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Nilpotency of a kernel of the Quillen map

By

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1. Introduction

Let G be a finite group, p a prime and $\mathscr{A}$ the set of all elementary abelian p subgroups of G. In [Q1], Quillen considered the natural map

$$\Phi: \mathbf{H}^*(G; \mathbf{Z}/p) \to \bigoplus_{A \in \mathscr{A}} \mathbf{H}^*(A; \mathbf{Z}/p)$$

and showed that $\ker \Phi \subset \sqrt{0}$.

In this paper we considered a certain central extension of groups

$$0 \to (\mathbf{Z}/2)^n \to G \to (\mathbf{Z}/2)^n \to 1$$

for any positive integer n and showed that there is an element $t_n \in \mathbf{H}^1(G)$ satisfying

1. $t_n \in \ker \Phi$
2. $t_n{}^n \neq 0$ and $t_n{}^{n+1} = 0$
3. $t_n{}^n \in \ker \left(\mathbf{H}^*(G) \to \bigoplus_{H \in \mathscr{H}} \mathbf{H}^*(H) \right)$,

where $\mathscr{H}$ denotes the set of all proper subgroups of G.

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Throughout this paper, all coefficient fields of cohomology groups are $\mathbf{Z}/2$ except commented otherwise.

2. Cohomology algebra of a certain group

Given an positive integer n, let V and W be isomorphic to $(\mathbf{Z}/2)^n$, whose bases are $\{e_i\}$ and $\{e_i'\}$. It is well known that $\mathbf{H}^*(\mathbf{Z}/2) \cong \mathbf{Z}/2[t]$, where $t \in \mathbf{H}^1(\mathbf{Z}/2)$ is the dual basis of $\mathbf{Z}/2$. Hence cohomology algebras of V and W are $\mathbf{Z}/2[s_1, \dots, s_n]$ and $\mathbf{Z}/2[t_1, \dots, t_n]$, where $\{s_i\}$ and $\{t_i\}$ are dual bases of $\{e_i\}$ and $\{e_i'\}$.

We consider the following central extension of groups.

$$0 \to V \to G \to W \to 1\,, \tag{2.1}$$

which is classified by $\sum (t_i{}^2 + t_i t_{i-1}) \otimes e_i \in \mathbf{H}^2(W; V)$. Then the transgression is

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determined by $\tau(s_i) = t_i^2 + t_i t_{i-1}$ and $\tau(s_1) = t_1^2$ in the Serre spectral sequence associated with this central extension.

The E_2 term of the Serre spectral sequence is a tensor product of cohomology algebras of V and W. We require the following lemma to obtain its E_3 term.

Lemma 2.1. *$\{t_1^2, t_2^2 + t_2 t_1, \ldots, t_n^2 + t_n t_{n-1}\}$ is a regular sequence. Therefore the E_3 term of the Serre spectral sequence is obtained as follows.*

$$E_3 \cong \mathbf{Z}/2[s_1^2, \ldots, s_n^2] \otimes \mathbf{Z}/2[t_1, \ldots, t_n]/(\tau(s_1), \ldots, \tau(s_n)) .$$

Proof. Multiplying t_1^2 in $\mathbf{Z}/2[t_1, \ldots, t_n]$ is obviously a monomorphism. If we take an element

$$\begin{aligned} f &\in \mathbf{Z}/2[t_1, \ldots, t_n]/(t_1^2, \ldots, t_i^2 + t_i t_{i-1}) \\ &\cong \mathbf{Z}/2[t_1, \ldots, t_i]/(t_1^2, \ldots, t_i^2 + t_i t_{i-1}) \otimes \mathbf{Z}/2[t_{i+1}, \ldots, t_n] \end{aligned}$$

to be $(t_{i+1}^2 + t_{i+1} t_i) \cdot f = 0$, we can conclude that f equals to zero by considering the highest degree of t_{i+1}. Q.E.D.

The E_3 term of the Serre spectral sequence of our group extension (2.1) is determined as above. Indeed, this E_3 term is isomorphic to the E_∞ term.

