neq 0, \text{ respectively}. \tag{1.2}$$

Throughout this paper, the coefficients of differential operators are assumed to be functions of the class C^∞.

Before the statement of results, let us explain our motivation. Concerning the following operator

$$L_1=D_t^2+D_x^2+g(x)D_y^2,$$

where we assume $xg'(x)\geqq 0$ in addition to (1.2), Kusuoka and Strook [4] have shown that L_1 is hypoelliptic if and only if

$$\lim_{x\to 0}|x \log g(x)|=0. \tag{1.3}$$

We remark that (1.3) allows the infinite degeneracy of $g(x)$ at $x=0$. For example,

$$L_2=D_t^2+D_x^2+e^{-|x|^{-\sigma}}D_y^2$$

is hypoelliptic if and only if $\sigma<1$. As a generalization of L_2, Morimoto [6] has considered the following operator

$$L_3=D_t^2+t^{2k}D_x^2+e^{-|x|^{-\sigma}}D_y^2,$$

where k is a non-negative integer. In [6] it was proved that L_3 is hypoelliptic if $\sigma<1/(k+1)$. See also page 2 of [2]. However in case of $k\geqq 1$, we can not see that $\sigma<1/(k+1)$ is necessary for L_3 to be hypoelliptic. The above result concerning L_3 comes from the following fact. Let P be a differential operator of the form

$$P=\sum_{k,l=1}^{d} a_{kl}(x)D_{x_k}D_{x_l} \qquad \text{in} \quad \boldsymbol{R}^d,$$

Communicated by Prof. N. Iwasaki, April 11, 1990

where a matrix (a_{kl}) is non-negative definite for each $x \in \boldsymbol{R}^d$. Morimoto [6] has shown that P is hypoelliptic in Ω (open subset of $\boldsymbol{R}^d$) if P satisfies the following estimate: for any $\varepsilon>0$ and for any compact subset K of Ω there exists a constant C such that

$$\|(\log \Lambda)u\|^2 \leqq \varepsilon Re(Pu, u)+C\|u\|^2 \quad \text{for} \quad u \in C_0^\infty(K). \tag{1.4}$$

Also [6] has shown that (1.4) is necessary for P to be hypoelliptic if P can be written in the following form

$$P=D_{x_1}^2+b(x', D_{x'}) \quad \text{in} \quad \boldsymbol{R}^d, \tag{1.5}$$

where $b(x', D_{x'})$ is a formally selfadjoint operator of second order satisfying

$$(b(x', D_{x'})v, v) \geqq -\text{const.}\|v\|^2 \quad \text{for} \quad v \in C_0^\infty(\boldsymbol{R}^{d-1}). \tag{1.6}$$

L_3 satisfies (1.4) if and only if $\sigma<1/(k+1)$. See Proposition 4 in [6]. In case of $k=0$, L_3 can be written in the form (1.5) and satisfies (1.6). So we see that $\sigma<1$ is necessary for L_3 to be hypoelliptic. However in case of $k \geqq 1$, L_3 can not be written in the form (1.5). Therefore, our information about the hypoellipticity of L_3 is not complete. So we consider the operator (1.1) generalizing L_3. Our first result is the following theorem.

Theorem 1. *Let L be the operator* (1.1) *satisfying* (1.2). *Assume moreover that*

$$\lim_{t \to 0} |t \log f(t)|=0, \tag{1.7}$$

$$\lim_{x \to 0} |x \log g(x)|=0. \tag{1.8}$$

Then L is hypoelliptic.

Theorem 1 shows that L_3 is hypoelliptic if $\sigma<1$ for any positive integer k. If we assume $tf'(t) \geqq 0$ in addition to (1.2), (1.7) is necessary for L to be hypoelliptic. Indeed, let P be an operator which we obtain from L by the change of variables $x=\varphi(z)$, where φ satisfies $\varphi'(z)=g(\varphi(z))$ and $\varphi(0)=1$. Applying Theorem 3 in [6], we see that P satisfies (1.4) for $\Omega=\{(t, z, y): z>0\}$. The condition (1.7) follows from (1.4).

Let us continue our argument. Hoshiro [3] has considered the following operator

$$L=D_t^2+f(t)D_x^2+g(t)D_y^2, \tag{1.9}$$

where $f(0)=g(0)=0$ and $f(t), g(t)>0$ for $t \neq 0$. In [3] it was shown that the hypoellipticity of L is decided by the combination of the vanishing order of $f(t)$ and $g(t)$ at $t=0$. See Theorem 1 and 2 in [3]. So we next consider

$$L=D_t^2+f(t)D_x^2+g(t, x)D_y^2 \tag{1.10}$$

generalizing (1.1) and (1.9). We assume that $g(t, x) \geqq 0$ and

$$f(0)=0, \quad f(t)>0 \quad \text{for} \quad t \neq 0. \tag{1.11}$$

Our second result is the following theorem.

Theorem 2. *Let L be the operator* (1.10) *satisfying* (1.11) *and the following con-*

ditions.

(A.1) *There exists a function* $G(t)$ *such that* $g(t, x) \leqq G(t)$, $tG'(t) \geqq 0$ *and*

$$\lim_{t \to 0} \sqrt{G(t)} \, |t \log f(t)| = 0 .$$

(A.2) L *is hypoelliptic in* $\{(t, x, y) \in \boldsymbol{R}^3 : (t, x) \neq (0, 0)\}$.

Then L *is hypoelliptic.*

Theorem 2 has the following corollary, which yields Theorem 2 in Hoshiro [3].

