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Equations of evolution on the Heisenberg group II

By

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1. Introduction

In [O_2] we have discussed the well-posedness of Cauchy problem for p-parabolic operators on the Heisenberg group. In this paper, we are concerned with the well-posedness for the hyperbolic operators of higher order on the Heisenberg group $\mathbf{H}^n$. Namely, we consider the following operators:

$$P = \partial_t^m + \sum_{j=1}^{m} a_j \partial_t^{m-j},$$

where a_j are the homogeneous right invariant differential operators of order j on $\mathbf{H}^n$. Our question is whether the hyperbolicity of the operator

$$\pi(P(\zeta)) = (i\zeta)^m + \sum_{j=1}^{m} \pi(a_j)(i\zeta)^{m-j}$$

for every non-trivial irreducible unitary representation π on the Heisenberg group implies the well-posedness for the Cauchy problem for P.

For the p-parabolic operators, the corresponding problem has a positive answer. ([O_2]). But for the hyperbolic operator we know that the problem as above has no solution unless some extra conditions are imposed. Our main result says that if Fourier transform of $i^m P\delta$ is strictly hyperbolic of non-degenerate type, $\pi(P(\zeta))$ is hyperbolic type and it satisfies some commutativity relations among its coefficients for every irreducible unitary representation π on the Heisenberg group, then there exist a neighborhood $U \subset \mathbf{H}^n$ of the origin and a positive number T such that for any $f \in C_0^\infty((-T, T) \times U)$ with support in $\{t \geq 0\}$, there is a solution $u(x, t) \in \mathscr{D}'((-T, T) \times U)$ such that

$$\begin{cases} Pu = f \text{ in } (-T, T) \times \mathbf{H}^n \\ \operatorname{supp} u \subset [0, T] \times \mathbf{H}^n . \end{cases} \tag{1.1}$$

We say briefly that the Cauchy problem for P is solvable at the origin if the above property holds.

To make a situation clear, in the last section, we shall give several examples. We note that the idea of the characterization of operators having some property by such implicit conditions goes back to V. V. Grušin's one in the

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study of hypoellipticity. For the wave equation on the Heisenberg group, there are the related works [N] and [T].

2. Statement of results

Recall that the $2n+1$-dimensional Heisenberg algebra h_n is the Lie algebra with generators X_i', X_i'', $i=1, \ldots, n$, X_0 satisfying the commutation relations

$$[X_i', X_j''] = \delta_{i,j} X_0 \,, \qquad [X_i', X_0] = [X_i'', X_0] = 0.$$

The Heisenberg group $\mathbf{H}^n$ is the unique simply-connected Lie group having h_n as its Lie algebra. The group $\mathbf{H}^n$ has a group of dilations $\{\delta_r\}$ defined by

$$\delta_r(X_i') = rX_i' \,, \qquad \delta_r(X_i'') = rX_i'' \,, \qquad \delta_r(X_0) = r^2 X_0 \,.$$

We shall use exponential coordinates

$$(x', x'', x_0) \in \mathbf{R}^{2n+1} \mapsto \exp(x'X' + x''X'' - x_0 X_0) \,.$$

Hereafter, as a set we identify the group $\mathbf{H}^n$ with its corresponding Lie algebra h_n.

Let $\mathbf{H}^n = \mathbf{R}^n \oplus \mathbf{R}^n \oplus \mathbf{R}$ and we let (x', x'', x_0) denote the components of a vector x in $\mathbf{H}^n$. Then, the bracket operation is given by

$$[x, y] = (0, 0, \langle x'', y' \rangle - \langle x', y'' \rangle)$$

and the formula for multiplication is

$$x \cdot y = x + y + \frac{1}{2}[x, y] \,.$$

We consider the operators of evolution on $\mathbf{H}^n$

$$P = \partial_t^m + \textstyle\sum_{j=1}^m a_j \partial_t^{m-j}$$

$$a_j = \textstyle\sum_{\langle I \rangle = j} c_I X^I \,,$$

where $\sqrt{-1}^{|I|} c_I \in \mathbf{R}$, $\langle I \rangle = N +$ (times of 0 appearing in $I = (i_1, \ldots, i_N)$) and the right invariant vector fields

$$X^I = X_{i_1} \ldots X_{i_N}:$$

$$X_0 = \frac{\partial}{\partial x_0} \,, \qquad X_j = \frac{\partial}{\partial x_j'} - \frac{x_j''}{2} \frac{\partial}{\partial x_0} \,, \qquad X_{-j} = \frac{\partial}{\partial x_j''} + \frac{x_j'}{2} \frac{\partial}{\partial x_0} \,, \qquad (j = 1, \ldots, n) \,.$$

There are two classes of irreducible unitary representations, as follows from the Stone-von Neuman theorem:

(1) A family of 1-dimensional representations which map X_0 to 0. They are parametrized by $\xi = (\xi', \xi'') \in \mathbf{R}^{2n}$, and are given by

$$\pi_\xi(x', x'', x_0) = e^{i(x'\xi' + x''\xi'')} \,, \qquad \xi \in \mathbf{R}^{2n},$$

i.e.,

$$\pi_\xi(X_i) = \sqrt{-1}\xi_i' \,, \qquad \pi_\xi(X_{-i}) = \sqrt{-1}\xi_i'' \,, \qquad \pi_\xi(X_0) = 0 \,.$$

(2) A family parametrized by $\lambda \in \mathbf{R}\backslash\{0\}$ acting on $L^2(\mathbf{R}^n)$ which map X_0 to a nonzero scalar. They are given by

$$[\pi_\lambda(x', x'', x_0)v](y) = e^{i\lambda(\langle x'', y\rangle - x_0 + \langle x', x''\rangle/2)}v(y + x') \qquad \text{for } v \in L^2(\mathbf{R}^n),$$

i.e.,

$$\pi_\lambda(X_i) = \frac{\partial}{\partial y_i}, \qquad \pi_\lambda(X_{-i}) = \sqrt{-1}\lambda y_i, \qquad \pi_\lambda(X_0) = \sqrt{-1}\lambda.$$

We introduce the following two generalized symbol of P, according to two family of irreducible representation on $\mathbf{H}^n$: for $\zeta \in \mathbf{C}$, and $\xi \in \mathbf{R}^{2n}$,

$$p_m(\zeta, \xi) = (i\zeta)^m + \sum_1^m \pi_\xi(a_j)(i\zeta)^{m-j}$$

and $\zeta \in \mathbf{C}$ and $\lambda \in \mathbf{R}\backslash 0$,

$$\mathscr{P}(\zeta, \lambda) = (i\zeta)^m + \sum_1^m \pi_\lambda(a_j)(i\zeta)^{m-j}: \mathscr{S}(\mathbf{R}^n) \to \mathscr{S}(\mathbf{R}^n).$$