Lemma 2.2. *In the E_3 term of the Serre spectral sequence, $d_3(s_i^2 \otimes 1) = 0$. Thus the spectral sequence is collapsed at this term.*

Proof. $\{s_i^2 \otimes 1\}$ generate $E_3^{0,*}$ and these elements are $\{Sq^1(s_i)\}$ in $\mathbf{H}^*(V)$. Therefore,

$$d_3(s_i^2 \otimes 1) = \tau(Sq^1(s_i)) = Sq^1(\tau(s_i)) = \begin{cases} 0 & i = 1 , \\ 1 \otimes t_i t_{i-1}(t_i + t_{i-1}) & \text{otherwise} . \end{cases}$$

$1 \otimes t_i t_{i-1}(t_i + t_{i-1}) = (1 \otimes t_{i-1})\tau(s_i)$ are equal to zero in the E_3 term. It is easy to say $d_r(s_i^2 \otimes 1) = 0$ for $r > 3$ because of a degree reason. Q.E.D.

Since all relations in the E_∞ term are those of $\mathbf{H}^*(W)$, the E_∞ term is isomorphic to $\mathbf{H}^*(G)$.

Theorem 2.3. *Cohomology algebra of G is isomorphic to the following algebra.*

$$\mathbf{Z}/2[\alpha_1, \ldots, \alpha_n] \otimes \mathbf{Z}/2[t_1, \ldots, t_n]/(t_1^2, t_2^2 + t_2 t_1, \ldots, t_n^2 + t_n t_{n-1}) ,$$

where α_i is projected to s_i^2 under the induced homomorphism of the inclusion of groups $V \to G$, and t_i is obtained from the cohomology group of W under the induced homomorphism of the projection of groups $G \to W$.

α_i's in the cohomology of G are indeed obtained by means of real representations of G. We now consider following commutative diagrams of central extentions of groups.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V & \longrightarrow & G & \longrightarrow & W & \longrightarrow & 1 \\ & & \downarrow & & \downarrow{\scriptstyle \pi_1} & & \downarrow & & \\ 0 & \longrightarrow & \mathbf{Z}/2\{e_1\} & \longrightarrow & \mathbf{Z}/4 & \longrightarrow & \mathbf{Z}/2\{e'_1\} & \longrightarrow & 0 \end{array} \qquad (2.2)$$

and

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V & \longrightarrow & G & \longrightarrow & W & \longrightarrow & 1 \\ & & \downarrow & & \downarrow{\scriptstyle \pi_i} & & \downarrow & & \\ 0 & \longrightarrow & \mathbf{Z}/2\{e_i\} & \longrightarrow & D_4 & \longrightarrow & \mathbf{Z}/2\{e_i', e_{i-1}'\} & \longrightarrow & 1\,, \end{array} \tag{2.3}$$

where D_4 is a dihedral group of order 8. Then we can obtain elements α_i as the Euler classes of real representations:

$$\rho_1 = i \circ \pi_1 \colon G \to \mathbf{Z}/4 \to SO(2)$$

$$\rho_i = j \circ \pi_i \colon G \to D_4 \to O(2)\,,$$

where two inclusions i and j are defined as usual:

$$\begin{aligned} i \colon \mathbf{Z}/4\{g\} &\hookrightarrow U(1) \hookrightarrow SO(2) \\ g &\mapsto \sqrt{-1} \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} j \colon D_4\{g, h \colon g^4 = 1, h^2 = 1, (gh)^2 = 1\} &\hookrightarrow O(2) \\ g &\mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ h &\mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned} \tag{2.5}$$

Commutative diagrams (2.2, 2.3) of central extensions are classified by the following elements.