Corollary 3. *Let* L *be an operator of the form*

$$L = D_t^2 + f(t) D_x^2 + g(t) h(x) D_y^2$$

satisfying

$$\begin{cases} f(0) = g(0) = h(0) = 0 \\ f(t), \ g(t), \ h(x) > 0 \quad \text{for} \quad t \neq 0, \quad x \neq 0 \\ t f'(t), \ t g'(t) \geqq 0 \end{cases} \tag{1.12}$$

and

$$\begin{cases} \lim_{t \to 0} \sqrt{g(t)} \, |t \log f(t)| = 0 \\ \lim_{t \to 0} \sqrt{f(t)} \, |t \log g(t)| = 0 \\ \lim_{x \to 0} |x \log h(x)| = 0 . \end{cases} \tag{1.13}$$

Then L *is hypoelliptic.*

Proof of Corollary 3. In view of Theorem 2, it suffices to show that (A.1) and (A.2) are satisfied. (A.1) follows from (1.13). We see that (A.2) is also satisfied by Theorem 3 in [3] and Corollary 2 in [6]. Q.E.D.

Let us consider the following examples.

Example 1. Let σ, δ be constants. Theorem 1 shows that

$$L = D_t^2 + e^{-|t|^{-\sigma}} D_x^2 + e^{-|x|^{-\delta}} D_y^2$$

is hypoelliptic if $\sigma < 1$ and $\delta < 1$. In case of $\sigma \geqq 1$, L is not hypoelliptic for any $\delta > 0$.

Example 2. Let σ and δ be constants and k be a positive integer. Corollary 3 shows that

$$L = D_t^2 + t^{2k} D_x^2 + e^{-|t|^{-\sigma} - |x|^{-\delta}} D_y^2$$

is hypoelliptic if $\sigma < k+1$ and $\delta < 1$. If $\sigma \geqq k+1$ and $\delta = 0$, L is not hypoelliptic. Cf. Example 1 in [3].

Example 3. Let ω, σ and δ be positive constants. Corollary 3 shows that

$$L = D_t^2 + e^{-|t|^{-\omega}} D_x^2 + e^{-|t|^{-\sigma} - |x|^{-\delta}} D_y^2$$

is hypoelliptic if $\delta < 1$.

The plan of this paper is as follows. In section 2, we prepare basic facts to prove Theorem 1 and 2. In section 3 and 4, we explain our microlocal energy method and complete the proof of Theorem 1. The proof of Theorem 2 will be given in section 5. Finally in section 6, we prove the propositions in section 3.

Acknowledgements. The author would like to express his gratitude to Professors N. Iwasaki and Y. Morimoto for helpful encouragements.

§ 2. Preliminaries

We begin this section by preparing the following Sobolev spaces which are necessary for the proof of Theorem 1 and 2.

Definition. We denote by $H^{k,l,j}$ $(-\infty<k, l, j<\infty)$ the space of all distributions $u\in\mathcal{S}'(\boldsymbol{R}^3)$ satisfying

$$\iiint|\hat{u}(\tau, \xi, \eta)|^2\langle\tau\rangle^{2k}\langle\xi\rangle^{2l}\langle\eta\rangle^{2j}d\tau d\xi d\eta<\infty\,,$$

where $\hat{u}$ is Fourier transform of u and $\langle\tau\rangle=(1+\tau^2)^{1/2}$. Moreover, $v\in H^{k,l,\infty}$ means that $v\in H^{k,l,j}$ for any j.

Remark. $H^{k,l,j}$ is a Hilbert space with the following inner product:

$$(u, v)_{H^{k,l,j}}=\iiint\hat{u}(\tau, \xi, \eta)\overline{\hat{v}(\tau, \xi, \eta)}\langle\tau\rangle^{2k}\langle\xi\rangle^{2l}\langle\eta\rangle^{2j}d\tau d\xi d\eta\,.$$

We say that a distribution u is locally of the class $H^{k,l,j}$ at (t_0, x_0, y_0) if there exists a function $\phi\in C_0^\infty$ with $\phi=1$ in a neighborhood of (t_0, x_0, y_0) such that $\phi u\in H^{k,l,j}$.

If u is a distribution and $(t_0, x_0, y_0)\in\boldsymbol{R}^3$, there exists three real numbers (k, l, j) such that $u\in H^{k,l,j}$ at (t_0, x_0, y_0). Let L be the operator stated in Theorem 1. If $u\in H^{k,l,j}$ and $Lu\in C^\infty$ at (t_0, x_0, y_0), we have $u\in H^{k+2,l-2,j-2}$ at (t_0, x_0, y_0). In case of $t_0=x_0=0$, the above fact is shown in the following way. Let $\phi_1\in C_0^\infty$ be a function with $\phi_1=1$ in a neighborhood of $(0, 0, y_0)$ such that $\phi_1 u\in H^{k,l,j}$ and $\phi_1 Lu\in C_0^\infty$. Choose $\phi(t, x, y)=\chi(t)\psi(x, y)$ so that $\phi\Subset\phi_1$, i.e., $\phi_1=1$ in a neighborhood of the support of ϕ. Here χ and ψ are equal to 1 in neighborhoods of $t=0$ and $(x, y)=(0, y_0)$, respectively. Then the right hand side of the equation

$$D_t^2(\phi u)=[D_t^2, \chi(t)]\psi u+\phi Lu-\phi(f(t)D_x^2+g(x)D_y^2)u$$

is of the class $H^{k,l-2,j-2}$. In fact, the second and third terms belong to C_0^∞ and $H^{k,l-2,j-2}$, respectively. The first one is of the class C_0^∞ because of the hypoellipticity of L in $\{t\neq0\}$. Hence we see that $\phi u\in H^{k+2,l-2,j-2}$. Repeating these arguments, we see that u is locally of the class $H^{k+2d,l-2d,j-2d}$ for any positive integer d.

In the case where $Lu\in C^\infty$ at $(0, 0, y_0)$, the partial Fourier transform of u at $(0, 0, y_0)$, i.e.,

$$\chi(t)(\psi u)^\wedge(t\,;\, \xi, \eta)=(2\pi)^{-1}\int e^{it\tau}\widehat{\chi\psi u}(\tau, \xi, \eta)d\tau$$

is of the class $C_0^\infty(-1, 1)$ with respect to t for almost every (ξ, η).