Denoting the Fourier transfomation of u with respect to the variables t, x by $\hat{u}$,

$$\hat{u} = \int_{\mathbf{R} \times \mathbf{H}^n} e^{i(-\langle t, \zeta\rangle + \langle \xi'', x'\rangle + \langle \xi', x''\rangle - \lambda x_0)}u(t, x)dtdx,$$

Let us denote the dual variables of t, (x', x''), x_0 by τ, $\xi = (\xi', \xi'')$, λ, respectively. We shall assume the following three conditions on these symbols:

(H-1) All roots $\zeta = \zeta_j(\xi)$ of the characteristic equation

$$p_m(\zeta, \xi) = 0$$

are real distinct if $\xi = (\xi', \xi'') \in \mathbf{R}^{2n}\backslash 0$, and if a root ζ_{j_0} takes the value 0 at some point $\xi \neq 0$, then the root ζ_{j_0} is identically zero,

(H-2) For any $\zeta \in \{\zeta \in \mathbf{C}; \operatorname{Im} \zeta < 0\}$ and any λ, $|\lambda| = 1$, the operator $\mathscr{P}(\zeta, \lambda)$ and $\mathscr{P}^*(\zeta, \lambda)$ are injective on the space $\mathscr{S}(\mathbf{R}^n)$ and

(H-3) The symbol $p(\zeta, \xi)$ is real-valued and the commutator

$$[\mathscr{P}(\zeta, \lambda), \partial_\zeta \mathscr{P}(\zeta, \lambda)] = 0$$

if $\zeta \in \mathbf{R}$ and $\lambda = \pm 1$.

Now, we can state our main result.

Theorem 2.1. *Suppose that the conditions* (H-1)–(H-3) *hold. Then, the Cauchy problem for P is solvable at the origin.*

For the well-posedness of (1.1), it is necessary that all the root ζ of $p_m(\zeta, \xi) = 0$ are real, which is almost an immeadiate consequence of Lax-Mizohata theorem. The condition (H-2) is also necessary for (1.1) to be well-posed. This is essentially proved in Theorem 1 of [O_1] (c.f. § 7). On the other hand, the non-zero condition of the roots of $p_m(\zeta, \xi) = 0$, except at most one root, is necessary in our formulation. Without this condition the modified statement is not true. In fact, we shall show that there are operators which satisfy all the other conditions but for which the Cauchy problem is not well-posed.

3. Estimate—elliptic type

We shall use the same notations as in [O_2]. Denoting the Fourier transfomation of u with respect to the variables t, x by $\hat{u}$,

$$\hat{u} = \int_{\mathbf{R}\times\mathbf{H}^n} e^{i(-\langle t,\zeta\rangle + \langle \xi'', x'\rangle + \langle \xi', x''\rangle - \lambda x_0)} u(t, x)\, dt dx ,$$

the symbol $p(\zeta, \xi, \lambda) = \widehat{P\delta}$ is a quasi-homogeneous polynomial in ζ, ξ, and λ of degree m and

$$\mathscr{P}(\zeta, \lambda) = \mathrm{op}_\lambda\,(p(\zeta, \cdot, \lambda)) ,$$

where

$$\mathrm{op}_\lambda\,(a) f(x') = (2\pi)^{-n} \int\int e^{i\langle x'-y', \eta'\rangle} a\left(\zeta, \lambda\left(\frac{x'+y'}{2}\right), \eta'\right) f(y')\, dy' d\eta' .$$

The quasi-homogeneous of p implies

$$\mathscr{P}(\zeta, \lambda) = |\lambda|^{m/2} \mathscr{P}(\zeta/|\lambda|^{1/2}, \pm 1) = |\lambda|^{m/2} \mathscr{P}_\pm(\zeta/|\lambda|^{1/2}) .$$

We recall the some function spaces, introduced in [M]. For a non-negative integer k, B_k will denote the closure of $\mathscr{S}(\mathbf{H}^n)$ in the norm $\|f\|_k = \sum_{|\alpha|\le k} \max |D^\alpha f|$. $\mathscr{M}$ is the space of distributions $u \in \mathscr{S}'(\mathbf{H}^n)$ such that there exists a positive integer k for which $qu \in B'_k$ for every polynomial $q(x)$ on $\mathbf{H}^n$. If $(u_j)_1^\infty$ is a sequence in $\mathscr{M}$ and $u \in \mathscr{M}$, then we say that u_j tends to u on $\mathscr{M}$ if and only if one can find k so that qu_j, $qu \in B'_k$ for all j and $qu_j \to qu$ weakly in B'_k when q is a polynomial.

For $s \in \mathbf{R}$, $\zeta \in \Gamma$ and $\mu > 0$, let us denote by $H_{s,\zeta,\mu}(\mathbf{R}^n)$ the space of distributions $u \in \mathscr{S}'(\mathbf{R}^n)$ for which $\Lambda^s u \in L^2(\mathbf{R}^n)$, where Λ^s means that

$$\Lambda^s u = \{1 + |\zeta|^2 + (|D_x|^2 + |x|^2\mu^2)\}^{s/2} u$$

and we write $\|\Lambda^s u\|_{L^2(\mathbf{R}^n)} = \|u\|_{s,\zeta,\mu}$.

First, we assume a stronger condition (H-1)$'$ than (H-1):

(H-1)$'$ the characteristic equation

$$p_m(\zeta, \xi) = 0$$

has non-zero real distinct roots $\zeta = \zeta_j(\xi)$, $1 \le j \le m$ if $\xi \in \mathbf{R}^{2n}\backslash 0$.

We shall construct a parametrix $\mathrm{op}_\lambda\,(q)$ of $\mathrm{op}_\lambda\,(p) = \mathscr{P}(\zeta, \lambda)$ when $\zeta \in \Gamma_{\theta,\varepsilon}$. Here $\Gamma_{\theta,\varepsilon}$ means

$$\Gamma_{\theta,\varepsilon} = \{\zeta \in \mathbf{C};\ \mathrm{Re}\,\zeta = -\theta\,\mathrm{Im}\,\zeta,\ \mathrm{Im}\,\zeta \le -\varepsilon\} , \qquad \theta \in \mathbf{R} .$$

Lemma 3.1. *Assume* (H-1)$'$ *is satisfied. For any* $\theta \in \mathbf{R}$ *and any positive number* ε, *there are positive constants* C *and* C_α *such that if* $\zeta \in \Gamma_{\theta,\varepsilon}$, $|\lambda| \ge 1$ *and* $\|\xi\|^2 \ge C|\lambda|$, *then*

$$|p(\zeta, \xi, \lambda)| \ge C^{-1}\{|\zeta| + (1 + |\xi| + |\lambda|^{1/2})\}^m$$

and

$$|\partial^{\alpha}_{\xi,\lambda}(1/p)| \le C_{\alpha}\{|\zeta| + (1 + |\xi| + |\lambda|^{1/2})\}^{-|\alpha|}/|p|$$

for any multi-index α.