$$\mathbf{H}^2(\mathbf{Z}/2\{e_1'\}; \mathbf{Z}/2\{e_1\}) \to \mathbf{H}^2(W; \mathbf{Z}/2\{e_1\}) \leftarrow \mathbf{H}^2(W; V)$$

$$t_1{}^2 \otimes e_1 \mapsto t_1{}^2 \otimes e_1 \leftarrow\!\shortmid \sum (t_i^2 + t_i t_{i-1}) \otimes e_i$$

and

$$\mathbf{H}^2(\mathbf{Z}/2\{e_i', e_{i-1}'\}; \mathbf{Z}/2\{e_i\}) \to \mathbf{H}^2(W; \mathbf{Z}/2\{e_i\}) \leftarrow \mathbf{H}^2(W; V)$$

$$(t_i{}^2 + t_i t_{i-1}) \otimes e_i \mapsto (t_i{}^2 + t_i t_{i-1}) \otimes e_i \leftarrow\!\shortmid \Sigma\, (t_i{}^2 + t_i t_{i-1}) \otimes e_i\,.$$

Proposition 2.4. *We can define α_i as $\rho_i{}^*(w_2)$. And since $\rho_i{}^*(w_1) = t_{i-1}$, the Steenrod operations over the cohomology algebra $\mathbf{H}^*(G)$ are completely determined by Wu formula as follows (see* [B]*).*

$$Sq^1(\alpha_i) = \alpha_i t_{i-1}$$

$$Sq^2(\alpha_i) = \alpha_i{}^2$$

$$Sq^1(t_i) = t_i{}^2\,.$$

Proof. Consider the following commutative diagrams of central extensions.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{Z}/2\{e_1\} & \longrightarrow & \mathbf{Z}/4 & \longrightarrow & \mathbf{Z}/2\{e'_1\} & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle i} & & \downarrow & & \\ 0 & \longrightarrow & \mathbf{Z}/2 & \longrightarrow & SO(2) & \xrightarrow{\text{double}} & SO(2) & \longrightarrow & 0 \end{array} \tag{2.6}$$

and

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{Z}/2\{e_i\} & \longrightarrow & D_4 & \longrightarrow & \mathbf{Z}/2\{e'_i, e'_{i-1}\} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow{\scriptstyle j} & & \downarrow & & \\ 1 & \longrightarrow & SO(2) & \longrightarrow & O(2) & \xrightarrow{\Delta} & \mathbf{Z}/2\{u\} & \longrightarrow & 1\,. \end{array} \tag{2.7}$$

In the former diagram (2.6), "double" means a double folding homomorphism which maps a matrix A to A^2. In the latter diagram (2.7), homomorphisms are defined as follows.

$$\begin{array}{ccccccccc} 0 & \rightarrow & \mathbf{Z}/2\{e_i\} & \rightarrow & D_4 & \rightarrow & \mathbf{Z}/2\{e'_i, e'_{i-1}\} & \rightarrow & 1 \\ & & e_i & \mapsto & g^2 & & & & \\ & & & & g & \mapsto & e'_i & & \\ & & & & h & \mapsto & e'_{i-1} & & \end{array} \tag{2.8}$$

and

$$\begin{array}{ccccc} \mathbf{Z}/2\{e'_i, e'_{i-1}\} & \ni & e'_i & & e'_{i-1} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{Z}/2\{u\} & \ni & 0 & & u\;. \end{array} \tag{2.9}$$

The former exact sequence (2.8) follows from the following diagram of central extensions.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{Z}/2\{e_i\} & \longrightarrow & \mathbf{Z}/4 & \longrightarrow & \mathbf{Z}/2\{e'_i\} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbf{Z}/2\{e_i\} & \longrightarrow & D_4 & \longrightarrow & \mathbf{Z}/2\{e'_i, e'_{i-1}\} & \longrightarrow & 1\,, \end{array} \tag{2.10}$$

which is classified by the following elements.

$$\mathbf{H}^2(\mathbf{Z}/2\{e'_i, e'_{i-1}\}; \mathbf{Z}/2\{e_i\}) \rightarrow \mathbf{H}^2(\mathbf{Z}/2\{e'_i\}; \mathbf{Z}/2\{e_i\})$$

$$(t_i^2 + t_i t_{i-1}) \otimes e_i \mapsto t_i^2 \otimes e_i\,.$$

The latter (2.9) follows from the following diagram.

$$
\begin{array}{ccccccc}
D_4 & \xrightarrow{j} & O(2) & \xrightarrow{\Delta} & \{+1, -1\} & \cong & \mathbf{Z}/2\{u\} \\
g & \mapsto & \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & \mapsto & +1 & \mapsto & 0 \\
h & \mapsto & \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \mapsto & -1 & \mapsto & u,
\end{array}
$$

where Δ maps a matrix to its determinant.

