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On local integrability conditions for nowhere-zero complex vector fields

By

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§ 1. Introduction

Let X be a nowhere-zero complex vector field, with C^∞ coefficients, in an open set Ω in R^{n+1}. We shall say that X is *locally integrable* at a point P in Ω if the homogeneous equation

$$Xu = 0 \tag{1.1}$$

has C^1 solutions $u_1, u_2, \dots, u_n$ in a neighborhood U of P such that $du_1 \wedge du_2 \wedge \cdots \wedge du_n \neq 0$ in U (cf. Lewy [5], Treves [12] and Jacobowitz-Treves [4]). When X is *locally integrable* at every point in Ω, we shall say that X is *locally integrable* in Ω. It is evident that X is *locally integrable* in Ω if $X = \overline{X}$ or X is real analytic in Ω. But Nirenberg [9] gave a vector field in R^2 which is not *locally integrable* at the origin; he proved that the equation

$$\partial u/\partial t + it(1 + t\phi(t, x))\partial u/\partial x = 0$$

admits the only constant C^1 solutions in every neighborhood of the origin where $\phi(t, x)$ is realvalued, even with respect to t and satisfies certain elaborate conditions. We note that $\partial/\partial t + it(1 + t\phi(t, x))\partial/\partial x$ is a non-solvable operator. Now, we may assume that X locally takes the following form:

$$X = \partial/\partial t + i \sum_{j=1}^{n} a^j(t, \chi)\partial/\partial x_j, \qquad \chi = (x_1, \dots, x_n) \in R^n$$

where all the $a^j(t, \chi)$ are realvalued. Then, it is said that X satisfies the solvability condition $(\mathscr{P})$ at P if there exists a neighborhood ω such that for every $\xi \in R^n$ and every $\chi_0 \in R^n$ the function $t \to \sum_{j=1}^{n} a^j(t, \chi_0)\xi_j$ does not change sign in the set $\{t \in R^1; (t, \chi_0) \in \omega\}$. From Treves [13], it follows that X is *locally integrable* at P if X satisfies $(\mathscr{P})$ at P. Considering these results, particularly we are concerned with the non-solvable vector fields in R^2 of the following form:

$$\partial/\partial t + ia(t, x)\partial/\partial x$$

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where $a(t, x)$ is a realvalued C^∞ function having the property: $ta(t, x) > 0$ for $t \neq 0$. Throughout this paper the operator $\partial/\partial t + ia(t, x)\partial/\partial x$ having this property shall be denoted by L. It becomes into the subject whether L is *locally integrable* at a point on the x axis. There are a few results on *local integrability* of L ([8], [9], [10] and [12], for example). Now we shall state our results. First, we set the following definition:

$f_{\mathrm{odd}}(t, x) =$ *the odd part* of $f(t, x)$ with respect to t and

$f_{\mathrm{even}}(t, x) =$ *the even part* of $f(t, x)$ with respect to t for a function $f(t, x)$.

A domain D in $\{(t, x); t \in R, x \in R\}$ is called a *flag domain if* $\mathrm{D} \subset \{(t, x); t > 0\}$ *and* $\partial\mathrm{D}$ *is a simple closed curve such that* $\partial\mathrm{D} \cap \{(t, x); t = 0\}$ *is a line with positive length.*

Now, as a necessary condition for L to be *locally integrable*, we get the following

Theorem A. *If, for every neighborhood* U *of a point* P *on the x axis, there is a flag domain* D *in* U *such that either* $\partial[\mathrm{D} \cap \{a_{\mathrm{even}}(t, x) > 0\}]$ *constituting of a finite number of rectifiable Jordan curves or* $\partial[\mathrm{D} \cap \{a_{\mathrm{even}}(t, x) < 0\}]$ *constituting of a finite number of rectifiable Jordan curves is included in* D, *then* L *is not locally integrable at* P.

Namely, *it is necessary that there exists a neighborhood* U *of* P *such that no flag domain* D *in* U *satisfies that either* $\partial[\mathrm{D} \cap \{a_{\mathrm{even}}(t, x) > 0\}]$ *constituting of a finite number of rectifiable Jordan curves or* $\partial[\mathrm{D} \cap \{a_{\mathrm{even}}(t, x) < 0\}]$ *constituting of a finite number of rectifiable Jordan curves is included in* D.

We see that Nirenberg's example does not satisfy this necessary condition.

We note that the condition above is not a sufficient one; because the following theorem holds: let c be a positive constant and both of $\{a_n\}$ and $\{b_n\}$ $(n = 1, 2, \dots.)$ positive sequences decreasing to 0 such that $a_n > b_n > a_{n+1}$ for every $n \in N$. Let both of $\{a'_n\}$ and $\{b'_n\}$ $(n = 1, 2, \dots.)$ be negative sequences increasing to 0 such that $a'_n < b'_n < a'_{n+1}$ for every $n \in N$. Then we set V_n, W_n, V'_n and W'_n $(n \in N)$ as follows:

$$V_1 = \{(t, x); 0 < t < c, b_1 \leqq x - x_0 < a_1\},$$

$$V_k = \{(t, x); 0 < t < c, b_k \leqq x - x_0 \leqq a_k\} \qquad (k = 2, 3, \dots),$$

$$W_j = \{(t, x); 0 < t < c, a_{j+1} < x - x_0 < b_j\} \qquad (j = 1, 2, \dots),$$

$$V'_1 = \{(t, x); 0 < t < c, b'_1 \geqq x - x_0 > a'_1\},$$

$$V'_k = \{(t, x); 0 < t < c, b'_k \geqq x - x_0 \geqq a'_k\} \qquad (k = 2, 3, \dots),$$

$$W'_j = \{(t, x); 0 < t < c, a'_{j+1} > x - x_0 > b'_j\} \qquad (j = 1, 2, \dots).$$

