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Stable retraction of CW-complexes whose cohomologies are small exterior algebras

By

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Introduction

In [Cohen-Peterson 84], F. R. Cohen and F. P. Peterson studied three cell complexes $X = S^{a_1} \cup e^{a_2} \cup e^{a_1+a_2}$ satisfying the following conditions:

SR1 $H^*(X; \mathbf{F}_2) \cong \Lambda(u_{a_1}, u_{a_2})$ where $\deg u_{a_i} = a_i$ for $i = 1$, 2 and $a_2 > a_1$,

SR2 $Sq^{a_2-a_1}u_{a_1} = u_{a_2}$,

SR3 $Y = S^{a_1} \cup e^{a_2}$ is a stable retract of X.

They proved that if such a space X exists, then $a_2 - a_1 = 2^t$ for $t = 0$, 1, 2, or 3 and $\binom{a_1}{2^t} \equiv 1 \bmod 2$.

In this note we determine necessary and sufficient conditions for the existance of such a three cell complex in terms of the degrees of the generators of the cohomology.

Theorem. *There exists a CW-complex X satisfying the above three conditions if and only if* $a_2 - a_1 = 2^t$ *and* $a_1 \equiv -1 \bmod 2^{t+1}$ *for* $t = 0$, 1, 2, *or* 3.

Let $St_k(\mathbf{F}^n)$ be the Stiefel manifold of orthonormal k-frames in $\mathbf{F}^n$ and $d = \dim_{\mathbf{R}} \mathbf{F}$ where $\mathbf{F} = \mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$. For $t = 0$, 1, and 2, the condition "$a_1 \equiv -1 \bmod 2^{t+1}$" corresponds to the exsistence of a nontrivial action of a Steenrod operation on the generators of the cohomology of the Stiefel manifolds $St_2(\mathbf{F}^{(a_1+d+1)/d})$ for $\mathbf{F} = \mathbf{R}$, $\mathbf{C}$ and $\mathbf{H}$, respectively.

When $t = 3$, as an application of the main theorem, we obtain the following corollary.

Corollary. E_6/F_4 *never retracts stably to its* 17-*skeleton. Thus* E_6/F_4 *is not stably parallelizable.*

The paper is organized as follows.

In § 1, we obtain a refinement of a result in [Cohen-Peterson84] and prove the "only if" part.

In § 2, in order to prove the "if" part, we construct a stably parallelizable smooth manifold which satisfies the condition 1 and 2 above. Since this manifold is stably parallelizable, it is stably reducible [Atiyah61]. Since it has only one

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top dimensional cell $e^{a_1+a_2}$, it retracts stably to $Y = S^{a_1} \cup e^{a_2}$ and the "if" part is proved.

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1. Proof of the necessary conditions

In this section, we extend a result in [Cohen-Peterson84] and prove the "only if" part of the Main theorem.

Suppose a three-cell complex X satisfies the three conditions 1, 2, and 3. Let $ret: \sum^r X \to \sum^r Y$ be the given retraction and let $\overline{ret}: X \to \Omega^r \sum^r Y \to QY$ be its adjoint, where $QY = \underset{\overrightarrow{n}}{\operatorname{colim}}\, \Omega^n \sum^n Y$. In $H_*(X; \mathbf{F}_2)$ let x_{a_1}, x_{a_2} and $x_{a_1+a_2}$ be the duals of u_{a_1}, u_{a_2}, and $u_{a_1} \cdot u_{a_2}$. We denote $\tilde{x}_{a_i} = \overline{ret}_* (x_{a_i})$ for $i = 1, 2$. Since $\overline{ret}_*: H_*(X; \mathbf{F}_2) \to H_*(QY; \mathbf{F}_2)$ is a map of coalgebras over the Steenrod algebra, $\overline{ret}_* (x_{a_1+a_2}) - \tilde{x}_{a_1} \cdot \tilde{x}_{a_2}$ is primitive in $H_*(QY; \mathbf{F}_2)$. It is well-known that $H_*(QY; \mathbf{F}_2)$ is a primitively generated free commutative Hopf algebra generated by $\{Q_I(x) | x \in \tilde{H}_*(Y; \mathbf{F}_2)\}$, where $Q_I(x)$ is the iterated Dyer-Lashof homology operation on x for an admissible sequence I. Thus

$$\overline{ret}_* (x_{a_1+a_2}) = \tilde{x}_{a_1} \cdot \tilde{x}_{a_2} + \sum \alpha_I Q_I(\tilde{x}_{a_1}) + \sum \alpha_J Q_J(\tilde{x}_{a_2})$$