Remark. The hypoellipticity of L in $\{(t, x)\neq(0, 0)\}$ has been shown in [6], under the assumptions (1.7) and (1.8). So we may assume $t_0=x_0=0$.

Now the proof of Theorem 1 is reduced to the following proposition.

Proposition 2.1. *Let L be the operator* (1.1) *satisfying* (1.2), (1.7) *and* (1.8). *Assume that $Lu\in C^\infty$ at* $(0, 0, y_0)$. *Then we have the following two claims.*

(i) *If $u\in H^{0,l,j}$ at $(0, 0, y_0)$, then $u\in H^{0,l+1/2,j-2}$ at $(0, 0, y_0)$.*

(ii) *If $u\in H^{0,0,j}$ at $(0, 0, y_0)$, then $u\in H^{0,0,\infty}$ at $(0, 0, y_0)$.*

From the above proposition and the previous arguments, we have $u\in H^{k,l,j}$ at $(0, 0, y_0)$ for any (k, l, j) if $Lu\in C^\infty$ at $(0, 0, y_0)$. Hence $u\in C^\infty$ at $(0, 0, y_0)$. The proof of Proposition 2.1 will be given in section 3 and 4.

§ 3. Microlocal energy method (1)

In this section, we prove Proposition 2.1-(i). Let us prepare the microlocal energy of distributions used in section 4 of [3], after some refinements. The use of the method here is slightly different from that in [3]. Choose first a sequence $\Psi_N\in C_0^\infty(\boldsymbol{R})$ $(N=1, 2, \cdots)$ with $\Psi_N=1$ in $\{x: |x|\leqq r'\}$ and $\Psi_N=0$ in $\{x: |x|\geqq r\}$ $(0<r'<r<1)$ satisfying

$$|D^{\mu+\nu}\Psi_N|\leqq C_\nu(CN)^\mu \qquad \text{for} \quad \mu\leqq N,$$

where C_ν and C are constants independent of N. Our microlocalizers $\{\alpha_n(\xi), \beta_n(x, y)\}$ are defined in such a way that

$$\alpha_n(\xi)=\Psi_{N_n}\left(\frac{\xi}{n}-1\right)+\Psi_{N_n}\left(\frac{\xi}{n}+1\right), \qquad \beta_n(x, y)=\Psi_{N_n}(x)\Psi_{N_n}(y-y_0),$$

where $N_n=[\log n]+1$. Our microlocal energy is

$$S_{n,m}^M u=\sum_{p+|q|\leq N_n}\|c_{pq}^n\alpha_n^{(p)}(D_x)(\beta_{n(q)}u)\|_{H^{0,0,m}}^2, \qquad u\in\mathcal{S}'(\boldsymbol{R}^3)$$

with $c_{pq}^n=n^p(M\log n)^{-p-|q|}$. Here $\alpha_n^{(p)}=\partial_\xi^p\alpha_n$ and $\beta_{n(q)}=D_x^{q_1}D_y^{q_2}\beta_n$ with $q=(q_1, q_2)$. We have now the following propositions whose proofs will be given in section 6.

Proposition 3.1. *Let $u\in H^{0,l,j}$ at $(0, 0, y_0)$ for some l, j. Then there exists a function $\chi(t)\in C_0^\infty$ with $\chi=1$ in a neighborhood of 0 and a positive constant M_0 such that*

$$S_{n,m}^M(\chi u)=O(n^{-2l}) \qquad for \quad m\leqq j, \quad M\geqq M_0.$$

Proposition 3.2. *Let $u\in H^{0,l-1,m}$ at $(0, 0, y_0)$. If there exists a function $\chi(t)\in C_0^\infty$ with $\chi=1$ in a neighborhood of 0 and a positive consant M such that*

$$S_{n,m}^M(\chi u)=O(n^{-2l}),$$

then for any $l'<l$ we have $u\in H^{0,l',m}$ at $(0, 0, y_0)$.

Remark. In general, $\phi_n=O(n^{-k})$ means that there exists a constant B such that $|\phi_n|\leqq Bn^{-k}$, when n is large.

Roughly speaking, these two propositions imply that the decreasing (or increasing) order of the microlocal energy $S_{n,m}^{M}(\chi u)$ indicates the regularity of u with respect to x-variable at $(0, 0, y_0)$.

Let us now begin the proof of Proposition 2.1-(i). The hypoellipticity of L in $\{t\neq 0\}$ enables us to know that the right hand side of the equation

$$\phi L(\chi u)=\phi[D_t^2, \chi(t)]u+\chi\phi Lu$$

is of the class C_0^∞ if $Lu\in C^\infty$ at $(0, 0, y_0)$, where $\chi(t)\in C_0^\infty$ and $\phi(x, y)\in C_0^\infty$ have their supports in small neighborhoods of $t=0$ and $(x, y)=(0, y_0)$, respectively. So we have

$$\phi Lv=h\,, \tag{3.1}$$

where $v=\chi u$ and $h\in C_0^\infty$.

Assume that $p+|q|\leqq N_n$ and $r>0$ is chosen sufficiently small so that $\beta_n\in\phi$. Let us operate $\alpha_n^{(p)}\beta_{n(q)}$ to the both sides of (3.1), namely,

$$\alpha_n^{(p)}\beta_{n(q)}Lv=\alpha_n^{(p)}\beta_{n(q)}h\,. \tag{3.2}$$

The asymptotic expansion gives

$$(Lv_{n,p,q}, v_{n,p,q})_m=\sum_{k=1}^{5} b_k \tag{3.3}$$

with

$$b_1=-\sum_{\nu=1,2}(-1)^\nu(\nu!)^{-1}(L_0^{(\nu)}v_{n,p,q+(\nu,0)}, v_{n,p,q})_m\,,$$

