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The approximation of the Schrödinger operators with penetrable wall potentials in terms of short range Hamiltonians

Dedicated to Professor Teruo Ikebe on his sixtieth birthday

By

Shin-ichi SHIMADA

§ 1. Introduction, Results

In our previous paper (Ikebe-Shimada [2]), we considered the Schrödinger operator with a penetrable wall potential in $\mathbf{R}^3$ formally given by

$$H_{\text{formal}} = -\Delta + q(x)\delta(|x| - a),$$

where $q(x)$ is real and continuous on $S_a = \{x \in \mathbf{R}^3; |x| = a\}$ $(a > 0)$ and δ denotes the one-dimensional delta function. As a rigorous selfadjoint realization of the formal expression H_{formal}, we adopted the selfadjoint operator H which is uniquely determined by the quadratic form h (which is to be associated with H_{formal})

$$h[u, v] = (\nabla u, \nabla v) + (q\gamma_a u, \gamma_a v)_{L_2(S_a)} \qquad (= (H_{\text{formal}} u, v)),$$

$$\text{Dom}[h] = H^1(\mathbf{R}^3)$$

(I-S[2, Theorem 1.4]). Here γ_a is the trace operator from $H^1(\mathbf{R}^3)$ to $L_2(S_a)$, Dom$[h]$ denotes the form domain of h, $(\ , \ )$ means the $L_2(\mathbf{R}^3)$ inner product, $(\ , \ )_{L_2(S_a)}$ the $L_2(S_a)$ inner product, and $H^m(G)$ the Sobolev space of order m over G. If $G = \mathbf{R}^3$, we regard $H^m(\mathbf{R}^3)$ as the Hilbert space with the inner product $(\ , \ )_{H^m}$ defined by

$$(u, v)_{H^m} = \int_{\mathbf{R}^3} (1 + |\xi|^2)^m (\mathscr{F}u)(\xi)\overline{(\mathscr{F}v)(\xi)} d\xi,$$

where $\mathscr{F}$ is the ordinary Fourier transform defined by

$$(\mathscr{F}u)(\xi) = (2\pi)^{-3/2} \int_{\mathbf{R}^3} e^{-i\xi \cdot x} u(x) dx.$$

More precisely, it is seen that

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$$Hu = -\Delta u \qquad \text{for any } u \in \mathrm{Dom}(H),$$

$$\mathrm{Dom}(H) = \{u; u \in H^1(\mathbf{R}^3), u \in H^2(\{x; |x| < a\}), u \in H^2(\{x; |x| > a\}),$$

$$q(x)(\gamma_a u)(x) - \left\{\frac{\partial u}{\partial n_+}(x) + \frac{\partial u}{\partial n_-}(x)\right\}\Big|_{S_a} = 0\},$$

where n_+ (n_-) denotes the outward (inward) normal to S_a.

In this paper, we shall show how to approximate H by short range Hamiltonians $H_\varepsilon = -\Delta + Q_\varepsilon$ in the norm resolvent sense (convergence of the resolvent with the uniform operator topology), where the potential $Q_\varepsilon(x)$ converges to $q(x)\delta(|x| - a)$ as $\varepsilon \downarrow 0$ in the distribution sense (see Theorem 1). Let us take $\rho(r)$ satisfying the following properties:

$$\text{(1.1)} \quad \begin{cases} \rho(r) \geq 0 \quad \text{for all } r \in \mathbf{R}, \quad \rho(r) \in C_0^\infty(\mathbf{R}), \quad \mathrm{supp}\, \rho \subset [-1, 1], \\ \displaystyle\int_{-\infty}^{+\infty} \rho(r) dr = 1, \end{cases}$$

where $C_0^\infty(G)$ is the set of all infinitely continuously differentiable functions with compact support in G and supp means support. Define $Q_\varepsilon(x)$ by

$$Q_\varepsilon(x) = \frac{1}{\varepsilon}\rho\left(\frac{|x| - a}{\varepsilon}\right)q(a\omega_x) \qquad \left(\omega_x = \frac{x}{|x|}\right).$$

Then we have the next theorem easily.