Proof. Since the principal symbol of p is p_m, the quasi-homogeneity and the condition (H-1)′ yield to the assertions.

Let Γ be a subset of $\mathbf{C}$ and $\mu \in \mathbf{R}$. Then we define the class S^{μ}_{Γ} which consists of the functions $a(\zeta, \xi)$ such that
1) $a(\zeta_0, \xi) \in C^{\infty}(\mathbf{R}^{2n})$ for any fixed $\zeta_0 \in \Gamma$ and
2) for any multi-index α of dimension $2n$, there exists a positive constant C such that

$$|D^{\alpha}_{\xi} a(\zeta, \xi)| \le C\langle\!\langle \xi \rangle\!\rangle^{\mu-|\alpha|}$$

for $\zeta \in \Gamma$, $\xi \in \mathbf{R}^{2n}$, where $\langle\!\langle \xi \rangle\!\rangle = (1 + |\zeta|^2 + |\xi|^2)^{1/2}$.
By the same argument in the section 4 in [O_2], we can construct the parametrix of p. Let $p_{\pm} = p(\zeta, \xi, \pm 1)$.

Proposition 3.2. *Suppose that P satisfies* (H-1)′. *Then for any $\theta \in \mathbf{R}$ and any $\varepsilon > 0$, one can find $q_{\pm} \in S^{-m}_{\Gamma_{\theta,\varepsilon}}$ such that if $\zeta \in \Gamma_{\theta,\varepsilon}$,*

$$p_{\pm} \# q_{\pm} - 1 \in S^{-\infty}_{\Gamma_{\theta,\varepsilon}}, \qquad q_{\pm} \# p_{\pm} - 1 \in S^{-\infty}_{\Gamma_{\theta,\varepsilon}}.$$

Proposition 3.3. *Suppose that* (H-1)′ *is satisfied. Then for any $\theta \in \mathbf{R}$, $\varepsilon > 0$ and $s \in \mathbf{R}$, there is a constant C such that when $\zeta \in \Gamma_{\theta,\varepsilon}$,*

$$\|u\|_{s+m,\zeta,1} \le C\{\|\mathscr{P}_{\pm}(\zeta)u\|_{s,\zeta,1} + \|u\|_{s,\zeta,1}\}$$

for any $u \in \mathscr{S}(\mathbf{R}^n)$.

Proposition 3.4. *Suppose that* (H-1)′ *and* (H-2) *are satisfied. Then, for any $\theta \in \mathbf{R}$, $\varepsilon > 0$ and $s \in \mathbf{R}$, there are constants C independent of ζ and λ such that*

$$\|u\|_{s+m,\zeta,|\lambda|} \le C\|\mathscr{P}(\zeta, \lambda)u\|_{s,\zeta,|\lambda|}$$

and

$$\|u\|_{s+m,\zeta,|\lambda|} \le C\|\mathscr{P}^*(\zeta, \lambda)u\|_{s,\zeta,|\lambda|}$$

for any $u \in \mathscr{S}(\mathbf{R}^n)$, $\zeta \in \Gamma_{\theta,\varepsilon}$ and $|\lambda| \ge 1$.

Proof. By the continuity argument as in the proof of Lemma 5 in [O_2], we have

$$\|u\|_{s+m,\zeta_0,1} \le C\|\mathscr{P}_{\pm}(\zeta_0)u\|_{s,\zeta_0,1}\,.$$

Multiplying both sides of this inequality by $|\lambda|^{s+m}$ and changing the variable y by $\tilde{y}|\lambda|^{1/2}$, we have the desired inequality. As for the adjoint operator, the inequality is obtained similarly.

Hence, for every $\zeta \in \mathbf{C}$, $\zeta \to \mathscr{P}^{-1}_{\pm}$ is analytic with values in $L^2(\mathbf{R}^n)$ except for the real line. But in order to obtain the estimate in $\zeta \in \Gamma_{A,B} = \{\mathrm{Im}\,\zeta \le$

$-B - A\log|\zeta|\}$, where A and B are positive constants, we must investigate the behavior of $\mathscr{P}_{\pm}^{-1}$ when $\operatorname{Im}\zeta$ approaches to zero. To overcome this difficulty, we consider $\mathscr{P}_{\pm}(\zeta)u = 0$ as a nonlinear eigenvalue problem. For the simplicity, we may only consider the operator $\mathscr{P}_{+}(\zeta)$, denoted by $\mathscr{P}(\zeta)$. By Proposition 3.4, we can apply the abstract theory in [LR] to the operator $\mathscr{P}$. In fact, since the operator a_j: $\mathscr{S}(\mathbf{R}^n) \to \mathscr{S}(\mathbf{R}^n)$ is closable, we denote its closure in $L^2(\mathbf{R}^n)$ by A_j.

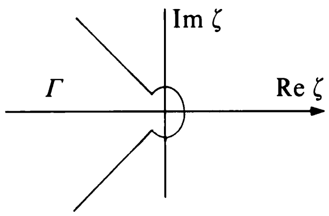
By our hypothesis, $\pi_{\xi}(a_m)$ is elliptic. Thus, from the result in [R], [S], it follows that there is a positive constant τ_0 and a non-zero constant σ such that for $\tilde{A}_m = \sigma A_m + \tau_0$, the domain of the operator $\tilde{A}_m$ is

$$D(\tilde{A}_m) = \{u \in \mathscr{S}'(\mathbf{R}^n);\ x^\alpha D^\beta u \in L^2, \text{ for any } |\alpha| + |\beta| \le m\}$$

which we denote by $\mathscr{H}^m(\mathbf{R}^n)$. By the same argument in the proof of Proposition 3.4, we see that (H-1)′ and (H-2) implies $(\tilde{A}_m - \zeta)^{-1}$ exists and

$$\|(\tilde{A}_m - \zeta)^{-1}\|_{L^2} = O(1/|\zeta|) \qquad \text{for } \zeta \in \Gamma \text{ as } \zeta \to \infty\,,$$

where Γ is an open cone with vertex 0 in $\mathbf{C}$ containing $(-\infty, 0]$ such that



.