$$b_2=2(g(x)D_yv_{n,p,q+(0,1)}, v_{n,p,q})_m\,,$$

$$b_3=-(g(x)v_{n,p,q+(0,2)}, v_{n,p,q})_m\,,$$

$$b_4=-(r_{n,p}(\beta_{n(q)}D_y^2v), v_{n,p,q})_m\,,$$

$$b_5=(h_{n,p,q}, v_{n,p,q})_m\,,$$

where

$$v_{n,p,q}=\alpha_n^{(p)}\beta_{n(q)}v,\qquad h_{n,p,q}=\alpha_n^{(p)}\beta_{n(q)}h\,,\qquad L_0=D_t^2+f(t)D_x^2,$$

$$L_0^{(\nu)}=2f(t)D_x^{2-\nu},\qquad r_{n,p}=[\alpha_n^{(p)}(D_x), g(x)],$$

$$(a, b)_m=(a, b)_{H^{0,0,m}}\,,\qquad m=j-2\,.$$

We are going to estimate the each terms on the right hand side of (3.3).

Lemma 3.3. *For any $\varepsilon>0$ there exists a constant N_0 which is independent of (p, q) such that*

$$(\log n)^2\|v_{n,p,q}\|_m^2\leqq\varepsilon(L_0v_{n,p,q}, v_{n,p,q})_m \qquad \textit{for}\quad n\geqq N_0,$$

where $\|*\|_m$ means $\|*\|_{H^{0,0,m}}$.

Proof of Lemma 3.3. From (1.7), it follows that for any $\varepsilon>0$ there exists a constant N_0 such that

$$(\log|\xi|)^2\int|\varphi(t)|^2dt \leqq \varepsilon\Big(\int|\varphi'(t)|^2dt+\xi^2\int f(t)|\varphi(t)|^2dt\Big) \tag{3.4}$$

for $\varphi\in C_0^\infty(-1, 1)$, $|\xi|\geqq N_0$. See Proposition 3.1 in [3]. Taking

$$\varphi(t)=\hat{v_{n,p,q}}(t;\ \xi,\ \eta)=\alpha_n^{(p)}(\xi)(\beta_{n(q)}v)\hat{}(t;\ \xi,\ \eta),$$

multiplying the both sides of (3.4) by $\langle\eta\rangle^{2m}$ and integrating with respect to (ξ, η), we have the desired estimate. Q.E.D.

For a convenience, we assume that $g(x)\leqq 1$. Let K be an arbitrary positive constant. Then we have

$$\begin{aligned} |b_2| &= 2|(g(x)D_yv_{n,p,q+(0,1)},\ v_{n,p,q})_m| \\ &\leqq K^{-1}(\log n)^{-2}(g(x)D_yv_{n,p,q+(0,1)},\ D_yv_{n,p,q+(0,1)})_m+K(\log n)^2\|v_{n,p,q}\|_m^2 \\ &\leqq K^{-1}(\log n)^{-2}(Lv_{n,p,q+(0,1)},\ v_{n,p,q+(0,1)})_m+\frac{1}{8}(Lv_{n,p,q},\ v_{n,p,q})_m, \end{aligned} \tag{3.5}$$

$$\begin{aligned} |b_3| &= |(g(x)v_{n,p,q+(0,2)},\ v_{n,p,q})_m| \\ &\leqq K^{-1}(\log n)^{-2}\|v_{n,p,q+(0,2)}\|_m^2+K(\log n)^2\|v_{n,p,q}\|_m^2 \\ &\leqq K^{-1}(\log n)^{-4}(Lv_{n,p,q+(0,2)},\ v_{n,p,q+(0,2)})_m+\frac{1}{8}(Lv_{n,p,q},\ v_{n,p,q})_m. \end{aligned} \tag{3.6}$$

From the Parseval's formula we have

$$\begin{aligned} &|(L_0^{(\nu)}v_{n,p,q+(\nu,0)},\ v_{n,p,q})_m| \\ &\leqq 2\iiint f(t)|\xi|^{2-\nu}|\hat{v_{n,p,q+(\nu,0)}}(t;\ \xi,\ \eta)\overline{\hat{v_{n,p,q}}(t;\ \xi,\ \eta)}|\langle\eta\rangle^{2m}dtd\xi d\eta \\ &\leqq (1-r)^{-\nu}n^{-\nu}\Big\{\iiint f(t)\xi^2|\hat{v_{n,p,q+(\nu,0)}}(t;\ \xi,\ \eta)|^2\langle\eta\rangle^{2m}dtd\xi d\eta \\ &\quad+\iiint f(t)\xi^2|\hat{v_{n,p,q}}(t;\ \xi,\ \eta)|^2\langle\eta\rangle^{2m}dtd\xi d\eta\Big\} \\ &\leqq K^{-1}(\log n)^{-2\nu}(Lv_{n,p,q+(\nu,0)},\ v_{n,p,q+(\nu,0)})_m+\frac{1}{8}(Lv_{n,p,q},\ v_{n,p,q})_m. \end{aligned}$$

Therefore,

$$\begin{aligned} |b_1| \leqq &\sum_{\nu=1,2}K^{-1}(\log n)^{-2\nu}(Lv_{n,p,q+(\nu,0)},\ v_{n,p,q+(\nu,0)})_m \\ &+\frac{1}{4}(Lv_{n,p,q},\ v_{n,p,q})_m. \end{aligned} \tag{3.7}$$

We estimate b_4 and b_5 in the following way:

$$|b_4|\leqq K\|r_{n,p}(\beta_{n(q)}D_y^2v)\|_m^2+K^{-1}\|v_{n,p,q}\|_m^2, \tag{3.8}$$

$$|b_5|\leqq K\|h_{n,p,q}\|_m^2+K^{-1}\|v_{n,p,q}\|_m^2. \tag{3.9}$$

From (3.5)-(3.9), we have

(3.10)
$$\frac{1}{2}(Lv_{n,p,q}, v_{n,p,q})_m$$
$$\leqq K^{-1}\sum_{|\nu|=1,2}(\log n)^{-2|\nu|}(Lv_{n,p,q+\nu}, v_{n,p,q+\nu})_m+K\|h_{n,p,q}\|_m^2$$
$$+2K^{-1}\|v_{n,p,q}\|_m^2+K\|r_{n,p}(\beta_{n(q)}D_y^2v)\|_m^2.$$

