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A remark on mutually disjoint irreducible decompositions of the regular representation of a group

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In the previous paper [3] we construct continuously many, mutually disjoint irreducible decompositions of the regular representation of a certain discrete group. We treated a restricted direct product $G=\prod'_{\alpha\in A}G_\alpha$, with discrete topology, of countably infinite family of finite groups G_α, $\alpha\in A$. Our main tools to construct irreducible decompositions are infinite tensor products of representations and choosing orthonormal bases of each representation spaces. The diversity of decompositions is based on the variety of choices of bases of Hilbert spaces. In this paper we apply this method to a not necessarily countable case and obtain many mutually disjoint decompositions more than cardinality of continuity.

As we know the restricted direct product group of an infinite family of finite groups is non-type I, if non-abelian factors are infinite. Assume that each factor G_α is not abelian, then we can actually construct $2^{|A|}$ mutually disjoint decompositions here.

If $|A|$ is more than countably infinite and G_α's are not abelian, then almost all representation spaces appearing in the decompositions are not separable. So we use direct integrals of non-separable Hilbert spaces.

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§ 1. Notations and preliminaries

Let G_α, $\alpha\in A$, be an infinite family of finite groups and $G:=\prod'_{\alpha\in A}G_\alpha$ be a restricted direct product group of G_α's with discrete topology.

We follow the notations in [3] and recall here some of them. Denote by L the left regular representation of G on $L^2(G)$. When G is not countable, Hilbert space $L^2(G)$ is not separable. So we treat non-separable representations and its decompositions.

Let Ξ_α be a complete family of representatives of $\widehat{G}_\alpha$, the unitary dual of G_α. For a representation $\xi_\alpha\in\Xi_\alpha$, denote by $V(\xi_\alpha)$ the representation space

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of ξ_α and by $\boldsymbol{B}(\xi_\alpha)$ the space of all bounded linear operators of $V(\xi_\alpha)$, and put $d(\xi_\alpha)=\dim V(\xi_\alpha)$. We consider $\boldsymbol{B}(\xi_\alpha)$ as a Hilbert space with an inner product $\langle u, v\rangle=(1/(d(\xi_\alpha))\mathrm{Tr}(uv^*)(u, v\in\boldsymbol{B}(\xi_\alpha))$. Let μ_α be a normalized measure on Ξ_α such that $\mu_\alpha(\{\xi_\alpha\})=|G_\alpha|^{-1}d(\xi_\alpha)^2$ $(\xi_\alpha\in\Xi_\alpha)$. Let Ξ be the direct product of Ξ_α, $\alpha\in A$ (not a restricted one) and μ the product measure of μ_α's on it (cf. [6, Chap. 2, § 8]): $\Xi=\prod_{\alpha\in A}\Xi_\alpha$, $\mu=\prod_{\alpha\in A}\mu_\alpha$.

For each $\xi=(\xi_\alpha)\in\Xi$, we take the reference vector $1(\xi)=(1_{V(\xi_\alpha)})_{\alpha\in A}$, and define an infinite tensor product $\mathcal{H}(\xi)$ of Hilbert spaces $\boldsymbol{B}(\xi_\alpha)$:

$$\mathcal{H}(\xi)=\otimes_{\alpha\in A}\{\boldsymbol{B}(\xi_\alpha), 1_{V(\xi_\alpha)}\}=\otimes_{\alpha\in A}^{1(\xi)}\boldsymbol{B}(\xi_\alpha),$$

where $1_{V(\xi_\alpha)}$ denotes the identity operator on $V(\xi_\alpha)$. We denote by ρ_ξ the representation on $\mathcal{H}(\xi)$: for $g=(g_\alpha)\in G$ and a decomposable element $\phi=\otimes_{\alpha\in A}^{1(\xi)}\phi_{\xi_\alpha}\in\mathcal{H}(\xi)$,

$$\rho_\xi(g)\phi=\otimes_{\alpha\in A}^{1(\xi)}(\xi_\alpha(g_\alpha)\cdot\phi_{\xi_\alpha}).$$

Let ϕ_h be a vector field on the measure space (Ξ, μ) corresponding to $h=(h_\alpha)\in G$ given as $\phi_h(\xi)=\otimes_{\alpha\in A}^{1(\xi)}\xi_\alpha(h_\alpha)\in\mathcal{H}(\xi)$, for $\xi\in\Xi$.