Therefore, we can define the fractional powers of $\tilde{A}_m$.

It is seen that

$$\sigma\mathscr{P}(\zeta) + \tau_0 = (1 + \zeta A_{m-1}\tilde{A}_m^{-1} + \cdots + \zeta^{m-1}A_1\tilde{A}_m^{-1} + \zeta^m\tilde{A}_m^{-1})\,\tilde{A}_m.$$

Since $A_j\tilde{A}_m^{-1}$ is a compact operator,

$$\mathscr{P}(\zeta)\tilde{A}_m^{-1} = I + \text{a compact operator and}$$

$$\tilde{A}_m^{-1}\mathscr{P}(\zeta) = I + \text{a compact operator}\,.$$

Hence $\mathscr{P}(\zeta)$ is a Fredholm operator in $L^2(\mathbf{R}^{2n})$ with index 0. Moreover, we have

Lemma 3.5. *$\zeta \mapsto \mathscr{P}(\zeta)^{-1}$ is a meromorphic function in the whole plane* $\mathbf{C}$ *with values in* $\mathscr{L}(L^2(\mathbf{R}^n), D(A_m))$.

Proof. To linearize $\mathscr{P}(\zeta)$ in ζ, we introduce the system

$$\mathscr{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & 1 & & 0 \\ \vdots & \vdots & 0 & \ddots & \\ 0 & 0 & 0 & & 1 \\ -A_m & -A_{m-1} & -A_{m-2} & \cdots & -A_1 \end{bmatrix}.$$

Then, $\mathscr{A}$ is a closed operator in the Hilbert space X

$$X = D(\tilde{A}_m^{(m-1)/m}) \times D(\tilde{A}_m^{(m-2)/m}) \times \cdots \times D(\tilde{A}_m^{1/m}) \times L^2(\mathbf{R}^n)$$

with domain

$$D(\mathscr{A}) = D(\tilde{A}_m) \times D(\tilde{A}_m^{(m-1)/m}) \times \cdots \times D(\tilde{A}_m^{1/m})$$

since the operator $A_j \tilde{A}_m^{-j/m}$ is extendable to a bounded operator on $L^2(\mathbf{R}^n)$ for any j.

It is seen that $\mathscr{A} - \zeta$ is a Fredholm operator with index $= 0$ for any $\zeta \in \mathbf{C}$. Hence the mapping $\zeta \mapsto (\mathscr{A} - \zeta)^{-1}$ is meromorphic in $\mathbf{C}$ with values in $\mathscr{L}(X)$. Denoting $(\mathscr{A} - \zeta)^{-1} = (b_{ij}(\zeta))_{1 \le i,j \le m}$, then $\mathscr{P}^{-1} = -b_{1,m}$. Hence $\mathscr{P}(\zeta)^{-1}$ is also meromorphic in $\mathbf{C}$. From Proposition 3.4, it follows that all the poles $\{\zeta_j\}$ lie on the real line.

$$\mathscr{P}(\zeta)^{-1} = \sum_{k=1}^{r_j} \frac{Q_{j,k}}{(\zeta - \zeta_j)^k} + S(\zeta)\,,$$

where $S(\zeta)$ is holomorphic in a neighborhood of ζ_j and

$$Q_{j,k} = \frac{1}{2\pi i} \int_{|\zeta - \zeta_j| = \varepsilon_j} (\zeta - \zeta_j)^{k-1} \mathscr{P}(\zeta)^{-1} d\zeta\,,$$

where ε_j is a sufficiently small positive number.

We note that $D(A_m) = D(\tilde{A}_m)$ and that if $u \in D(A_m)$ and $A_m u = 0$, then $u \in \mathscr{S}(\mathbf{R}^n)$. We can realize the fractional power $\tilde{A}_m^{-1/m}$ as pseudo-differential operator (c.f. [R]), so that the above argument is also valid if we replace the base space $L^2(\mathbf{R}^n)$ by $\mathscr{H}^k(\mathbf{R}^n)$. ($k$ is a positive integer.) Summing up, we have

Theorem 3.6. *Suppose that* (H-1)$'$ *and* (H-2) *are satified. For any* $R > 0$ *and any* $k \in \mathbf{N}$, *there are constants* C *and* N *such that*

$$\|u\|_{m+k,\tilde{\zeta},1} \le C\tilde{\gamma}^{-N} \|\mathscr{P}_{\pm}(\tilde{\zeta})u\|_{k,\tilde{\zeta},1}$$

if $|\tilde{\zeta}| \le R$ *and* $\operatorname{Im}\tilde{\zeta} = -\tilde{\gamma} < 0$, $\tilde{\zeta} = \zeta/|\lambda|^{1/2}$.

If the condition (H-1) is satisfied instead of (H-1)$'$, $p(\zeta, \xi, \lambda)$ is a polynomial of order $m-1$ with respect to ξ and the terms of total order less than $m-1$ have the order at most $m-2$ with respect to ξ. Moreover, $\zeta^{-1}\pi_\xi(a_{m-1})$ is elliptic and $a_m = 0$. Therefore, the preceding argument, slightly modified, implies the followings results.

Proposition 3.7. *Suppose that* P *satisfies* (H-1). *Then for any* $\theta \in \mathbf{R}$ *and any* $\varepsilon > 0$, *one can find* $q \in S_{\Gamma_{\theta,\varepsilon}}^{-m+1,0}$ *such that if* $\zeta \in \Gamma_{\theta,\varepsilon}$,

$$p \# q - 1 \in A_{\Gamma_{\theta,\varepsilon}}^{0,\infty}\,, \qquad q \# p - 1 \in S_{\Gamma_{\theta,\varepsilon}}^{0,\infty}\,.$$

Theorem 3.8. *Suppose that* (H-1) *and* (H-2) *are satisfied. For any* $R > 0$ *and any* $k \in \mathbf{N}$, *there are constants* C *and* N *such that*

$$\|u\|_{m-1+k,\tilde{\zeta},1} \le C\tilde{\gamma}^{-N} \|\mathscr{P}_{\pm}(\tilde{\zeta})u\|_{k,\tilde{\zeta},1}$$

if $|\tilde{\zeta}| \le R$ *and* $\operatorname{Im}\tilde{\zeta} = -\tilde{\gamma} < 0$, $\tilde{\zeta} = \zeta/|\lambda|^{1/2}$.