Let us now observe that $c_{pq}^n(\log n)^{-|\nu|}=M^{|\nu|}c_{p,q+\nu}^n$. From (3.10) we see that

(3.11)
$$\frac{1}{2}(Lw_{n,p,q}, w_{n,p,q})_m$$
$$\leqq K^{-1}M^4\sum_{|\nu|=1,2}(Lw_{n,p,q+\nu}, w_{n,p,q+\nu})_m+K\|c_{pq}^nh_{n,p,q}\|_m^2$$
$$+2K^{-1}\|w_{n,p,q}\|_m^2+K\|c_{pq}^nr_{n,p}(\beta_{n(q)}D_y^2v)\|_m^2,$$

where $w_{n,p,q}=c_{pq}^nv_{n,p,q}$.

Lemma 3.4. *If we choose M sufficiently large, then*

$$\sum_{p+|q|\leqq N_n}\|c_{pq}^nr_{n,p}(\beta_{n(q)}D_y^2v)\|_m^2=O(n^{-2l-2}).$$

Proof of Lemma 3.4. Writing the symbol by the oscillatory integral together with the fact that $(1-r)n\leqq|\xi|\leqq(1+r)n$ for $\xi\in\operatorname{supp}\alpha_n$, we see that

$$\|r_{n,p}\|_{H^{0,l,m}\to H^{0,0,m}}\leqq\text{const.}\,|r_{n,p}|_d^{(l)}$$
$$\leqq\text{const.}\,|\alpha_n^{(p+1)}|_{d_1}^{(l)}\,|g_{(1)}(x)|_{d_2}^{(0)}$$
$$\leqq\text{const.}\,(CN_n)^pn^{-p-l-1},$$

where $|a|_d^{(l)}$ denotes the seminorm in $S_{1,0}^l$, i.e.,

$$|a|_d^{(l)}=\max_{\mu+\nu\leqq d}\sup|a_{(\mu)}^{(\nu)}(x,\xi)|\langle\xi\rangle^{-l+\nu}.$$

See also page 58 of [5]. Therefore,

$$\|r_{n,p}(\beta_{n(q)}D_y^2v)\|_m\leqq\|r_{n,p}\|_{H^{0,l,m}\to H^{0,0,m}}\|\beta_{n(q)}\phi D_y^2v\|_{H^{0,l,m}}$$
$$\leqq\text{const.}\,|r_{n,p}|_d^{(l)}\,|\beta_{n(q)}|_{d_3}^{(0)}\|\phi D_y^2v\|_{H^{0,l,m}}$$
$$\leqq\text{const.}(CN_n)^{p+|q|}n^{-p-l-1}.$$

Thus we have

$$\|c_{pq}^nr_{n,p}(\beta_{n(q)}D_y^2v)\|_m\leqq\text{const.}(2CM^{-1})^{p+|q|}n^{-l-1}.$$

In case of $2CM^{-1}<1$, we have

$$\sum_{p+|q|\leqq N_n}\|c_{pq}^nr_{n,p}(\beta_{n(q)}D_y^2v)\|_m^2\leqq\text{const.}\ n^{-2l-2}\sum_{p,q}(2CM^{-1})^{2p+2|q|}=O(n^{-2l-2}).$$

Q.E.D.

Let us sum up the both sides of (3.11) with respect to (p, q) satisfying $p+|q|\leqq N_n-2$. From (3.11) and Lemma 3.4 we see that

(3.12) $$\frac{1}{2}W_n^M \leqq K^{-1}M^4 W_n^M + KS_{n,m}^M h + 2K^{-1}S_{n,m}^M v + O(n^{-2l-2}),$$

where

$$W_n^M = \sum_{p+|q|\leqq N_n} (Lw_{n,p,q}, w_{n,p,q})_m.$$

To establish (3.12), we used

$$\sum_{N_n-2\leqq p+|q|\leqq N_n} (Lw_{n,p,q}, w_{n,p,q})_m = O(n^{-2l-2})$$

for sufficiently large M. Cf. Lemma 1 in [2]. The first term on the right hand side of (3.12) is absorbed into the left hand side by taking K sufficiently large. Since $h \in C_0^\infty$, $S_{n,m}^M h = O(n^{-2l-2})$. Thus we have

(3.13) $$\frac{1}{4}W_n^M \leqq 2K^{-1}S_{n,m}^M v + O(n^{-2l-2}).$$

By Poincaré's inequality,

$$(Lw_{n,p,q}, w_{n,p,q})_m \geqq \|D_t w_{n,p,q}\|_m^2 \geqq \delta \|w_{n,p,q}\|_m^2$$

holds for some constant $\delta > 0$. Therefore, $W_n^M \geqq \delta S_{n,m}^M v$. By taking K sufficiently large, we obtain from (3.13)

$$S_{n,m}^M v = O(n^{-2l-2}).$$

In view of Proposition 3.2, the proof is completed.

§ 4. Microlocal energy method (2)

In this section we prove Proposition 2.1-(ii). Here we rely on the microlocal energy method again. But our argument in this section is not so delicate as previous one. Indeed, the microlocal energy prepared here is quite simple. Let $\Psi \in C_0^\infty(\boldsymbol{R})$ be a function satisfying $\Psi = 1$ in $\{|x| \leqq r'\}$ and $\Psi = 0$ in $\{|x| \geqq r\}$, where we assume that $0 < r' < r < 1$. Our microlocalizers $\{\alpha_n(\eta), \beta(y)\}$ are defined in such a way that

$$\alpha_n(\eta) = \Psi\left(\frac{\eta}{n} - 1\right) + \Psi\left(\frac{\eta}{n} + 1\right), \qquad \beta(y) = \Psi(y - y_0).$$

Our microlocal energy is

$$S_{N,n}u = \sum_{p+q \leqq N} \|c_{pq}^n \alpha_n^{(p)}(D_y)(\beta_{(q)}u)\|_{L_2}^2, \qquad u \in \mathcal{S}'(\boldsymbol{R}^3)$$

with $c_{pq}^n = n^{(1/2)(p-q)}$. We have now the following proposition.