We fix an element $\xi=(\xi_\alpha)\in\Xi$. Let $N(\xi_\alpha)=\{1, 2, \cdots, d(\xi_\alpha)\}$, and define a measure ν_{ξ_α} on $N(\xi_\alpha)$ by $\nu_{\xi_\alpha}(\{m\})=1/d(\xi_\alpha)$ for $m\in N(\xi_\alpha)$. Let $(K(\xi), \nu_\xi)$ be the direct product of $(N(\xi_\alpha), \nu_{\xi_\alpha})$: $K(\xi)=\prod_{\alpha\in A}N(\xi_\alpha)$, $\nu_\xi=\prod_{\alpha\in A}\nu_{\xi_\alpha}$.

Fix an orthonormal basis $\{v_{\xi_\alpha,j}|1\leqq j\leqq d(\xi_\alpha)\}$ of $V(\xi_\alpha)$ for each ξ_α, and denote by $\mathcal{O}(\xi)$ the system of orthonormal bases of $V(\xi_\alpha)$, $\alpha\in A$. For $\kappa=(\kappa_\alpha)\in K(\xi)$, we take the infinite tensor product Hilbert space $\boldsymbol{V}(\kappa)=\boldsymbol{V}(\xi, \mathcal{O}(\xi); \kappa)=\otimes_{\alpha\in A}\{V(\xi_\alpha), v_{\xi_\alpha,\kappa_\alpha}\}$ with a reference vector $v(\kappa)=(v_{\xi_\alpha,\kappa_\alpha})_{\alpha\in A}$ consisting of $v_{\xi_\alpha,\kappa_\alpha}$ from the system $\mathcal{O}(\xi)$.

Now we define an irreducible representation $\rho_{\xi,\kappa}=\rho_{\xi,\kappa}^{\mathcal{O}(\xi)}$ of G on $\boldsymbol{V}(\kappa)$: for $g=(g_\alpha)\in G$ and a decomposable vector $v=\otimes_{\alpha\in A}^{v(\kappa)}v_\alpha\in\boldsymbol{V}(\kappa)$,

$$\rho_{\xi,\kappa}(g)v=\otimes_{\alpha\in A}^{v(\kappa)}(\xi_\alpha(g_\alpha)v_\alpha).$$

§ 2. An irreducible decomposition of the regular representation

First we obtain the central factor decomposition of the regular representation of G and then starting from it we can construct an irreducible decomposition in a similar way as in [3].

Theorem 2.1. *The central factor decomposition of the left regular representation L of G on $L^2(G)$ is given by the following direct integral:*

$$\{L, L^2(G)\}\simeq^{\{\phi_h\}}\int_{\Xi}^{\oplus}\{\rho_\xi, \mathcal{H}(\xi)\}d\mu(\xi).$$

A natural equivalence is given by

$$L^2(G)\ni\delta_h\leftrightarrow\phi_h\in{}^{\{\phi_h\}}\int_{\Xi}^{\oplus}\mathcal{H}(\xi)d\mu(\xi)\,.$$

The factor representations coming into here are of type II_1 or type I finite.