Theorem 1. *Let $q(x)\delta(|x| - a)$ be the distribution belonging to $\mathscr{E}'(\mathbf{R}^3)$ defined by*

$$\langle q\delta(|\cdot| - a), \varphi\rangle = \int_{S_a} q(x)\varphi(x)dS_x \qquad \text{for any } \varphi \in \mathscr{E}(\mathbf{R}^3).$$

Then $Q_\varepsilon(x) \to q(x)\delta(|x| - a)$ as $\varepsilon \downarrow 0$ in $\mathscr{E}'(\mathbf{R}^3)$, where dS_x denotes the measure induced on S_a by the Lebesgue measure dx, $\mathscr{E}(\mathbf{R}^3)$ the Fréchet space of C^∞-functions, and $\mathscr{E}'(\mathbf{R}^3)$ the dual space of $\mathscr{E}(\mathbf{R}^3)$ (cf. Schwartz [6, Chap. III]).

Let H_0 be the selfadjoint operator defined by $H_0 = -\Delta$, $\mathrm{Dom}(H_0) = H^2(\mathbf{R}^3)$. Then $H_\varepsilon = H_0 + Q_\varepsilon$ also becomes a selfadjoint operator with $\mathrm{Dom}(H_\varepsilon) = H^2(\mathbf{R}^3)$ by Kato [3, Chap. V, Theorem 5.4]. Let $R(z) = (H - z)^{-1}$ and $R_\varepsilon(z) = (H_\varepsilon - z)^{-1}$ be the resolvents of H and H_ε, respectively. Then we shall prove the following

Theorem 2. *For sufficiently large z such that $\mathrm{Im}\, z \neq 0$, $R_\varepsilon(z)$ converges to $R(z)$ as $\varepsilon \downarrow 0$ in $\mathbf{B}(L_2(\mathbf{R}^3), H^1(\mathbf{R}^3))$ with the uniform operator topology, where $\mathbf{B}(X, Y)$ denotes the Banach space of bounded linear operators on X to Y ($\mathbf{B}(X) = \mathbf{B}(X, X)$).*

By this theorem and Kato [3, Chap. VIII, Cor. 1.4], we have

Theorem 3. *H_ε converges to H as $\varepsilon \downarrow 0$ in the norm resolvent sense.*

Another way of selfadjoint realization of H_{formal} and the related approximation problem will be found in Antoine-Gesztesy-Shabani [1].

§ 2. Preliminary Lemmas

Lemma 1. *Let r be positive and $u \in \mathscr{S}(\mathbf{R}^3)$. Then*

$$\|u(r\cdot)\|_{L_2(S_1)} \le \frac{1}{\sqrt{r}} \|\nabla u\| \le \frac{1}{\sqrt{r}} \|u\|_{H^1}\,, \tag{2.1}$$

where $\|u\| = \sqrt{(u, u)}$, $\|u\|_{L_2(S_a)} = \sqrt{(u, u)_{L_2(S_a)}}$, *and* $\|u\|_{H^m} = \sqrt{(u, u)_{H^m}}$. $\mathscr{S}(\mathbf{R}^3)$ *denotes the set of functions which together with all their derivatives fall off faster than the inverse of any polynomial.*

For the proof, see I-S[2, Lemma 1.3].

Lemma 2. *Let $u \in \mathscr{S}(\mathbf{R}^3)$. Then*

$$\|u(r\cdot) - u(r'\cdot)\|_{L_2(S_1)} \le \frac{|r - r'|^{1/2}}{\min(r, r')} \|\nabla u\|\,. \tag{2.2}$$

Proof. We have only to show the lemma in the case $0 < r' < r$. By Schwarz' inequality we have for any $\omega \in S_1$

$$\begin{aligned} |u(r\omega) - u(r'\omega)|^2 &= \left| \int_{r'}^{r} \frac{\partial u}{\partial \rho}(\rho\omega) d\rho \right|^2 \\ &\le (r - r') \int_{r'}^{r} \left| \frac{\partial u}{\partial \rho}(\rho\omega) \right|^2 d\rho \\ &\le \frac{(r - r')}{r'^2} \int_{r'}^{r} \rho^2 \left| \frac{\partial u}{\partial \rho}(\rho\omega) \right|^2 d\rho\,. \end{aligned} \tag{2.3}$$

Integrating the both sides of (2.3) with respect to ω over S_1 yields

$$\begin{aligned} \|u(r\cdot) - u(r'\cdot)\|_{L_2(S_1)}^2 &\le \frac{(r - r')}{r'^2} \int_{r' \le |x| \le r} \left| \frac{\partial u}{\partial \rho}(x) \right|^2 dx \\ &\le \frac{(r - r')}{r'^2} \left\| \frac{\partial u}{\partial \rho} \right\|^2 . \end{aligned} \tag{2.4}$$

(2.2) follows from (2.4) and $\left| \frac{\partial u}{\partial \rho}(x) \right| \le |\nabla u(x)|$. Q.E.D.