4. Estimate—hyperbolic type

In § 3, we have constructed the parametrix of p in the domain $\Gamma_{\theta,\varepsilon}$, which enables us to obtain the estimate in

$$\{\zeta \in \mathbf{C};\ |\zeta|/|\lambda|^{1/2} < R, \operatorname{Im}\zeta < 0\}.$$

In this section, we shall prove an uniform estimate in the complementary domain

$$\{\zeta \in \mathbf{C};\ |\zeta|/|\lambda|^{1/2} \geq R, \operatorname{Im}\zeta < 0\}.$$

More precisely, we work in the domain

$$\Gamma = \{\zeta \in \mathbf{C};\ -\operatorname{Im}\zeta \geq B + A \log|\zeta|\},$$

where A and B are some constants. Then, we have

Theorem 4.1. *For* $\tilde{\zeta} = \zeta/|\lambda|^{1/2}$, $\operatorname{Im}\zeta < 0$, *there are positive constants* R *and* C *such that if* $|\tilde{\zeta}| > R$, *then*

$$\|u\|_{m-1,\tilde{\zeta},1} \leq C\|\mathscr{P}_{\pm}u\|_{s,\tilde{\zeta},1}$$

for any $u \in \mathscr{S}(\mathbf{R}^n)$.

For the proof, we make some preparations. When $\mu \in \mathbf{R}$ and k is a non-negative integer, $S_{\Gamma}^{\mu,k}$ is the space of C^{∞} functions a on $\mathbf{R}^{2n}$ with a parameter $\zeta = \sigma - i\gamma \in \Gamma$ satisfying the following inequalities: for any α, there exist positive constants C_{α} such that

$$|D_{\xi}^{\alpha}a(\zeta,\xi)| \leq C_{\alpha}\langle\!\langle\xi\rangle\!\rangle^{\mu}\gamma^{-2k-|\alpha|}$$

for any $\zeta \in \Gamma_{A,B}$, any $\xi \in \mathbf{R}^{2n}$, and

$$S_{\Gamma}^{\mu,\infty} = \bigcap_{k} S_{\Gamma}^{\mu,k}.$$

We recall the composition formula.

Lemma 4.2. *Assume that* $a \in S_{\Gamma}^{\mu,k}$ *and* $b \in S_{\Gamma}^{\mu,k'}$. *Then* $a \# b \in S_{\Gamma}^{\mu+\mu',k+k'}$ *and*

$$a \# b \sim \sum_{0}^{\infty} \{(i\sigma(D_{\xi}, D_{\eta})/2)^{k}a(\zeta,\xi)b(\zeta,\eta)/k!\}|_{\xi=\eta},$$

where σ *is the bilinear form* $\langle x'', y'\rangle - \langle x', y''\rangle$ *on* $\mathbf{H}^n$.

Proof. Let g be a Riemannian metric in $\mathbf{R}_{\xi}^{2n}$:

$$g = (d\xi'^2 + d\xi''^2)/|\gamma|.$$

Then the result follows from Theorem 18.5.4 in [H_2].

Recall that $p(\zeta,\xi,\lambda) = \widehat{P\delta_{t,x}}$.

Lemma 4.3. *Suppose that* (H-1) *holds. Then, one can find a positive constant* R *such that if* $\zeta \in \Gamma$, $|\operatorname{Im}\zeta| \geq R|\lambda|^{1/2}$, *then*

$$|p(\zeta, \xi, \lambda)| \geq C^{-1}\gamma(|\zeta| + |\xi| + |\lambda|^{1/2})^{m-1}$$

and

$$|\partial_\xi^\alpha \partial_\lambda^\beta (1/p)| \leq C_0 C^{|\alpha+\beta|}(\alpha+\beta)!\gamma^{-|\alpha+\beta|}/|p| .$$

Proof. Since $p(\zeta, \xi, 0)$ is strictly hyperbolic, $p_\pm = p(\zeta, \xi, \pm 1)$ is hyperbolic i.e. there is a constant R such that for every root $\zeta = \mu_{j,\pm}(\xi)$ of $p_\pm = 0$,

$$|\mathrm{Im}\, \mu_{j,\pm}(\xi)| \leq R .$$

Since

$$p(\zeta/|\lambda|^{1/2}, \xi/|\lambda|^{1/2}, \lambda/|\lambda|^{1/2}) = \prod_{j=1}^m (\zeta/|\lambda|^{1/2} - \mu_{j,\pm}(\xi/|\lambda|^{1/2}))$$

and for some large constant $c > 0$, if $|\xi| > c|\lambda|^{1/2}$ and $|\mathrm{Im}\,\zeta| > c|\lambda|^{1/2}$, then

$$|\mu_i(\xi, \lambda) - \mu_j(\xi, \lambda)| \geq \delta(|\xi| + |\lambda|^{1/2}), \qquad \delta > 0,$$

we see that if $|\xi| > c|\lambda|^{1/2}$ and $|\mathrm{Im}\,\zeta| > c|\lambda|^{1/2}$, then

$$|p(\zeta, \xi, \lambda)| \geq C^{-1}\gamma(|\zeta| + |\xi| + |\lambda|^{1/2})^{m-1} .$$

On the other hand, if we choose R large enough, in the region $\{|\mathrm{Im}\,\zeta| > R|\lambda|^{1/2}, |\xi| \leq c|\lambda|^{1/2}\}$, $p(\zeta, \xi, \lambda)$ does not vanish. Therefore, we get the elliptic estimate stronger than the hyperbolic estimate. This finishes the proof of the first half of the statement of Lemma. Using the inequalities:

$$\left|\frac{p^{(\alpha)}}{\alpha! p}\right| \leq C|\mathrm{Im}\,\zeta|^{-|\alpha|},$$

and the Faá di Bruno formula, we obtain the second half of the result.