Then the proposition follows immediately, because it is well known that $w_2 \in \mathbf{H}^*(BO(2))$ maps to s_i^2 under the induced homomorphism of the inclusion $\mathbf{Z}/2\{e_i\} \hookrightarrow U(1) \cong SO(2) \hookrightarrow O(2)$. In case of ρ_1, $\rho_1{}^*(w_1) = 0$ because ρ_1 is factored through $SO(2)$. On the other hand, we can conclude that $\rho_i{}^*(w_1) = t_{i-1}$ because t_{i-1} maps to w_1 under the determinant homomorphism $\Delta\colon O(2) \to \mathbf{Z}/2\{e'_{i-1}\}$. These arguments also determine the Steenrod square operations for the fact that $Sq^1(w_2) = w_1 w_2$ in $\mathbf{H}^*(BO(2))$. Q.E.D.

3. Nilpotency of a kernel of the Quillen map

We defined a group G in the previous section. This group has the following properties. For brevity's sake, let $\mathscr{H}$ denote the set of all proper subgroups of G and $\mathscr{A}$ denote the set of all elementary abelian subgroups of G.

Theorem 3.1. 1. $\ker\left(\mathbf{H}^*(G) \to \bigoplus_{A \in \mathscr{A}} \mathbf{H}^*(A)\right)$ *contains an element of height* n *nilpotency.*

2. $\ker\left(\mathbf{H}^*(G) \to \bigoplus_{H \in \mathscr{H}} \mathbf{H}^*(H)\right)$ *is not trivial.*

To prove this theorem, we consider the following properties of G.

Lemma 3.2. *V is exactly the center of G.*

Proof. Since we defined G by a central extension (2.1), V is off course included by $Z(G)$, the center of G. Now recall diagrams (2.2, 2.3) in order to prove the inverse inclusion. But for our use, we have to modify and merge diagrams for π_1 and π_2 as follows.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & V & \longrightarrow & G & \longrightarrow & W & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow{\scriptstyle \pi_{1,2}} & & \downarrow & & \\
0 & \longrightarrow & \mathbf{Z}/2\{e_1, e_2\} & \longrightarrow & P & \longrightarrow & \mathbf{Z}/2\{e'_1, e'_2\} & \longrightarrow & 1.
\end{array}
$$

This diagram of central extensions is classified by following elements.

$$
\mathbf{H}^2(\mathbf{Z}/2\{e'_1, e'_2\}; \mathbf{Z}/2\{e_1, e_2\}) \to \mathbf{H}^2(W; \mathbf{Z}/2\{e_1, e_2\}) \leftarrow \mathbf{H}^2(W; V)
$$

$$
t_1{}^2 \otimes e_1 + (t_2{}^2 + t_2 t_1) \otimes e_2 \mapsto t_1{}^2 \otimes e_1 + (t_2{}^2 + t_2 t_1) \otimes e_2 \leftarrow\!\shortmid \sum (t_i{}^2 + t_i t_{i-1}) \otimes e_i .
$$

Since the center of a dihedral group D_4 is $\mathbf{Z}/2$ and $\pi_i\colon G \to D_4$ is an epimorphism, a restriction of this projection map to $Z(G)$ is a map from $Z(G)$ to $\mathbf{Z}/2\{e_i\}$. We must determine the center of P in order to carry out the same method for $\mathbf{Z}/2\{e_1, e_2\}$. For this purpose, we need the following diagram of central extensions.

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathbf{Z}/2 & = & \mathbf{Z}/2 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathbf{Z}/2\{e_1, e_2\} & \longrightarrow & P & \longrightarrow & \mathbf{Z}/2\{e_1', e_2'\} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathbf{Z}/2\{e_2\} & \longrightarrow & D_4 & \longrightarrow & \mathbf{Z}/2\{e_1', e_2'\} & \longrightarrow & 1\,. \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 1 & & & &
\end{array}
$$

Since the center of D_4 is $\mathbf{Z}/2$ and a double folding homomorphism from P to D_4 is an epimorphism, the center of P is exactly $\mathbf{Z}/2\{e_1, e_2\}$.