Let us define the Fourier transform $\mathscr{F}_{S_a}$ on $L_2(S_a)$ by

$$(\mathscr{F}_{S_a} u)(\xi) = (2\pi)^{-3/2} \int_{S_a} e^{-i\xi\cdot x} u(x) dS_x \qquad (\xi \in \mathbf{R}^3)\,. \tag{2.5}$$

Let us introduce the weighted L_2 space $L_2^s(\mathbf{R}^3)$ defined by

$$L_2^s(\mathbf{R}^3) = \{u(x); (1 + |x|^2)^{s/2} u(x) \in L_2(\mathbf{R}^3)\}$$

with the norm $\|u\|_{L_2^s(\mathbf{R}^3)} = \|(1 + |\cdot|^2)^{s/2} u\|$. Then we have the next

Lemma 3. *Let $s > 1/2$. Then there exists a constant $C = C(a, s)$ such that*

$$\|\mathscr{F}_{S_a} u\|_{L_2^{-s}(\mathbf{R}^3)} \le C\|u\|_{L_2(S_a)} \qquad \text{for any } u \in L_2(S_a)\,. \tag{2.6}$$

For the proof, see e.g. Mochizuki [5, p. 16]. We also need the following continuity lemma with respect to the radial direction.

Lemma 4. *Let r and r' be positive. Then we have for any $u \in L_2(S_a)$*

$$\|(\mathscr{F}_{S_1} u)(r\cdot) - (\mathscr{F}_{S_1} u)(r'\cdot)\|_{L_2^{-1}(\mathbf{R}^3)} \le \frac{|r - r'|^{1/2}}{\min(r, r')} \|u\|_{L_2(S_1)}\,. \tag{2.7}$$

Proof. (cf. Kuroda [4, §2.3, Theorem 3]) Consider the linear functional $V(f)$ on $L_2^1(\mathbf{R}^3)$ defined by

$$V(f) = \int_{\mathbf{R}^3} d\xi f(\xi) \overline{\{(\mathscr{F}_{S_1} u)(r\xi) - (\mathscr{F}_{S_1} u)(r'\xi)\}} \qquad \text{for } f \in L_2^1(\mathbf{R}^3)\,.$$

For any $f \in \mathscr{S}(\mathbf{R}^3)$, we have by (2.5) and Fubini's theorem

$$V(f) = \int_{S_1} d\omega \{(\mathscr{F}^* f)(r\omega) - (\mathscr{F}^* f)(r'\omega)\} \overline{u(\omega)}$$

($\mathscr{F}^*$: inverse Fourier transform). Thus, by Schwarz' inequality and Lemma 2 we obtain

$$\begin{aligned} |V(f)| &\le \|(\mathscr{F}^* f)(r\cdot) - (\mathscr{F}^* f)(r'\cdot)\|_{L_2(S_1)} \|u\|_{L_2(S_1)} \\ &\le \frac{|r - r'|^{1/2}}{\min(r, r')} \|\nabla(\mathscr{F}^* f)\| \, \|u\|_{L_2(S_1)} \\ &= \frac{|r - r'|^{1/2}}{\min(r, r')} \| |\cdot| f(\cdot)\| \, \|u\|_{L_2(S_1)} \\ &\le \frac{|r - r'|^{1/2}}{\min(r, r')} \|u\|_{L_2(S_1)} \|f\|_{L_2^1(\mathbf{R}^3)}\,. \end{aligned} \tag{2.8}$$

Since $\mathscr{S}(\mathbf{R}^3)$ is dense in $L_2^1(\mathbf{R}^3)$, (2.7) follows from (2.8). Q.E.D.

§3. Proof of Theorem 2

Let $R_0(z) = (H_0 - z)^{-1}$ be the resolvent of H_0. Let us define the integral operator T_κ depending on a complex parameter κ by

$$(T_\kappa u)(x) = \frac{-1}{4\pi} \int_{S_a} \frac{e^{i\kappa|x-y|}}{|x - y|} q(y) u(y) dS_y \qquad (x \in \mathbf{R}^3)\,.$$

It is seen that if $\operatorname{Im} \kappa > 0$, T_κ is a bounded operator from $L_2(S_a)$ to $H^1(\mathbf{R}^3)$ (I-S[2, Lemma 2.6]).