Lemma 4.4. *Suppose that P satisfies* (H-1). *Then one can find $q_\pm \in S_\Gamma^{-m+1,0}$ such that*

$$p_\pm \# q_\pm - 1 \in S_\Gamma^{0,1}, \qquad q_\pm \# p_\pm - 1 \in S_\Gamma^{0,1} .$$

Proof. Let $\chi(t, \xi) \in C^\infty(\mathbf{R}^{2n+1})$ such that

$$\chi = \begin{cases} 1 & \text{if } |t| + |\xi| \geq 2 \\ 0 & \text{if } |t| + |\xi| \leq 1 . \end{cases}$$

and $\psi(\zeta, \xi) = \chi(|\mathrm{Im}\,\zeta|/R, \xi/R)$ for R is large enough. We may consider the case $+$. Let

$$g(\zeta, \xi) = \psi(\zeta, \xi)/p_+(\zeta, \xi) .$$

Then $g \in S_\Gamma^{-m+1,0}$. Since $p_+ g - 1 = \psi - 1 \in S_\Gamma^{0,\infty}$, from Lemma 4.2, it follows that

$$p_+ \# g - 1 \in S_\Gamma^{0,1} .$$

Proposition 4.5. *Suppose that P satisfies* (H-1). *For $\tilde{\zeta} = \zeta/|\lambda|^{1/2}$, there is a positive constant R_1 such that if $|\mathrm{Im}\,\tilde{\zeta}| > R_1$ and* $\mathrm{Im}\,\zeta < 0$, *then*

$$-\operatorname{Im}\tilde{\zeta}\|u\|_{s+m-1,\tilde{\zeta},1} \le C_s\|\mathscr{P}_\pm u\|_{s,\tilde{\zeta},1}$$

for any $u \in \mathscr{S}(\mathbf{R}^n)$.

Proof. If $r \in S_\Gamma^{0,1}$, then there is a positive constant C such that

$$\|\mathrm{op}_1(r)u\|_{s,\tilde{\zeta},1} \le C|\operatorname{Im}\tilde{\zeta}|^{-1}\|u\|_{s,\tilde{\zeta},1}.$$

Therefore, from Lemma 4.4 and its proof, we obtain the desired estimate.

For the case that $\operatorname{Im}\tilde{\zeta}$ is bounded but $\operatorname{Re}\tilde{\zeta}$ large enough, the hypothesis (H-3) plays an important role.

Proposition 4.6. *Suppose that* (H-1) *and* (H-3) *are satisfied. For any* $s \in \mathbf{R}$, *there are constants* C, N *and* R_2 *such that*

$$-\operatorname{Im}\tilde{\zeta}\|u\|_{m-1,\tilde{\zeta},1} \le C\|\mathscr{P}u\|_{0,\tilde{\zeta},1}$$

if $|\operatorname{Re}\tilde{\zeta}| \ge R_2$ *and* $-N \le \operatorname{Im}\zeta < 0$.

Proof. For simplicity we only consider the case $\lambda = 1$. Let $\mathscr{P} = \mathscr{P}_+$ and $\mathscr{Q} = \partial_\zeta\mathscr{P}$. Recall that p_m be the principal part of $p(\zeta, \xi)$. We consider the following form:

$$-2i\operatorname{Im}(\mathscr{P}u, \mathscr{Q}u)_{L^2} = ((\mathscr{Q}^*\mathscr{P} - \mathscr{P}^*\mathscr{Q})u, u)_{L^2}.$$

Since the symbol p is real-valued for real ζ, we see that

$$\mathscr{P}^*(\zeta) = \mathscr{P}(\bar{\zeta}).$$

Also (H-3) implies that

$$-\{\mathscr{Q}^*\mathscr{P} - \mathscr{P}^*\mathscr{Q}\} = i(\operatorname{Im}\zeta)\mathscr{R},$$

where $\mathscr{R}$ is an operator of degree $2m-2$. In fact, the top order term of $\mathscr{Q}^*\mathscr{P} - \mathscr{P}^*\mathscr{Q}$ is

$$\begin{aligned}
2i\operatorname{Im}(\overline{\partial_\zeta p_m}p_m) &= 2i\operatorname{Im}\left[\overline{\left(\frac{p_m'}{p_m}\right)}\right]|p_m|^2 \\
&= 2i\operatorname{Im}\left[\sum_{j=1}^m \frac{1}{\bar{\zeta} - \lambda_j}\right]|p_m|^2 \\
&= 2i\operatorname{Im}\left[\sum_{j=1}^m \frac{\zeta - \lambda_j}{|\zeta - \lambda_j|^2}\right]|p_m|^2 \\
&= 2i\operatorname{Im}\zeta\sum_{k=1}^m \prod_{j\ne k}|\zeta - \lambda_j(\xi)|^2.
\end{aligned}$$

Here, λ_j denotes the simple real roots of $p_m(\tilde{\zeta}, \xi) = 0$. Moreover, by the product formula, Lemma 4.1, the remaining terms of $\mathscr{Q}^*\mathscr{P} - \mathscr{P}^*\mathscr{Q}$ is estimated by $\mathcal{O}(|(\zeta, \xi)|^{2m-3})$ as $|(\zeta, \xi)| \to \infty$. Therefore, there are positive constants C_j, $j = 1, 2$, such that for the full symbol $r(\zeta, \xi)$ of the operator $\mathscr{R}$ satisfies the inequality

$$r(\zeta, \xi) \geq C_1 |(\zeta, \xi)|^{2m-2} - C_2 .$$

Hence, if $|\mathrm{Re}\, \zeta|$ ia large enough,

$$r(\zeta, \xi) \geq \frac{1}{2} C_1 |(\zeta, \xi)|^{2m-2}$$

for any $\xi \in \mathbf{R}^{2n}$. It is easy to see that for any $\varepsilon > 0$, there is a constant C, C_ε and R_2 such that

$$-\mathrm{Im}\, \tilde{\zeta} \|u\|^2_{m-1,\tilde{\zeta},1} \leq C \|\mathscr{P}u\|_0 \|\mathscr{Q}u\|_0 \leq \{C_\varepsilon |\mathrm{Im}\, \tilde{\zeta}|^{-1} \|\mathscr{P}u\|_0^2 + \varepsilon |\mathrm{Im}\, \tilde{\zeta}| \, \|u\|^2_{m-1,\tilde{\zeta},1}\}$$

for any $u \in \mathscr{S}(\mathbf{R}^n)$ and any $|\mathrm{Re}\, \tilde{\zeta}| \geq R_2$. We get the desired inequality.

Summing up, by quasi-homogeneity, we have

Theorem 4.7. *Suppose that* (H-1) *and* (H-3) *are satisfied. There are constants* C, N *and* R *such that*

$$-\mathrm{Im}\, \zeta \|u\|_{m-1,\zeta,\lambda} \leq C \|\mathscr{P}u\|_{0,\zeta,\lambda}$$

if $|\zeta|/|\lambda|^{1/2} \geq R$, $\mathrm{Im}\, \zeta < 0$ *and* $|\lambda| \geq 1$.

For the bounded λ, we have

Proposition 4.8. *Suppose that* (H-1) *holds. There is a positive number* B *such that*

$$-\mathrm{Im}\, \zeta \|u\|_{s+m-1,\zeta,\lambda} \leq C \|\mathscr{P}u\|_{s,\zeta,\lambda}$$

for $|\lambda| \leq 1$, $\mathrm{Im}\, \zeta \leq -B$ *and* $u \in \mathscr{S}(\mathbf{R}^n)$.