Now we define a map k:

$$k\colon G \to P \times \prod_{i>2} D_4$$

$$x \mapsto (\pi_{1,2}(x), \pi_3(x), \dots, \pi_n(x))\,.$$

This map k is indeed a monomorphism.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & V & \longrightarrow & G & \longrightarrow & W & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle k} & & \downarrow & & \\
0 & \longrightarrow & \prod_{i\geq 1} \mathbf{Z}/2\{e_i\} & \longrightarrow & P \times \prod_{i>2} D_4 & \longrightarrow & \prod_{i>1} \mathbf{Z}/2\{e_i', e_{i-1}'\} & \longrightarrow & 1\,.
\end{array}
$$

In the above diagram of central extensions, left vertical arrow is an isomorphism and right vertical arrow is a monomorphism owing to their definitions. Then the middle one is also a monomorphism by means of a snake lemma.

k maps $Z(G)$ one-to-one into $Z(P) \times \prod_{i>2} Z(D_4)$ because of its definition. Therefore V includes $Z(G)$ because:

$$Z(G) \overset{k}{\subset} Z(P) \times \prod_{i>2} Z(D_4) \cong \mathbf{Z}/2\{e_1, e_2\} \times \prod_{i>2} \mathbf{Z}/2\{e_i\} \cong V.$$

We can conclude here that the center of G is exactly V. Q.E.D.

Lemma 3.3. *V is a unique maximal elementary abelian subgroup of G.*

Proof. Suppose G has a maximal elementary abelian subgroup A. Without loss of generality, A can be assumed including V, because all elements of V are commutative with all elements of G. Then the following diagram holds.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V & \longrightarrow & A & \longrightarrow & A/V & \longrightarrow & 0 \\ & & \| & & \cap & & \downarrow & & \\ 0 & \longrightarrow & V & \longrightarrow & G & \longrightarrow & W & \longrightarrow & 1. \end{array}$$

Right vertical arrow is a monomorphism by means of a snake lemma. This fact implies that there is at least one $t_i \in \mathbf{H}^1(W)$ which dose not equal to zero in a cohomology of A/V.

Since A is a direct product of V and A/V, the transgression is trivial in the Serre spectral sequence converging to a cohomology of A:

$$\tau(s_1) = t_1{}^2 = 0, \quad \tau(s_2) = t_2{}^2 + t_2 t_1 = 0, \ \ldots, \ \tau(s_n) = t_n{}^2 + t_n t_{n-1} = 0.$$

Hence

$$t_1 = t_2 = \cdots = t_n = 0,$$

because $\mathbf{H}^*(A/V)$ is an integral domain. If $A \neq V$, this contradicts the fact for the cohomology of A/V stated above. Q.E.D.

Remark. We have $\sqrt{0} = \mathrm{Im}\,\{\mathbf{H}^+(W) \to \mathbf{H}^+(G)\}$ and $\mathbf{H}^*(G)/\sqrt{0} = \mathbf{Z}/2[\alpha_1, \ldots, \alpha_n]$. Hence, because of [Q1], $\sqrt{0}$ corresponds to the conjugacy class of maximal elementary abelian subgroups, which is obviously V. On the other hand, since V is the center of G, V is actually fixed by the conjugation.

We have proved the former part of the theorem 3.1 because t_n has such a property that we want.