Lemma 5. *Let ε, s, and z be such that $0 < \varepsilon \le a/2$, $1/2 < s < 1$, and $z \in \mathbb{C}\backslash[0, \infty)$, respectively. Then there exists a constant $C_1 = C_1(s)$ (independent of ε and z) such that*

$$(3.1) \qquad \|R_0(z)Q_\varepsilon\|_{\mathbf{B}(H^1(\mathbf{R}^3))} \le C_1 \left[\sup_{\xi \in \mathbf{R}^3} \left\{ \frac{(1 + |\xi|^2)^{1+s}}{||\xi|^2 - z|^2} \right\} \right]^{1/2},$$

where $\|\cdot\|_{\mathbf{B}(X,Y)}$ denotes the norm of $\mathbf{B}(X, Y)$.

Proof. By (1.1) it holds that

$$(3.2) \qquad \begin{cases} \dfrac{1}{\varepsilon}\rho\left(\dfrac{r-a}{\varepsilon}\right) \ge 0 \quad \text{for all } r \in \mathbf{R}, \quad \operatorname{supp} \dfrac{1}{\varepsilon}\rho\left(\dfrac{r-a}{\varepsilon}\right) \subset [a-\varepsilon, a+\varepsilon], \\ \displaystyle\int_{a-\varepsilon}^{a+\varepsilon} \frac{1}{\varepsilon}\rho\left(\frac{r-a}{\varepsilon}\right) dr = 1. \end{cases}$$

For any $u \in \mathscr{S}(\mathbf{R}^3)$ we have by Fubini's theorem and (2.5)

$$\begin{aligned}(3.3) \quad (\mathscr{F}R_0(z)Q_\varepsilon u)(\xi) &= (2\pi)^{-3/2} \int_{\mathbf{R}^3} dx e^{-i\xi\cdot x} \\ &\quad \times \int_{\mathbf{R}^3} dy \frac{e^{i\sqrt{z}|x-y|}}{4\pi|x-y|} \frac{1}{\varepsilon}\rho\left(\frac{|y|-a}{\varepsilon}\right) q(a\omega_y)u(y) \\ &= \int_{a-\varepsilon}^{a+\varepsilon} dr \frac{1}{\varepsilon}\rho\left(\frac{r-a}{\varepsilon}\right) r^2 \int_{S_1} d\omega q(a\omega)u(r\omega) \\ &\quad \times (2\pi)^{-3/2} \int_{\mathbf{R}^3} dx e^{-i\xi\cdot x} \frac{e^{i\sqrt{z}|x-r\omega|}}{4\pi|x-r\omega|} \\ &= \int_{a-\varepsilon}^{a+\varepsilon} dr \frac{1}{\varepsilon}\rho\left(\frac{r-a}{\varepsilon}\right) r^2 (2\pi)^{-3/2} \int_{S_1} d\omega \frac{e^{-i\xi\cdot r\omega}}{|\xi|^2 - z} q(a\omega)u(r\omega) \\ &= \frac{1}{|\xi|^2 - z} \int_{a-\varepsilon}^{a+\varepsilon} dr \frac{1}{\varepsilon}\rho\left(\frac{r-a}{\varepsilon}\right) r^2 [\mathscr{F}_{S_1}(q(a\cdot)u(r\cdot))](r\xi),\end{aligned}$$

where by $\sqrt{z}$ is meant the branch of square root of z with $\operatorname{Im}\sqrt{z} \ge 0$ and we have used the fact that

$$\mathscr{F}\left(\frac{e^{i\kappa|\cdot - y|}}{4\pi|\cdot - y|}\right)(\xi) = (2\pi)^{-3/2} \frac{e^{-i\xi\cdot y}}{|\xi|^2 - \kappa^2}.$$