Proposition 4.1. *Let $u \in H^{0,0,j}$ at $(0, 0, y_0)$ for some j and N be an arbitrary integer. Then $u \in H^{0,0,\infty}$ at $(0, 0, y_0)$ if and only if there exists a function $\chi(t, x) \in C_0^\infty$ with $\chi = 1$ in a neighborhood of $(t, x) = (0, 0)$ such that $S_{N,n}(\chi u)$ is rapidly decreasing as $n \to \infty$, i.e.,*

$$S_{N,n}(\chi u) = O(n^{-2s})$$

for any $s > 0$.

The proof of Proposition 4.1 can be given by the same argument as in section 6. See also page 111 of [5].

Let us now begin the proof of Proposition 2.1-(ii). The hypoellipticity of L in $\{(t, x)\neq(0, 0)\}$ enables us to know that the right hand side of the equation

$$\phi L(\chi_1\chi_2 u)=\phi[D_t^2, \chi_1]\chi_2 u+\phi f(t)[D_x^2, \chi_2]\chi_1 u+\chi_1\chi_2\phi L u$$

is of the class C_0^∞ if $Lu\in C^\infty$ at $(0, 0, y_0)$. Here $\chi_1(t)\in C_0^\infty$, $\chi_2(x)\in C_0^\infty$ and $\phi(y)\in C_0^\infty$ have their supports in small neighborhoods of $t=0$, $x=0$ and $y=y_0$, respectively. So we have

$$\phi L v=h\,, \tag{4.1}$$

where $v=\chi_1\chi_2 u$ and $h\in C_0^\infty$.

Assume that $r>0$ is chosen sufficiently small so that $\beta\Subset\phi$. Let us operate $\alpha_n^{(p)}\beta_{(q)}$ to the both sides of (4.1), namely,

$$\alpha_n^{(p)}\beta_{(q)}Lv=\alpha_n^{(p)}\beta_{(q)}h\,. \tag{4.2}$$

The asymptotic expansion gives

$$(Lv_{n,p,q}, v_{n,p,q})=-\sum_{\nu=1,2}(-1)^\nu(\nu!)^{-1}(L^{(\nu)}v_{n,p,q+\nu}, v_{n,p,q})+(h_{n,p,q}, v_{n,p,q})\,, \tag{4.3}$$

where $v_{n,p,q}=\alpha_n^{(p)}\beta_{(q)}v$, $h_{n,p,q}=\alpha_n^{(p)}\beta_{(q)}h$ and $L^{(\nu)}=2g(x)D_y^{2-\nu}$. Therefore,

$$(Lv_{n,p,q}, v_{n,p,q})\leqq\sum_{\nu=1,2}|(L^{(\nu)}v_{n,p,q+\nu}, v_{n,p,q})|+K\|h_{n,p,q}\|^2+K^{-1}\|v_{n,p,q}\|^2\,, \tag{4.4}$$

where K is an arbitrary positive constant.

We are going to estimate the first term on the right hand side of (4.4). From the Parseval's formula, we have

$$\begin{aligned}|(L^{(\nu)}v_{n,p,q+\nu}, v_{n,p,q})|&\leqq 2\iiint g(x)|\eta|^{2-\nu}|\hat{v}_{n,p,q+\nu}(t, x\,;\,\eta)\overline{\hat{v}_{n,p,q}(t, x\,;\,\eta)}|\,dtdxd\eta\\&\leqq 4\iiint g(x)|\eta|^{2-2\nu}|\hat{v}_{n,p,q+\nu}(t, x\,;\,\eta)|^2dtdxd\eta\\&\quad+\frac{1}{4}\iiint g(x)\eta^2|\hat{v}_{n,p,q}(t, x\,;\,\eta)|^2dtdxd\eta\,,\end{aligned}$$

where $\hat{v}_{n,p,q}(t, x\,;\,\eta)$ denotes the partial Fourier transform of $v_{n,p,q}$ with respect to y. Since $\hat{v}_{n,p,q}(t, x\,;\,\eta)=\alpha_n^{(p)}(\eta)(\beta_{(q)}v)^\wedge(t, x\,;\,\eta)$ and $(1-r)n\leqq|\eta|\leqq(1+r)n$ for $\eta\in\operatorname{supp}\alpha_n$, we have

$$|(L^{(\nu)}v_{n,p,q+\nu}, v_{n,p,q})|\leqq 4(1-r)^{-2\nu}n^{-2\nu}(Lv_{n,p,q+\nu}, v_{n,p,q+\nu})+\frac{1}{4}(Lv_{n,p,q}, v_{n,p,q})\,. \tag{4.5}$$

Therefore,

$$\begin{aligned}\frac{1}{2}(Lv_{n,p,q}, v_{n,p,q})\leqq 4(1-r)^{-4}&\sum_{\nu=1,2}n^{-2\nu}(Lv_{n,p,q+\nu}, v_{n,p,q+\nu})\\&+K\|h_{n,p,q}\|^2+K^{-1}\|v_{n,p,q}\|^2\,.\end{aligned} \tag{4.6}$$

Let us now observe that $c_{pq}^n n^{-(1/2)\nu}=c_{p,q+\nu}^n$. From (4.6) we see that

(4.7)
$$\frac{1}{2}(Lw_{n,p,q}, w_{n,p,q}) \leqq 4(1-r)^{-4}n^{-1}\sum_{\nu=1,2}(Lw_{n,p,q+\nu}, w_{n,p,q+\nu}) + K\|c_{pq}^{n}h_{n,p,q}\|^{2} + K^{-1}\|w_{n,p,q}\|^{2},$$

where $w_{n,p,q} = c_{pq}^{n}v_{n,p,q}$.