Theorem 2.2. *(i) Fix $\xi=(\xi_\alpha)\in\Xi$. Take a system $\mathcal{O}(\xi)=\{v_{\xi_\alpha,j}|1\leqq j\leqq d(\xi_\alpha)\}_{\alpha\in A}$ of orthonormal bases of $V(\xi_\alpha)$'s. Then an irreducible decomposition of the factor representation $\{\rho_\xi,\mathcal{H}(\xi)\}$ is given by*

$$\{\rho_\xi,\mathcal{H}(\xi)\}\simeq{}^{\{F_h(\xi)\}}\int_{K(\xi)}^{\oplus}\{\rho_{\xi,\varepsilon}^{\mathcal{O}(\xi)},\ V(\xi,\mathcal{O}(\xi);\ \kappa)\}d\nu_\xi(\kappa)$$

A natural equlvalence mapping is given by

$$\mathcal{H}(\xi)\ni\varphi_h(\xi)\leftrightarrow F_h(\xi)(\cdot)\in{}^{\{F_h(\xi)\}}\int_{K(\xi)}^{\oplus}V(\xi,\mathcal{O}(\xi);\ \kappa)d\nu_\xi(\kappa)\,.$$

(ii) Fix a system $\mathcal{O}(\xi)=\{v_{\xi_\alpha,j}|1\leqq j\leqq d(\xi_\alpha)\}_{\alpha\in A}$ of orthonormal bases of $V(\xi_\alpha)$'s for each $\xi=(\xi_\alpha)\in\Xi$. An irreducible decomposition of the left regular representation L is given by the following direct integral:

$$\{L,L^2(G)\}\simeq{}^{\{\phi_h\}}\int_{\Xi}^{\oplus}\left\{{}^{\{F_h(\xi)\}}\int_{K(\xi)}^{\oplus}\{\rho_{\xi,\kappa}^{\mathcal{O}(\xi)},\ V(\xi,\mathcal{O}(\xi);\ \kappa)\}d\nu_\xi(\kappa)\right\}d\mu(\xi)\,.$$

Two irreducible components $\rho_{\xi,\kappa}^{\mathcal{O}(\xi)}$ and $\rho_{\xi',\kappa'}^{\mathcal{O}(\xi')}$ are mutually equivalent if and only if $\xi=\xi'$ and $\kappa_\alpha=\kappa'_\alpha$ except a finite number of $\alpha\in A$.

Note that decompositions in the above two theorems are direct integrals of non-separable Hilbert spaces in general. Such direct integrals are studied in [4], [5].

§ 3. Cardinality of mutually disjoint irreducible decompositions

First we decompose each representation $\{\rho_\xi,\mathcal{H}(\xi)\}$ into irreducibles. There exist infinitely many, mutually disjoint decompositions and we can evaluate the cardinality of these decompositions. Next assembling these decompositions for each $\xi\in\Xi$, we evaluate the cardinality of mutually disjoint irreducible decompositions of the regular representation.

Theorem 3.1. *Fix an element $\xi=(\xi_\alpha)\in\Xi=\prod_{\alpha\in A}\Xi_\alpha$. Assume that the cardinality of $\{\alpha\in A|d(\xi_\alpha)\geqq2\}$ is d. Then there exist at least 2^d mutually disjoint irreducible decompositions of the factor representation $\{\rho_\xi,\mathcal{H}(\xi)\}$.*

Proof. Fix two systems $\mathcal{O}(\xi)^i$, $i=1,2$, of orthonormal bases $\{v_{\xi_\alpha,j}^i|1\leqq j\leqq d(\xi_\alpha)\}_{\alpha\in A}$, such that, for a fixed $\varepsilon>0$,

$$|\langle v^1_{\xi_\alpha,j},\ v^2_{\xi_\alpha,j'}\rangle|\leqq 1-\varepsilon \qquad (1\leqq j, j'\leqq d(\xi_\alpha))$$

for any α with $d(\xi_\alpha)\geqq 2$.