Thus, we have by Schwarz' inequality, Fubini's theorem and (3.2)

$$\begin{aligned}(3.4) \quad \|R_0(z)Q_\varepsilon u\|_{H^1}^2 &= \int_{\mathbf{R}^3} d\xi (1 + |\xi|^2) \\ &\quad \times \left| \frac{1}{|\xi|^2 - z} \int_{a-\varepsilon}^{a+\varepsilon} dr \frac{1}{\varepsilon}\rho\left(\frac{r-a}{\varepsilon}\right) r^2 [\mathscr{F}_{S_1}(q(a\cdot)u(r\cdot))](r\xi) \right|^2 \\ &\le \int_{\mathbf{R}^3} d\xi \frac{(1 + |\xi|^2)}{||\xi|^2 - z|^2} \left(\int_{a-\varepsilon}^{a+\varepsilon} dr \frac{1}{\varepsilon}\rho\left(\frac{r-a}{\varepsilon}\right) r^4 \right)\end{aligned}$$

$$\times\left(\int_{a-\varepsilon}^{a+\varepsilon} dr\frac{1}{\varepsilon}\rho\left(\frac{r-a}{\varepsilon}\right)|[\mathscr{F}_{S_1}(q(a\cdot)u(r\cdot))](r\xi)|^2\right)$$

$$\leq (a+\varepsilon)^4 \sup_{\xi\in\mathbf{R}^3}\left\{\frac{(1+|\xi|^2)^{1+s}}{||\xi|^2-z|^2}\right\}\int_{a-\varepsilon}^{a+\varepsilon} dr\frac{1}{\varepsilon}\rho\left(\frac{r-a}{\varepsilon}\right)$$

$$\times\int_{\mathbf{R}^3} d\xi(1+|\xi|^2)^{-s}|[\mathscr{F}_{S_1}(q(a\cdot)u(r\cdot))](r\xi)|^2 .$$

By the change of variables $\zeta = r\xi$, we have

$$\int_{\mathbf{R}^3} d\xi(1+|\xi|^2)^{-s}|[\mathscr{F}_{S_1}(q(a\cdot)u(r\cdot))](r\xi)|^2 \tag{3.5}$$

$$= \int_{\mathbf{R}^3} d\zeta r^{-3}(1+r^{-2}|\zeta|^2)^{-s}|[\mathscr{F}_{S_1}(q(a\cdot)u(r\cdot))](\zeta)|^2 .$$

Thus, in view of the inequality

$$(1+r^{-2}|\zeta|^2)^{-s} \leq \max(r^{2s}, 1)(1+|\zeta|^2)^{-s} \qquad \text{if } s>0 \text{ and } r>0,$$

we have by Lemma 1 and Lemma 3

$$\int_{\mathbf{R}^3} d\xi(1+|\xi|^2)^{-s}|[\mathscr{F}_{S_1}(q(a\cdot)u(r\cdot))](r\xi)|^2 \tag{3.6}$$

$$\leq r^{-3}\max(r^{2s}, 1)\|\mathscr{F}_{S_1}(q(a\cdot)u(r\cdot))\|^2_{L_2^{-s}(\mathbf{R}^3)}$$

$$\leq r^{-3}\max(r^{2s}, 1)C(1, s)^2\|q(a\cdot)u(r\cdot)\|^2_{L_2(S_1)}$$

$$\leq r^{-4}\max(r^{2s}, 1)C(1, s)^2\left(\max_{x\in S_a}|q(x)|\right)^2\|u\|^2_{H^1},$$

where $C(1, s)$ is as given in Lemma 3. Therefore, since $0<\varepsilon\leq a/2$, we obtain by (3.4), (3.6), and (3.2)

$$\|R_0(z)Q_\varepsilon u\|^2_{H^1} \leq (a+\varepsilon)^4 \sup_{\xi\in\mathbf{R}^3}\left\{\frac{(1+|\xi|^2)^{1+s}}{||\xi|^2-z|^2}\right\}C(1, s)^2 \tag{3.7}$$

$$\times\left(\max_{x\in S_a}|q(x)|\right)^2\|u\|^2_{H^1}\int_{a-\varepsilon}^{a+\varepsilon} dr\frac{1}{\varepsilon}\rho\left(\frac{r-a}{\varepsilon}\right)r^{-4}\max(r^{2s}, 1)$$

$$\leq C_1(s)^2 \sup_{\xi\in\mathbf{R}^3}\left\{\frac{(1+|\xi|^2)^{1+s}}{||\xi|^2-z|^2}\right\}\|u\|^2_{H^1},$$

where $C_1(s)$ is a constant which is independent of ε such that $0<\varepsilon\leq a/2$. Since $R_0(z)$ is a bounded operator from $L_2(\mathbf{R}^3)$ to $H^1(\mathbf{R}^3)$ and $\mathscr{S}(\mathbf{R}^3)$ is dense in $H^1(\mathbf{R}^3)$, (3.1) follows from (3.7). Q.E.D.