By degree reason, there is only one possible homology operation appearing in this equation, i.e. $Q_{a_2-a_1}(\tilde{x}_{a_1})$. Thus

$$\overline{ret}_* (x_{a_1+a_2}) = \tilde{x}_{a_1} \cdot \tilde{x}_{a_2} + \alpha Q_{a_2-a_1}(\tilde{x}_{a_1})$$

for some $\alpha \in \mathbf{F}_2$. In order to determine α and a relation between a_1 and a_2, we apply Sq_*^n to this equation and get

$$Sq_*^n\, \overline{ret}_* (x_{a_1+a_2}) = Sq_*^n(\tilde{x}_{a_1} \cdot \tilde{x}_{a_2}) + \alpha Sq_*^n Q_{a_2-a_1}(\tilde{x}_{a_1}) \tag{1}$$

$Sq_*^n\, \overline{ret}_* (x_{a_1+a_2}) = \overline{ret}_*\, Sq_*^n(x_{a_1+a_2}) = 0$ and by Cartan formula $Sq_*^n(\tilde{x}_{a_1} \cdot \tilde{x}_{a_2}) = \tilde{x}_{a_1} \cdot Sq_*^n(\tilde{x}_{a_2})$. The last term of (1) can be computed by the Nishida relation [Nishida68].

Theorem 1.1 (Nishida). *For any space Z, in $H_*(QZ; \mathbf{F}_2)$, the following relations hold for any $x \in H_*(QZ; \mathbf{F}_2)$ and integers r and i.*

$$Sq_*^r Q_i(x) = \sum_j \binom{i+k-r}{r-2j} Q_{i-r+2j} Sq_*^j(x)$$

where $k = \deg x$.

In our case

$$Sq_*^n Q_{a_2-a_1}(\tilde{x}_{a_1}) = \binom{a_n - n}{n} Q_{a_2-a_1-n}(\tilde{x}_{a_1}).$$

Therefore

$$0 = \tilde{x}_{a_1} \cdot Sq_*^n(\tilde{x}_{a_2}) + \alpha \binom{a_2 - n}{n} Q_{a_2 - a_1 - n}(\tilde{x}_{a_1})$$

Since $Q_0(x) = x^2$ for any element x,

$$\alpha \binom{a_2 - n}{n} \equiv 0 \bmod 2 \qquad \text{if } n < a_2 - a_1$$

$$\alpha \binom{a_1}{a_2 - a_1} \equiv 1 \bmod 2 \qquad \text{if } n = a_2 - a_1$$

Thus $\alpha \equiv 1 \bmod 2$ and $\binom{a_1}{a_2 - a_1} \equiv 1 \bmod 2$. By the non-exsistence of the Hopf invariant one elements [Adams60], the condition **SR2** implies that $a_2 - a_1 = 2^t$ for $t = 0, 1, 2,$ or 3. Therefore

$$\binom{a_1 + 2^t - n}{n} \equiv 0 \bmod 2 \qquad \text{if } n < 2^t$$

$$\binom{a_1}{2^t} \equiv 1 \bmod 2$$

The next lemma follows from an elementary calculation of binomial coefficient.

Lemma 1.2. *If* $\binom{a - n}{n} \equiv 0 \bmod 2$ *for any* $n < 2^k$ *then* $a \equiv -1 \bmod 2^k$.

With this lemma, we obtain $a_1 \equiv -1 \bmod 2^t$.

2. Proof of the sufficient conditions and corollary

In this section we prove the exsistence of a three-cell complex satisfying **SR1**, **SR2** and **SR3** described in Introduction.

When $t = 0, 1,$ and 2, let $X = St_2(\mathbf{F}^{(a_1+d+1)/d})$ for $\mathbf{F} = \mathbf{R}$, $\mathbf{C}$ and $\mathbf{H}$, respectively. As stated in Introduction, X satisfies **SR1** and **SR2**. And it is well-known that $St_2(\mathbf{F}^{(a_1+d+1)/d})$ stably retracts to the truncated quasi-projective space $Q_{(a_1+d+1)/d,2} = S^{a_1} \cup e^{a_1+d}$ [James76]. Hence the cases $t = 0, 1,$ and 2 are proved.