Now we use the following fact. For any $s>0$ there exists a large number N such that

(4.8)
$$\sum_{N-2\leq p+q\leq N}(Lw_{n,p,q}, w_{n,p,q}) = O(n^{-2s}).$$

Let $s>0$ be an arbitral number. We choose N so that (4.8) holds. Summing up the both sides of (4.7) with respect to (p, q) satisfying $p+q \leqq N-2$, we have

(4.9)
$$\frac{1}{2}W_{N,n} \leqq 4(1-r)^{-4}n^{-1}W_{N,n} + KS_{N,n}h + K^{-1}S_{N,n}v + O(n^{-2s}),$$

where
$$W_{N,n} = \sum_{p+q\leq N}(Lw_{n,p,q}, w_{n,p,q}).$$

The first term on the right hand side of (4.8) is absorbed into the left hand side. Since $h \in C_0^\infty$, we see that $S_{N,n}h = O(n^{-2s})$. Therefore,

(4.10)
$$\frac{1}{4}W_{N,n} \leqq K^{-1}S_{N,n}v + O(n^{-2s}).$$

By Poincaré's inequality, $W_{N,n} \geqq \delta S_{N,n}v$ holds for some constant $\delta>0$. Taking K sufficiently large, we obtain from (4.10) that

$$S_{N,n}v = O(n^{-2s}).$$

In view of Proposition 4.1, the proof is completed.

§ 5. Proof of Theorem 2

The proof of Theorem 2 is also reduced to Proposition 2.1. We can prove Proposition 2.1-(ii) in the same way as in section 4. Here we shall prove (i). Our argument in this section is quite analogous to that in section 3. So we need only slight modification. Recall the equation (3.3) and replace $g(x)$ with $g(t, x)$. Then we have

(5.1)
$$(Lv_{n,p,q}, v_{n,p,q})_m = \sum_{k=1}^{5} b_k$$

with

$$b_1 = -\sum_{\nu=1,2}(-1)^{\nu}(\nu!)^{-1}(L_0^{(\nu)}v_{n,p,q+(\nu,0)}, v_{n,p,q})_m,$$

$$b_2 = 2(g(t, x)D_y v_{n,p,q+(0,1)}, v_{n,p,q})_m,$$

$$b_3 = -(g(t, x)v_{n,p,q+(0,2)}, v_{n,p,q})_m,$$

$$b_4 = -(r_{n,p}(\beta_{n(q)}D_y^2 v), v_{n,p,q})_m,$$

$$b_5 = (h_{n,p,q}, v_{n,p,q})_m,$$

where

$$L=D_t^2+f(t)D_x^2+g(t,\ x)D_y^2, \qquad r_{n,p}=[\alpha_n^{(p)}(D_x),\ g(t,\ x)].$$

We may estimate b_1, b_4 and b_5 in the same way as in section 3. See (3.7)-(3.9). To estimate b_2 and b_3, we prepare the following lemma.

Lemma 5.1. *Let $G(t)$ be the function which we stated in Theorem 2. Then for any $\varepsilon>0$ there exists a constant N_0 which is independent of (p, q) such that*

$$(\log n)^2(G(t)v_{n,p,q},\ v_{n,p,q})_m\leqq\varepsilon(L_0v_{n,p,q},\ v_{n,p,q})_m \qquad \textit{for} \quad n\geqq N_0.$$

Proof of Lemma 5.1. From (A.1), it follows that for any $\varepsilon>0$ there exists a constant N_0 such that

$$(5.2) \qquad (\log|\xi|)^2\int G(t)|\varphi(t)|^2dt\leqq\varepsilon\Big(\int|\varphi'(t)|^2dt+\xi^2\int f(t)|\varphi(t)|^2dt\Big)$$

for $\varphi\in C_0^\infty(-1,\ 1)$, $|\xi|\geqq N_0$. See Proposition 3.1 in [3]. Taking $\varphi(t)=\hat{v}_{n,p,q}(t;\ \xi,\ \eta)$, we obtain the desired estimate. Q.E.D.

Thus we have

$$\begin{aligned}
|b_2|&=2|(g(t,\ x)D_yv_{n,p,q+(0,1)},\ v_{n,p,q})_m|\\
&\leqq K^{-1}(\log n)^{-2}(g(t,\ x)D_yv_{n,p,q+(0,1)},\ D_yv_{n,p,q+(0,1)})_m\\
&\quad+K(\log n)^2(g(t,\ x)v_{n,p,q},\ v_{n,p,q})_m\\
&\leqq K^{-1}(\log n)^{-2}(Lv_{n,p,q+(0,1)},\ v_{n,p,q+(0,1)})_m\\
&\quad+K(\log n)^2(G(t)v_{n,p,q},\ v_{n,p,q})_m\\
&\leqq K^{-1}(\log n)^{-2}(Lv_{n,p,q+(0,1)},\ v_{n,p,q+(0,1)})_m\\
&\quad+\frac{1}{8}(Lv_{n,p,q},\ v_{n,p,q})_m,\\
|b_3|&=|(g(t,\ x)v_{n,p,q+(0,2)},\ v_{n,p,q})_m|\\
&\leqq K^{-1}(\log n)^{-2}(G(t)v_{n,p,q+(0,2)},\ v_{n,p,q+0,2)})_m\\
&\quad+K(\log n)^2(G(t)v_{n,p,q},\ v_{n,p,q})_m\\
&\leqq K^{-1}(\log n)^{-4}(Lv_{n,p,q+(0,2)},\ v_{n,p,q+(0,2)})_m\\
&\quad+\frac{1}{8}(Lv_{n,p,q},\ v_{n,p,q})_m.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{1}{2}(Lv_{n,p,q},\ v_{n,p,q})_m&\leqq K^{-1}\sum_{|\nu|=1,2}(\log n)^{-2|\nu|}(Lv_{n,p,q+\nu},\ v_{n,p,q+\nu})_m\\
&\quad+K\|h_{n,p,q}\|_m^2+2K^{-1}\|v_{n,p,q}\|_m^2+K\|r_{n,p}(\beta_{n(q)}D_y^2v)\|_m^2.
\end{aligned}$$

By the same argument as in section 3, we obtain

$$S_{n,m}^M(v)=O(n^{-2l-2}).$$

Q.E.D.