Lemma 6. *Let* ε, s, *and* z *be such that* $0<\varepsilon\leq a/2$, $1/2<s<1$, *and* $z\in\mathbf{C}\setminus[0,\infty)$, *respectively. Then there exists a constant* $C_2 = C_2(s, z)$ (*independent* *of* ε) *such that*

$$\|R_0(z)Q_\varepsilon + T_{\sqrt{z}}\gamma_a\|_{\mathbf{B}(H^1(\mathbf{R}^3))} \le \sqrt{\varepsilon}C_2 . \tag{3.8}$$

Proof. As we got (3.3), we have for any $u \in \mathscr{S}(\mathbf{R}^3)$

$$(\mathscr{F}T_{\sqrt{z}}\gamma_a u)(\xi) = \frac{-a^2}{|\xi|^2 - z}[\mathscr{F}_{S_1}(q(a\cdot)u(a\cdot))](a\xi) . \tag{3.9}$$

Thus, we have by (3.2), (3.3), and (3.9)

$$\begin{aligned} &[\mathscr{F}(R_0(z)Q_\varepsilon + T_{\sqrt{z}}\gamma_a)u](\xi) \\ &= \frac{1}{|\xi|^2 - z}\int_{a-\varepsilon}^{a+\varepsilon} dr\frac{1}{\varepsilon}\rho\left(\frac{r-a}{\varepsilon}\right)(r^2 - a^2)[\mathscr{F}_{S_1}(q(a\cdot)u(r\cdot))](r\xi) \\ &+ \frac{a^2}{|\xi|^2 - z}\int_{a-\varepsilon}^{a+\varepsilon} dr\frac{1}{\varepsilon}\rho\left(\frac{r-a}{\varepsilon}\right)[\mathscr{F}_{S_1}(q(a\cdot)(u(r\cdot) - u(a\cdot)))](r\xi) \\ &+ \frac{a^2}{|\xi|^2 - z}\int_{a-\varepsilon}^{a+\varepsilon} dr\frac{1}{\varepsilon}\rho\left(\frac{r-a}{\varepsilon}\right) \\ &\times \{[\mathscr{F}_{S_1}(q(a\cdot)u(a\cdot))](r\xi) - [\mathscr{F}_{S_1}(q(a\cdot)u(a\cdot))](a\xi)\} \\ &= I_1(\xi) + I_2(\xi) + I_3(\xi) . \end{aligned} \tag{3.10}$$

We shall estimate the $L^1_2(\mathbf{R}^3)$ norm of $I_j(\xi)$ $(j = 1, 2, 3)$. On replacing r^2 by $r^2 - a^2$ in (3.3), as we got (3.7), we have

$$\begin{aligned} \int_{\mathbf{R}^3} d\xi(1 + |\xi|^2)|I_1(\xi)|^2 &\le \frac{\varepsilon^2(2a+\varepsilon)^2}{(a-\varepsilon)^4}\max\{(a+\varepsilon)^{2s}, 1\} \\ &\times \sup_{\xi \in \mathbf{R}^3}\left\{\frac{(1 + |\xi|^2)^{1+s}}{||\xi|^2 - z|^2}\right\}C(1, s)^2\left(\max_{x \in S_a}|q(x)|\right)^2\|u\|^2_{H^1} \\ &\le \varepsilon^2\tilde{C}_1\|u\|^2_{H^1} , \end{aligned} \tag{3.11}$$

where $\tilde{C}_1$ is a constant which is independent of ε such that $0 < \varepsilon \le a/2$. Similarly, as we got (3.4), we have

$$\begin{aligned} \int_{\mathbf{R}^3} d\xi(1 + |\xi|^2)|I_2(\xi)|^2 &\le a^4 \sup_{\xi \in \mathbf{R}^3}\left\{\frac{(1 + |\xi|^2)^{1+s}}{||\xi|^2 - z|^2}\right\} \\ &\times \int_{a-\varepsilon}^{a+\varepsilon} dr\frac{1}{\varepsilon}\rho\left(\frac{r-a}{\varepsilon}\right)\int_{\mathbf{R}^3} d\xi(1 + |\xi|^2)^{-s}|[\mathscr{F}_{S_1}(q(a\cdot)(u(r\cdot) - u(a\cdot)))](r\xi)|^2 . \end{aligned} \tag{3.12}$$