Lemma 3.4.

$$t_n \in \ker\left(\mathbf{H}^*(G) \to \bigoplus_{A \in \mathscr{A}} \mathbf{H}^*(A)\right), \qquad t_n{}^n \neq 0, \qquad t_n{}^{n+1} = 0.$$

Proof. Lemma 3.2 means

$$\ker\left(\mathbf{H}^*(G) \to \bigoplus_{A \in \mathscr{A}} \mathbf{H}^*(A)\right) \cong \ker(\mathbf{H}^*(G) \to \mathbf{H}^*(V))$$

$$\cong \mathbf{Z}/2[t_1, \ldots, t_n]/(t_1{}^2, t_2{}^2 + t_2 t_1, \ldots, t_n{}^2 + t_n t_{n-1}).$$

$$t_n^2 = t_n t_{n-1} .$$

$$t_n^3 = t_n^2 t_{n-1} = t_n t_{n-1}^2 = t_n t_{n-1} t_{n-2} .$$

$$\vdots$$

$$t_n^n = t_n t_n^{n-1} = t_n^2 t_{n-1} \dots t_2 = t_n t_{n-1} \dots t_2 t_1 .$$

$$t_n^{n+1} = t_n t_n^n = t_n t_{n-1} \dots t_1^2 = 0 .$$

$t_n t_{n-1} \dots t_1 \in \mathbf{Z}/2[t_1, \dots, t_n]/(t_1^2, t_2^2 + t_2 t_1, \dots, t_n^2 + t_n t_{n-1}) \subset \mathbf{H}^n(G)$ is the only one element which does not equal to zero because

$$\mathbf{Z}/2[t_1, \dots, t_n]/(t_1^2, t_2^2 + t_2 t_1, \dots, t_n^2 + t_n t_{n-1}) \cong \Lambda_{\mathbf{z}/2}[t_1, \dots, t_n]$$

as vector spaces over $\mathbf{Z}/2$, which follows from the fact that $\{t_1^2, t_2^2 + t_2 t_1, \dots, t_n^2 + t_n t_{n-1}\}$ is a regular sequence. Q.E.D.

The rest of the theorem is proved as follows.

Proof of the theorem (3.1). For each H, a proper subgroup of G, we consider the following commutative diagram of central extensions.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V & \longrightarrow & G & \longrightarrow & W & \longrightarrow & 1 \\ & & \cup & \text{p.b.} & \cup & & \uparrow{\scriptstyle l} & & \\ 0 & \longrightarrow & V \cap H & \longrightarrow & H & \longrightarrow & H/(V \cap H) & \longrightarrow & 1 , \end{array}$$

where "p.b." means a pullback. Hence the right arrow of the diagram is a monomorphism.

(i) First consider $W \ncong H/(V \cap H)$ case. In this case, $\mathbf{H}^1(W)$ has an element $\varepsilon_1 t_1 + \dots + \varepsilon_{q-1} t_{q-1} + t_q$ that maps to zero under the induced homomorphism of l. Then $t_n t_{n-1} \dots t_1 \in \mathbf{H}^n(G)$ maps to zero in the cohomology $\mathbf{H}^n(H)$.

$$\begin{aligned} t_n t_{n-1} \dots t_1 &= t_n t_{n-1} \dots t_{q+1} t_q t_{q-1} \dots t_1 \\ &= t_n t_{n-1} \dots t_{q+1} (\varepsilon_1 t_1 + \dots + \varepsilon_{q-1} t_{q-1}) t_{q-1} \dots t_1 \\ &= \sum_{i=1}^{q-1} \varepsilon_i t_n t_{n-1} \dots t_{q+1} t_{q-1} \dots t_i^2 \dots t_1 \\ &= \sum_{i=1}^{q-1} \varepsilon_i t_n t_{n-1} \dots t_{q+1} t_{q-1} \dots t_2 t_1^2 \\ &= 0 . \end{aligned}$$

(ii) So we assume l is an isomorphism. Then the transgression of the Serre spectral sequence converging to $\mathbf{H}^*(G)$ is factored through $\mathbf{H}^*(V \cap H)$. But this contradicts the fact that the transgression $\tau\colon \mathbf{H}^1(V) \to \mathbf{H}^2(W)$ is a monomorphism because of its definition.

Therefore we can conclude that $t_n t_{n-1} \dots t_1 \in \bigcap_{H \in \mathscr{H}} \ker(\mathbf{H}^*(G) \to \mathbf{H}^*(H))$.

Q.E.D.

Added in proof: A part of the main theorem of this paper is independently proved by George S. Avrunin and Jon F. Carlson in [A-C], without consideration of the kernel of the Quillen map.

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