When $t = 3$, by assumption, $a_1 \equiv -1 \bmod 2^4$. So we can write $a_1 = 16n - 1$ and $a_2 = 16n - 7$ for some $n \geq 1$. We shall construct X as a total space of a unit sphere bundle of a $16n$-dimensional real orientable vector bundle over S^{a_2}.

For a homotopy class $[f] \in \pi_{16n+7}(BSO(16n))$, let X_f be the total space of the pull-back of the universal linear S^{16n-1}-bundle over S^{16n+7} by f.

Proposition 2.1. *There exists* $[f] \in \pi_{16n+7}(BSO(16n))$ *satisfying*

$$H^*(X_f; \mathbf{F}_2) \cong \Lambda(u_{16n-1}, u_{16n+7}), \qquad Sq^8 u_{16n-1} = u_{16n+7}.$$

Proof. It is obvious that, for an arbitrary element $[f] \in \pi_{16n+7}(BSO(16n))$,

$$H^*(X_f; \mathbf{F}_2) \cong \Lambda(u_{16n-1}, u_{16n+7})$$

as rings. In order to prove the nontriviality of the action of S^{q8}, we need the following results.

Proposition 2.2 ([Barratt-Mahowald]). *If* $l < 2k - 1$ *and* $k \geq 13$, *then*

$$\pi_l(BSO(k)) \cong \pi_l(BSO) \oplus \pi_l(SO(2k)/SO(k))\,.$$

Proposition 2.3 ([Hoo-Mahowald]). *If* $k \equiv -1 \bmod 16$, *then*

$$\pi_{k+7}(SO(2k)/SO(k)) \cong \mathbf{Z}/2\mathbf{Z}$$

$$\pi_{k+7}(SO(2k)/SO(k+1)) \cong \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$$

$$\pi_{k+6}(SO(2k)/SO(k)) \cong 0$$

Now consider the following diagram

$$\begin{array}{ccccc}
 & & & & BSO(k) \\
 & & & \overset{i}{\nearrow} & \uparrow \\
 & & & & SO(2k)/SO(k) \times SO(k) \\
 & & & & \uparrow \\
S^k & \xrightarrow{=} & SO(k+1)/SO(k) & \xrightarrow{i'} & SO(2k)/SO(k)
\end{array}$$

Let $k = 16n - 1$ and apply $\pi_*(-)$ to this diagram to get

$$\begin{array}{ccccc}
\pi_{16n+7}(BSO(16n)) & \xrightarrow{\partial} & \pi_{16n+6}(S^{16n-1}) & \xrightarrow{i_*} & \pi_{16n+6}(BSO(16n-1)) \\
 & & \uparrow & & \uparrow \cong \\
 & & & & \pi_{16n+6}(BSO) \oplus \pi_{16n+6}(SO(32n-2)/SO(16n-1) \\
 & & & & \uparrow \cong \\
 & & \pi_{16n+6}(S^{16n-1}) & \xrightarrow{i'_*} & \pi_{16n+6}(SO(32n-2)/SO(16n-1))
\end{array}$$

Note that $\pi_{16n+6}(BSO) = 0$. We claim that $i'_* = 0$ in this diagram. To prove this, consider the homotopy exact sequence for the fibration

$$S^k \xrightarrow{i'} SO(2k)/SO(k) \xrightarrow{p} SO(2k)/SO(k+1)$$

$$\cdots \to \pi_{k+7}(S^k) \xrightarrow{i'_*} \pi_{k+7}(SO(2k)/SO(k)) \xrightarrow{p_*} \pi_{k+7}(SO(2k)/SO(k+1))$$

$$\to \pi_{k+6}(S^k) \to \pi_{k+6}(SO(2k)/SO(k)) \to \cdots \tag{2}$$

When $k \equiv -1 \bmod 16$, by Proposition 2.3,

$$\cdots \to \pi_{k+7}(S^k) \xrightarrow{i'_*} \mathbf{Z}/2\mathbf{Z} \xrightarrow{p_*} \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z} \to \pi_{k+6}(S^k) \to 0$$

By a classical calculation of Toda's, $\pi_{k+6}(S^k) \cong \mathbf{Z}/2\mathbf{Z}$ in the stable range, $k > 6$. Since $k = 16n - 1$ and $n \geq 1$, p_* is monic and hence $i'_* = 0$.