From (3.6) and Lemma 2, it follows that

$$(3.13)\qquad \int_{\mathbf{R}^3} d\xi(1+|\xi|^2)^{-s}|[\mathscr{F}_{S_1}(q(a\cdot)(u(r\cdot)-u(a\cdot)))](r\xi)|^2$$

$$\leq r^{-3}\max(r^{2s},1)C(1,s)^2\|q(a\cdot)(u(r\cdot)-u(a\cdot))\|^2_{L_2(S_1)}$$

$$\leq r^{-3}\max(r^{2s},1)C(1,s)^2\left(\max_{x\in S_a}|q(x)|\right)^2\frac{|r-a|}{\{\min(r,a)\}^2}\|\nabla u\|^2$$

$$\leq \varepsilon\frac{\max\{(a+\varepsilon)^{2s},1\}}{(a-\varepsilon)^5}C(1,s)^2\left(\max_{x\in S_a}|q(x)|\right)^2\|u\|^2_{H^1},$$

if $r\in[a-\varepsilon,a+\varepsilon]$. Therefore, by (3.12), (3.13), and (3.2) we obtain

$$(3.14)\qquad \int_{\mathbf{R}^3} d\xi(1+|\xi|^2)|I_2(\xi)|^2\leq \varepsilon\tilde{C}_2\|u\|^2_{H^1},$$

where $\tilde{C}_2$ is a constant which is independent of ε such that $0<\varepsilon\leq a/2$. We shall proceed to estimate the $L_2^1(\mathbf{R}^3)$ norm of $I_3(\xi)$. By Schwarz' inequality and (3.2) we have

$$|I_3(\xi)|^2\leq\frac{a^4}{||\xi|^2-z|^2}\int_{a-\varepsilon}^{a+\varepsilon}dr\frac{1}{\varepsilon}\rho\left(\frac{r-a}{\varepsilon}\right)$$

$$\times|[\mathscr{F}_{S_1}(q(a\cdot)u(a\cdot))](r\xi)-[\mathscr{F}_{S_1}(q(a\cdot)u(a\cdot))](a\xi)|^2.$$

Thus, we have by Fubini's theorem

$$(3.15)\qquad \int_{\mathbf{R}^3} d\xi(1+|\xi|^2)|I_3(\xi)|^2\leq a^4\sup_{\xi\in\mathbf{R}^3}\left\{\frac{(1+|\xi|^2)^2}{||\xi|^2-z|^2}\right\}\int_{a-\varepsilon}^{a+\varepsilon}dr\frac{1}{\varepsilon}\rho\left(\frac{r-a}{\varepsilon}\right)$$

$$\times\int_{\mathbf{R}^3} d\xi(1+|\xi|^2)^{-1}|[\mathscr{F}_{S_1}(q(a\cdot)u(a\cdot))](r\xi)-[\mathscr{F}_{S_1}(q(a\cdot)u(a\cdot))](a\xi)|^2$$

$$=a^4\sup_{\xi\in\mathbf{R}^3}\left\{\frac{(1+|\xi|^2)^2}{||\xi|^2-z|^2}\right\}\int_{a-\varepsilon}^{a+\varepsilon}dr\frac{1}{\varepsilon}\rho\left(\frac{r-a}{\varepsilon}\right)$$

$$\times\|[\mathscr{F}_{S_1}(q(a\cdot)u(a\cdot))](r\cdot)-[\mathscr{F}_{S_1}(q(a\cdot)u(a\cdot))](a\cdot)\|^2_{L_2^{-1}(\mathbf{R}^3)}.$$

From Lemma 4 and Lemma 1 it follows that

$$(3.16)\qquad \|[\mathscr{F}_{S_1}(q(a\cdot)u(a\cdot))](r\cdot)-[\mathscr{F}_{S_1}(q(a\cdot)u(a\cdot))](a\cdot)\|^2_{L_2^{-1}(\mathbf{R}^3)}$$

$$\leq\frac{|r-a|}{\{\min(r,a)\}^2}\|q(a\cdot)u(a\cdot)\|^2_{L_2(S_1)}$$

$$\leq\frac{|r-a|}{\{\min(r,a)\}^2}\left(\max_{x\in S_a}|q(x)|\right)^2\frac{1}{a}\|u\|^2_{H^1}$$

$$\leq\frac{\varepsilon}{a(a-\varepsilon)^2}\left(\max_{x\in S_a}|q(x)|\right)^2\|u\|^2_{H^1}\qquad\text{if } r\in[a-\varepsilon,a+\varepsilon].$$