§6. Proofs of Proposition 3.1 and 3.2

Here we give the proofs of Proposition 3.1 and 3.2.

Proof of Proposition 3.1. Let $\phi_1 \in C_0^\infty$ be a function with $\phi_1 = 1$ in a neighborhood of $(0, 0, y_0)$ such that $\phi_1 u \in H^{0,l,j}$ and choose $\phi(t, x, y) = \chi(t)\psi(x, y)$ so that $\phi \Subset \phi_1$. Here χ and ψ are equal to 1 in a neighborhoods of $t=0$ and $(x, y) = (0, y_0)$, respectively. Take $r>0$ sufficiently small so that $\beta_n \Subset \psi$. We assume $p+|q| \leqq N_n$ and $v = \chi u$. Then we have

$$\begin{aligned} n^l \|\alpha_n^{(p)}\beta_{n(q)}v\|_m &= \|n^l \alpha_n^{(p)}(\xi)(\beta_{n(q)}v)^\wedge(t;\ \xi, \eta)\langle\eta\rangle^m\| \\ &\leqq \text{const.} \|\alpha_n^{(p)}(\xi)(\beta_{n(q)}v)^\wedge(t;\ \xi, \eta)\langle\xi\rangle^l\langle\eta\rangle^m\| \\ &\leqq \text{const.}\, n^{-p}(CN_n)^p \|\beta_{n(q)}\psi v\|_{H^{0,l,m}} \\ &\leqq \text{const.}\, n^{-p}(CN_n)^{p+|q|} \|\psi v\|_{H^{0,l,m}} \\ &\leqq \text{const.}\, n^{-p}(CN_n)^{p+|q|}. \end{aligned}$$

Recall that $(1-r)n \leqq |\xi| \leqq (1+r)n$ for $\xi \in \operatorname{supp} \alpha_n$. Therefore,

$$n^l \|c_{pq}^n \alpha_n^{(p)} \beta_{n(q)} v\|_m \leqq \text{const.} (2CM^{-1})^{p+|q|}.$$

If $2CM^{-1} < 1$, we see that

$$n^{2l} S_{n,m}^M(\chi u) = n^{2l} \sum_{p+|q| \leqq N_n} \|c_{pq}^n \alpha_n^{(p)} \beta_{n(q)} v\|_m^2 \leqq \text{const.} \sum_{p,q} (2CM^{-1})^{2p+2|q|}.$$

Since $\sum_{p,q} (2CM^{-1})^{2p+2|q|} < \infty$, the proof is completed.

Proof of Proposition 3.2. Let $\psi(x, y) \in C_0^\infty$ be a function with $\psi = 1$ in a neighborhood of $(0, y_0)$ such that $\psi \Subset \beta_n$. Then we have

$$\|\alpha_n(\psi v)\|_m = \|\alpha_n \psi \beta_n v\|_m \leqq \|\psi \alpha_n \beta_n v\|_m + \|\omega(\beta_n v)\|_m,$$

where $v = \chi u$ and $\omega = [\alpha_n, \psi]$. By the same argument as in the proof of Lemma 3.4, we see that

$$\|\omega(\beta_n v)\|_m = O(n^{-l}).$$

Therefore,

$$\|\alpha_n \psi v\|_m \leqq \text{const.} \|\alpha_n \beta_n v\|_m + O(n^{-l}).$$

Since $S_{n,m}^M v = O(n^{-2l})$, we have

$$\|\alpha_n \beta_n v\|_m = O(n^{-l}).$$

Therefore,

$$\|\alpha_n \psi v\|_m = O(n^{-l}).$$

Let us observe that $\sum_{n=1}^\infty \alpha_n(\xi)^2 n^{2s-1} \geqq \text{const.} \langle\xi\rangle^{2s}$ for any $s \in \boldsymbol{R}$. This fact can be seen by noticing that the number of n such that $\alpha_n(\xi) = 1$ is estimated from below by const.$\langle\xi\rangle$. Combining the above arguments, we see that

$$\iiint |\widehat{\phi v}(\tau, \xi, \eta)|^2 \langle\xi\rangle^{2l'} \langle\eta\rangle^{2m} d\tau d\xi d\eta \leqq \text{const.} \sum_{n=1}^{\infty} \|\alpha_n(\phi v)\|_m^2 n^{2l'-1} < \infty$$

if $l' < l$. Q.E.D.

Department of Mathematics
Osaka University

References

[1] V.S. Fedii, On a criterion for hypoellipticity, Math USSR Sb., **14** (1971), 15-45.
[2] T. Hoshiro, Microlocal energy method of Mizohata and hypoellipticity, J. Math. Kyoto Univ., **28** (1988), 1-12.
[3] T. Hoshiro, Hypoellipticity for infinitely degenerate elliptic and parabolic operators of second order, J. Math. Kyoto Univ., **28** (1988), 615-632.
[4] S. Kusuoka and C. Strook, Applications of the Malliavin calculus, part II, J. Fac. Sci. Univ. Tokyo Sect. IA. Math., **32** (1985), 1-76.
[5] S. Mizohata, On the Cauchy Problem, Academic Press, 1986.
[6] Y. Morimoto, A criterion for hypoellipticity of second order differential operators, Osaka J. Math., **24** (1987), 651-675.