Let $\Lambda=\{1, 2\}^A$ and take $\lambda=(\lambda_\alpha)_{\alpha\in A}$. We set systems of orthonormal bases $\mathcal{O}(\xi;\ \lambda)$ for $\lambda\in\Lambda$ by $\{v^{\lambda_\alpha}_{\xi_\alpha,j}|1\leqq j\leqq d(\xi_\alpha)\}_{\alpha\in A}$. Then remembering C.C. Moore's criterion of equivalence of infinite tensor product representations [2], [3, Lemma 4.1], we have $\rho^{\mathcal{O}(\xi;\lambda)}_{\xi,\kappa}\not\simeq\rho^{\mathcal{O}(\xi;\lambda')}_{\xi,\kappa'}$ for any κ, $\kappa'\in K(\xi)$, in case $\lambda_\alpha\neq\lambda'_\alpha$ for infinitely many α with $d(\xi_\alpha)\geqq 2$.

We set a maximal subset $\Lambda(\xi)$ of Λ which has the property that, for any two elements λ, $\lambda'\in\Lambda(\xi)$, $\lambda_\alpha\neq\lambda'_\alpha$ for infinitly many α with $d(\xi_\alpha)\geqq 2$. The set $\Lambda(\xi)$ is of cardinality 2^d. Therefore we see that there exist 2^d mutually disjoint irreducible decompositions parametrized by $\lambda\in\Lambda(\xi)$:

$$^{\{F_h(\xi)\}}\int^{\oplus}_{K(\xi)}\{\rho^{\mathcal{O}(\xi;\lambda)}_{\xi,\kappa},\ V(\xi,\ \mathcal{O}(\xi;\ \lambda);\ \kappa)\}\nu_\xi(\kappa)\,.$$

Thus we have constructed 2^d irreducible decompositions of the representation ρ_ξ in the central factor decomposition of the regular representation. Summing up over $\xi\in\Xi$. we get the next theorem.

Theorem 3.2. *Let a discrete group* $G=\prod'_{\alpha\in A}G_\alpha$ *be the restricted direct product group of finite groups* G_α. *Let* d *be the cardinality of* $A^{non}=\{\alpha\in A|$ G_α *is not commutative*$\}$.

Then there exist 2^d *mutually disjoint irreducible decompositions of the left regular representation* L *of* G *of the following type:*

$$\{L,\ L^2(G)\}\simeq{}^{\{\phi_h\}}\int^{\oplus}_{\Xi}\left\{{}^{\{F_h(\xi)\}}\int^{\oplus}_{K(\xi)}\{\rho^{\mathcal{O}(\xi)}_{\xi,\kappa},\ V(\xi,\ \mathcal{O}(\xi);\ \kappa)\}d\nu_\xi(\kappa)\right\}d\mu(\xi)\,.$$

Proof. Already 2^d mutually disjoint irreducible decompositions of ρ_ξ are constructed in Theorem 3.1 under an assumption on ξ. So it is sufficient for us to prove that the assumption of Theorem 3.1 holds for almost all $\xi\in\Xi$.

Put $X=\{\xi\in\Xi|\xi$ does not satisfy the assumption of Theorem 3.1$\}$. Note that $\xi\in X$ is characterized by the property "$d(\xi_\alpha)=1$ except a finite number of $\alpha\in A^{non}$". Let B be a countably infinite subset of A^{non} and $\Xi(B)=\prod_{\alpha\in B}\Xi_\alpha$. We define $X(B)\subset\Xi(B)$ as

$$X(B)=\{\zeta\in\Xi(B)|\exists\,\xi\in X \text{ such that } \zeta_\alpha=\xi_\alpha \text{ for } \forall\,\alpha\in B\}\,.$$

Since B is countable, $X(B)$ is measurable and of measure zero as a subset of $\Xi(B)$ (see [3, Proof of Theorem 4.2]). As a subset of a nullset $X(B)\times\prod_{\alpha\in A\setminus B}\Xi_\alpha$ of Ξ, X is measurable and of measure zero.

The proof of the theorem is now completed.

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