Therefore, by (3.15), (3.16), and (3.2) we obtain

$$\int_{\mathbf{R}^3} d\xi(1+|\xi|^2)|I_3(\xi)|^2 \leq \varepsilon\tilde{C}_3\|u\|_{H^1}^2\,, \tag{3.17}$$

where $\tilde{C}_3$ is a constant which is independent of ε such that $0<\varepsilon\leq a/2$. Since $T_{\sqrt{z}}\gamma_a$ is a bounded operator from $H^1(\mathbf{R}^3)$ to itself and $\mathscr{S}(\mathbf{R}^3)$ is dense in $H^1(\mathbf{R}^3)$, (3.8) follows from (3.10), (3.11), (3.14) and (3.17). Q.E.D.

We are now in a position to prove Theorem 2.

Proof of Theorem 2. First we remark that by the closed graph theorem $R_\varepsilon(z)$ and $R(z)$ are bounded operators from $L_2(\mathbf{R}^3)$ to $H^2(\mathbf{R}^3)$ and $H^1(\mathbf{R}^3)$, respectively. Let us recall that resolvent equations for the pairs (H_ε, H_0) and (H, H_0):

$$R_\varepsilon(z)-R_0(z)=-R_0(z)Q_\varepsilon R_\varepsilon(z) \qquad \text{(the second resolvent equation)}$$

and

$$R(z)-R_0(z)=T_{\sqrt{z}}\gamma_a R(z) \qquad \text{(I-S[2, Theorem 3.2])}\,.$$

Thus, we have

$$\begin{aligned} R_\varepsilon(z)-R(z) &= -R_0(z)Q_\varepsilon(R_\varepsilon(z)-R(z)) \\ &\quad -(R_0(z)Q_\varepsilon+T_{\sqrt{z}}\gamma_a)R(z)\,. \end{aligned} \tag{3.18}$$

Take $z\in\mathbf{C}\backslash[0,\infty)$ sufficiently large such that $\operatorname{Im} z\neq 0$ and

$$C_1(s)\left[\sup_{\xi\in\mathbf{R}^3}\left\{\frac{(1+|\xi|^2)^{1+s}}{||\xi|^2-z|^2}\right\}\right]^{1/2}<1/2\,,$$

which is possible because of $1/2<s<1$. Then, for any $u\in L_2(\mathbf{R}^3)$ we have by (3.18), Lemma 5 and Lemma 6

$$\begin{aligned} \|R_\varepsilon(z)u-R(z)u\|_{H^1} &\leq \|R_0(z)Q_\varepsilon(R_\varepsilon(z)u-R(z)u)\|_{H^1} \\ &\quad +\|(R_0(z)Q_\varepsilon+T_{\sqrt{z}}\gamma_a)R(z)u\|_{H^1} \\ &\leq \frac{1}{2}\|R_\varepsilon(z)u-R(z)u\|_{H^1}+\sqrt{\varepsilon}C_2\|R(z)u\|_{H^1}\,, \end{aligned}$$

and hence

$$\begin{aligned} \|R_\varepsilon(z)u-R(z)u\|_{H^1} &\leq 2\sqrt{\varepsilon}C_2\|R(z)u\|_{H^1} \\ &\leq 2\sqrt{\varepsilon}C_2\|R(z)\|_{\mathbf{B}(L_2(\mathbf{R}^3),H^1(\mathbf{R}^3))}\,\|u\|\,. \end{aligned} \tag{3.19}$$

The required result follows from (3.19). Q.E.D.

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References

[1] J. P. Antoine, F. Gesztesy and J. Shabani, Exactly solvable models of sphere interactions in quantum mechanics, J. Phys. A: Math. Gen., **20** (1987), 3687–3712.

[2] T. Ikebe and S. Shimada, Spectral and scattering theory for the Schrödinger operators with penetrable wall potentials, J. Math. Kyoto Univ., 31-1 (1991), 219–258.

[3] T. Kato, Perturbation Theory for Linear Operators, Springer, 1966.

[4] S. T. Kuroda, An Introduction to Scattering Theory, Lecture Notes Series No. 51, Aarhus Univ., 1978.

[5] K. Mochizuki, Scattering Theory for the Wave Equations, Kinokuniya, Tokyo, 1984 (in Japanese).

[6] L. Schwartz, Théorie des Distributions, Hermann, Paris, 1966.