Let $\sigma\colon S^{15} \to S^8$ be the Hopf map. We use the same notation to denote its suspensions in the homotopy groups of spheres, $\sigma \in \pi_{k+7}(S^k)$. By the above argument, $i'_*(\sigma) = 0$ thus $i_*(\sigma) = 0$ in the above diagram. Since the top row of this diagram is exact, there exists $[f] \in \pi_{16n+7}(BSO(16n))$ with $\partial[f] = \sigma$. Therefore $\partial(1) = \sigma$ in the top row of the following diagram, where 1 denotes the identity map on S^{16n+7}.

$$\begin{array}{ccccccc}
\cdots \to & \pi_{16n+7}(X_f) & \to & \pi_{16n+7}(S^{16n+7}) & \xrightarrow{\partial} & \pi_{16n+6}(S^{16n-1}) & \to \cdots \\
& \downarrow & & \downarrow f_* & & \| & \\
\cdots & \pi_{16n+7}(BSO(16n-1)) & \to & \pi_{16n+7}(BSO(16n)) & \xrightarrow{\partial} & \pi_{16n+6}(S^{16n-1}) & \to \cdots
\end{array}$$

Since $\partial(1)$ is the attaching map of e^{16n+7} to S^{16n-1} in X_f, it follows that $Sq^8 u_{16n-1} = u_{16n+7}$ in $H^*(X_f; \mathbf{F}_2) \cong \Lambda(u_{16n-1}, u_{16n+7})$. This completes the proof of Proposition 2.1.

In order to finish the proof of the main theorem, it remains to show that X_f in Proposition 2.1 retracts stably to its $(16n+7)$-skeleton. From the construction, it is clear that X_f has a homotopy type of a smooth manifold. Thus it is enough to prove that it is stably parallelizable. This follows from

Lemma 2.4. $\widetilde{KO}(X_f) = \widetilde{KO}^0(X_f) = 0$.

Proof. The E_2-term of the Atiyah-Hirzebruch-Serre spectral sequence for the fibration $S^{16n-1} \to X_f \to S^{16n+7}$ is

$$E_2 = H^*(S^{16n+7}; \widetilde{KO}^*(S^{16n-1}))$$

By degree reason, $E_2^{s,t} = 0$ if $s + t = 0$. Hence $\widetilde{KO}^0(X_f) = 0$.

Proof of Corollary. The mod 2 cohomology ring of E_6/F_4 is well-known [Araki61].

$$H^*(E_6/F_4; \mathbf{F}_2) \cong \Lambda(x_9, x_{17})$$

and $Sq^8 x_9 = x_{17}$. But $9 \not\equiv -1 \bmod 16$. By theorem, therefore, E_6/F_4 does not stably retracts onto its 17-skeleton. Since the top cell does not split off, E_6/F_4 is not stably parallelizable.

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References

[Adams60] J. F. Adams, On the non-existece of Hopf invariant one, Ann. of Math., **72** (1960), 20–104.

[Araki61] S. Araki, Cohomology modulo 2 of the compact exceptional groups E_6 and E_7, J. of Math. Osaka City Univ., **12** (1961), 43–65.

[Atiyah61] M. F. Atiyah, Thom complexes, Proc. London Math. Soc. (3), **11** (1961), 291–310.

[Barratt-Mahowald] H. G. Barratt and M. E. Mahowald, The metastable homotopy of $O(n)$, Bull. A.M.S., **70** (1964), 758–760.

[Cohen-Peterson84] F. R. Cohen and F. P. Peterson, Suspensions of Stiefel manifolds, Quart. J. of Math. Oxford (2), **35** (1984), 115–119.

[Hoo-Mahowald] C. S. Hoo and M. E. Mahowald, Some homotopy groups of Stiefel manifolds, Bull. A.M.S., **71** (1965), 661–667.

[James76] I. M. James, The topology of Stiefel manifolds, London Mathematical Society Lecture Notes Series 24, Cambridge University Press, Cambrdige, 1976.

[Nishida68] G. Nishida, Cohomology operations in iterated loop spaces, Proc. Japan Acad., **44** (1968), 104–109.