Proof. We can make the same reasoning as in the proof of Proposition 4.5 but in the region

$$\{(x, \xi, \zeta, \lambda); |\xi| + |x\lambda| > R \gg 1, \mathrm{Im}\, \zeta < -B, |\lambda| < 1\}$$

instead of

$$\{(x, \xi, \zeta, \lambda); |\xi| + |x||\lambda|^{1/2} > R \gg 1, \mathrm{Im}\, \zeta < -B'|\lambda|^{1/2}, |\lambda| > 1\} .$$

5. Proof of Theorem 2.1

Let $P_\zeta = (i\zeta)^m + \sum_{j=1}^m a_j(i\zeta)^{m-j}$. Then, we have the following basic inequality.

Theorem 5.1. *There are positive constants* A, B, C *and a positive integer* N *such that*

$$\sum_{j+\langle I \rangle \leq m-1} \gamma \|\zeta^j X^I u\| \leq C \sum_{|J| \leq N} \|X_0^J P_\zeta u\|$$

for $u \in \mathscr{S}(\mathbf{R}^n)$ *and* $\zeta \in \Gamma_{A,B}$.

Proof.

$$\begin{aligned}\|\pi_\lambda(X^I u)\varphi\| &= \|\pi_\lambda({}^t X^I)\pi_\lambda(u)\varphi\| \\ &\le C(1+|\lambda|)^N \|\pi_\lambda({}^t P)\pi_\lambda(u)\varphi\| \\ &= C(1+|\lambda|)^N \|\pi_\lambda(Pu)\,\varphi\|.\end{aligned}$$

We note that

$$\pi_\lambda(u)^* = \pi_\lambda(\bar{u}(-x)), \qquad \overline{\pi_\lambda(\bar{P})} = \pi_{-\lambda}(P),$$

$$\int_{\mathbf{H}^n} |u(x)|^2 dx = \int_{\mathbf{R}\setminus\{0\}} \mathrm{Tr}\,(\pi_\lambda(u)\pi_\lambda(u)^*)\, d\lambda.$$

If T is a Hilbert-Schmidt operator on a Hilbert space H with inner product $(\cdot, \cdot)_H$, then

$$\mathrm{Tr}\, T = \sum_{j=1}^{\infty} (T\varphi_j, T\varphi_j)_H ,$$

where $\{\varphi_j\}$ is an orthonormal system of H. Let $\{\varphi_j\}$ be an orthonormal system of $L^2(\mathbf{R}^n)$ such that each φ_j belongs to $\mathscr{S}(\mathbf{R}^n)$.

$$\begin{aligned}\|\pi_\lambda(X^J u)\varphi_j\| &= \|\pi_\lambda({}^t X^J)\pi_\lambda(u)\varphi_j\| \\ &\le C(1+|\lambda|)^p \|\pi_{-\lambda}(P^*)\pi_{-\lambda}(u)\varphi_j\| \\ &= C(1+|\lambda|)^p \|\pi_\lambda({}^t P)\pi_{-\lambda}(u)\varphi_j\| \\ &= C(1+|\lambda|)^p \|\pi_\lambda(Pu)\,\varphi_j\|.\end{aligned}$$

This inequality and the formula for Tr imply the desired estimate.

Proposition 5.2. *Let ζ be fixed with* $\mathrm{Im}\,\zeta < 0$. *If $P_\zeta u = 0$ for $u \in L^2$, then $u = 0$. The same property also holds for $P_\zeta^* u = 0$.*

Proof. Let $\chi \in \mathscr{S}(\mathbf{R})$ be any function such that $\operatorname{supp} \hat{\chi}(\lambda) \subset [a, b]$. We take a convolution u and χ with respect to the variable x_0;

$$v(x) = (u * \chi)(x) = \int u(x_1, \dots, x_{2n}, x_0 - z)\chi(z)dz .$$

We remark that if u is a solution of $P_\zeta u = 0$, then v is also a solution of the same equation. The energy inequality which we establish is persistence with the commutator argument between the molifier and v, as in the globally elliptic case (c.f. [M_2]) since we fix ζ. Hence we can conclude that $u * \chi = 0$ for any χ, which implies $u = 0$.

Let $\zeta \in \Gamma_{0,B}$. Here, B are chosen in Proposition 4.8 in the previous section. Let Y be the Hilbert space such that

$$Y = \{v \in \mathscr{S}'; (1 - X_0^2)^q v \in L^2(\mathbf{H}^n)\} .$$

with natural inner product $(\cdot, \cdot)_Y$, where $2q = N$. If for some $v \in Y$,

$$(P_\zeta u, v)_Y = 0$$

for any $u \in \mathscr{S}$, then we have

$$(u, (1 - X_0^2)^{2q} P_\zeta^* v)_{L^2} = 0 .$$

Hence, $(1 - X_0^2)^{2q} P_\zeta^* v = 0$ in $\mathscr{S}'$, $(1 + |\lambda|^2)^{2q} \widehat{P_\zeta^* v} = 0$, $\widehat{P_\zeta^* v} = 0$, and $P_\zeta^* v = 0$ for $v \in Y \subset L^2(\mathbf{H}^n)$. Then,

$$\langle \mathscr{P}^* \pi_\lambda(v)\varphi, \psi \rangle = \langle \pi_\lambda(v)\varphi, \mathscr{P}\psi \rangle = 0 .$$

Hence $\mathscr{P}^* \pi_\lambda(v)\varphi = 0$ in $\mathscr{S}'$. By Proposition 5.2, it follows that $\pi_\lambda(v)\varphi = 0$, which yields to $v = 0$. From the molifier argument in $\mathbf{H}^n$ (Proposition 6.4 in [M_2]), it follows that the estimate in Theorem 5.1 is holds for $u \in D(P_\zeta)$, where

$$D(P_\zeta) \subset X^* = \{u \in \mathscr{S}'; X^I u \in L^2(\mathbf{H}^n), \langle I \rangle \le m^*\} .$$

Hence the range P_ζ is closed. This means that for each $\zeta \in \Gamma$ the range $P_\zeta = Y$. Here, we take $m^* = m$ or $m - 1$, accordingly P satisfies the condition (H-1)$'$ or (H-1).

Let

$$\mathscr{A} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & 0 & 1 & & 0 \\ \vdots & \vdots & 0 & \ddots & \\ 0 & 0 & 0 & & 1 \\ -A_m & -A_{m-1} & -A_{m-2} & \dots & -A_1 \end{bmatrix}$$

and

$$\tilde{X} = \{L^2(\mathbf{R}^{2n+1})\}^m \text{ and } \tilde{Y} = H_N^{m-1} \times H_N^{m-2} \times \cdots \times H_N^0 ,$$

where

$$H_N^j = \{u \in \mathscr{S}'(\mathbf{R}^{2n+1}); (1 - X_0^2)^{2N} X^I u \in L^2, \forall I, |I| \le j\} .$$

Then, $\mathscr{A}$ is a closed operator from $\tilde{X}$ to $\tilde{Y}$ with its domain

$$D(\mathscr{A}) = \{u \in \tilde{X}; \mathscr{A}u \in \tilde{Y}\} .$$

Since the equation $(\zeta - \mathscr{A})U = F$ is equivalent to the equation $P_\zeta u = f$, $(\zeta - \mathscr{A})$ is a bijection from $D(\mathscr{A})$ to $\tilde{Y}$. Therefore, the inverse operator $(\zeta - \mathscr{A})^{-1}$ from $\tilde{Y}$ to $D(\mathscr{A})$ exists, is holomorphic and has its norm less than $C(-\operatorname{Im}\zeta)^{-1}$ in the domain Γ. Therefore, the operator $\mathscr{A}$ generate a distribution semi-group.

Let us denote the Laplace transform of f with support in $\{t \ge 0\}$ by $\tilde{f}$. Let $\chi(t)$ be a smooth function on $[0, T]$ such that

$$\chi = 1 \text{ if } 0 \le t \le T/3 \text{ and } = 0 \qquad \text{for } 2T/3 \le t \le T .$$

We extend the function $\chi(t)$ in the whole line $\mathbf{R}$ by 0 and denote it by the same letter. Put

$$U(x, t) = (2\pi i)^{-1} \int_{\mathscr{C}} e^{i\zeta t} \zeta^{-2} P_\zeta^{-1} \widetilde{\chi f}(\zeta) d\zeta ,$$

where $\mathscr{C} = \partial \Gamma_{A,B}$. Then the solution of the Cauchy problem

$$(*) \qquad \begin{cases} Pu = f \\ \operatorname{supp}_t u \subset [0, \infty) \end{cases}$$

in $[0, T/3]$ is given by

$$u = D_t^2 U(x, t)$$

(c.f. [C] and Chap. 8 in [F]).

6. Examples and Remarks

In this section, we consider the meaning of our condition (H-1)–(H-2).

Theorem 6.1. *Suppose that the Cauchy problem* (1.1) *is well-posed at the origin in the distribution's sense* (*c.f.* $[\mathrm{H}_1]$). *Then for any* ζ *with* $\operatorname{Im} \zeta < 0$,

$$\{u \in \mathscr{S}(\mathbf{R}^n);\ \mathscr{P}_\pm^*(\zeta) u = 0\} = \{0\} .$$

Proof. We may only consider the '+' case. Suppose that for some ζ_0 having negative imaginary part, there is a function $v \neq 0$ satisfying $\mathscr{P}_+^*(\zeta_0) v = 0$. We define a smooth function u by the following way. For $\phi \in C_0^\infty(\mathbf{H}^n)$,

$$\langle u_\lambda(t, \cdot), \phi \rangle = (\pi_\lambda(\phi) v, v) e^{it\zeta_0 |\lambda|^{1/2}} ,$$

which satisfies the equation $Pu = 0$ because if X is a right invariant vector field on $\mathbf{H}^n$,

$$\pi_\lambda({}^t X \phi) v = \pi_\lambda(X) \pi_\lambda(\phi) v .$$

Set

$$u_\rho(x, t) = \phi(x, t) \int u_{\rho\lambda}(x, t) g(\lambda) d\lambda ,$$

where $\phi \in C_0^\infty(\mathbf{R}^{2n+2})$ such that $\phi = 1$ near the origin and sufficiently small support and $g \in C_0^\infty(\mathbf{R}) \backslash \{0\}$ such that $g \geq 0$ with support contained in $(1, \infty)$. By integration by parts, it is seen that for any $J \in \mathbf{N}$,

$$P u_\rho = \mathcal{O}(\rho^{-J}) \text{ as } \rho \to \infty .$$

With this functon u_ρ, the same argument in the proof of Theorem 1 of $[\mathrm{O}_1]$ enables us to show that a priori inequality which follows from the well-posedness never holds.

Let us begin with good examples for our theory.

Example 1 ($m = 2, n = 1$).

$$P = \partial_t^2 - a(X_1^2 + X_{-1}^2) + ciX_0 ,$$

where a is a real positive number and c is a complex constant. It is well-known that the Cauchy problem for P is well-posed if and only if c is real and $|c| \leq a$. From the point of view in our theory, this is equivalent to the condition (H-2). By simple calculation, this is verified since the Hermite operator $-\partial_x^2 + x^2$ has the lowest eigenvalue 1.

Example 2 ($m = 3, n = 1$).

$$P = \partial_t\{\partial_t^2 - a(X_1^2 + X_{-1}^2) + c_1X_0\} + c_2iX_1X_0 + c_3iX_{-1}X_0 ,$$

where a is a real positive number and c_j are real numbes. The Cauchy problem for P is well-posed if and only if

$$a^{1/2}(a - |c_1|) \geq \sqrt{c_1^2 + c_2^2} .$$

Example 3 ($m = 4, n = 1$).

$$P = \{\partial_t^2 - a(X_1^2 + X_{-1}^2)\}\{\partial_t^2 - b(X_1^2 + X_{-1}^2)\} + cX_0^2 ,$$

where a and b are positive constants and c is a real constant. The Cauchy problem for P is well-posed if and only if

$$\frac{1}{4}(a - b)^2 \geq -c \geq -ab.$$

The following example shows the limitation of our theorem.

Example 4 ($m = 2, n = 1$).

$$P = -\partial_t^2 + 2\partial_t X_1 + X_{-1}^2 + cX_0$$

where c us a complex constant. It is easy to see that the Cauchy problem for this operator is well-posed if and only if $c = 0$. Let us examine the conditions in Theorem 2.1.

$$\mathscr{F}P\delta = \zeta^2 - 2\zeta\xi_1 - \xi_2^2 + ic .$$

Hence, the condition (H-1) is not satisfied for this operator. But it is easily verified that the condition (H-2) holds in this case for any constant c. This observation shows that without condition (H-1), the condition (H-2) does not implies the well-posedness.

From the Euclidean point of view, the similar examples as above have been considered in [IP], [Z], [O_1] and [H_1], respectively.

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