Theorem B. *Assume that* $a_{\mathrm{even}}(t, x)$ *is nonnegative. If* $a_{\mathrm{even}}(t, x) \equiv 0$ in $\bigcup_{j=1}^{\infty} \mathrm{V}_j \cup \mathrm{V}'_j$ *and* $a_{\mathrm{even}}(t, x) > 0$ in $\bigcup_{j=1}^{\infty} \mathrm{W}_j \cup \mathrm{W}'_j$, *provided that at least one of*

$$\lim_{n\to\infty} \frac{\iint_{W_n} a_{\text{even}}(t, x)\,dt\,dx}{b_n - a_{n+1}}$$

and

$$\lim_{n\to\infty} \frac{\iint_{W_n'} a_{\text{even}}(t, x)\,dt\,dx}{a_{n+1}' - b_n'}$$

is a positive constant, then L *is not locally integrable at* $P(0, x_0)$.

One can easily check that there is a neighborhood U of P such that no flag domain in U satisfies the condition of Theorem A under the assumption of Theorem B.

From the facts above, *we know that the form of existence of* supp a_{even} *affects the local integrability of L*; in case of non existence of supp a_{even}, we have an affirmative result (cf. [7], [8]):

Theorem C. *Assume that* $a_{\text{even}}(t, x) \equiv 0$. *Then* L *is locally integrable at every point on the x axis.*

Furthermore we obtain the following

Theorem D. *Let* $P(0, x_0)$ *be a point on the x axis and* $\beta(t, x)$ $\{1 - \alpha(t, x)\}/\{1 + \alpha(t, x)\}$ *where* $\alpha(t, x) \equiv 1 + i \int_0^t a_x(s, x)ds$.

Assume that $\beta(t, x)$ *can be extended as a* C^0 *function* $\tilde{\beta}(t_1, x_1)$ *which is defined in a neighborhood* U_0 *of the origin where*

$$t_1 = \int_0^t a(s, x)ds \qquad \text{and} \qquad x_1 = x - x_0\,.$$

Moreover, assume that the following conditions hold:

(i) $\sup|\tilde{\beta}(t_1, x_1)| < 1$.

(ii) $\sup|\tilde{\beta}(t_1, x_1)|C_p < 1$

where p is a fixed exponent such that $p > 2$ *and* C_p *stands for a positive constant satisfying*

$$\|Tg\|_p \leqq C_p\|g\|_p \qquad \textit{for } \forall g \in L_p(R^2)$$

where

$$Tg(z) \equiv (2\pi i)^{-1} \int \{g(\zeta) - g(z)\}/[(\zeta - z)^{-2}\}d\zeta d\bar{\zeta} \qquad \textit{and}$$

$$z = t_1 + ix_1\,.$$

(iii) $\tilde{\beta}(t_1, x_1)$ *has a distributional derivative* $\tilde{\beta}_z(t_1, x_1) \in L_p$.

Then, L *is locally integrable at* P.

We remark that, when $a_{\mathrm{add}}(t, x)$ vanishes of finite order, Theorem C follows also from Theorem D.

We note that, generally, for every point P on the x axis there exist a neighborhood U of P and a function $u \in C^1(U \cap \{t \geqq 0\})$ such that $Lu = 0$ and $du \neq 0$ in $U \cap \{t \geqq 0\}$.

Finally, we shall refer to existence of a certain relation between *local solvability* and *local integrability*; it does not seem that both of them have a relation each other. But, we claim that there exists a certain connection under the assumption that solutions mean C^∞: for simplicity, let X be a nowhere-zero complex vector field in R^2. Setting $X = \partial/\partial x_1 + b(x_1, x_2)\partial/\partial x_2$, we see that X is *locally integrable* at a point P if and only if the inhomogeneous equation $Xu = b_{x_1}$ has a solution in a neighborhood of P (see Ninomiya [7], Hörmander [3], and Treves [12] and [14]). Differently from two dimensional case, the situation in case of three more dimension vector fields X becomes more complicated and we shall find that there is a certain link between *local solvability* and *local integrability*.

§ 2. Proof of Theorem A

Assume that L is *locally integrable* at P. We use the method of Nirenberg [9]. Let u_1 be a C^1 solution of $Lu_1 = 0$ in a neighborhood U of $\mathrm{P}(0, x_0)$ such that $du_1 \neq 0$. Then, $(\partial u_1/\partial x)(0, x_0) \neq 0$. Let θ_0 be $\mathrm{Arg}(\partial u_1/\partial x)(0, x_0)$ and c a constant such that $0 < c < \pi/2$. Set $u = e^{i(c-\theta_0)}u_1$. Then, u is a C^1 solution of $Lu = 0$ in U such that both of $\mathrm{Re}\,\partial u/\partial x$ and $\mathrm{Im}\,\partial u/\partial x$ are positive at P. Therefore we may assume that both of $\mathrm{Re}\,\partial u/\partial x$ and $\mathrm{Im}\,\partial u/\partial x$ are positive in U, contracting U if necessary.

Then, we may suppose that there exists a flag domain D in U such that $\partial[D \cap \{a_{\mathrm{even}}(t, x) > 0\}]$ constituting of a finite number of rectifiable Jordan curves is included in D.

Furthermore we can assume that $D \cap \{a_{\mathrm{even}}(t, x) > 0\}$ is an open set ω obtained by removing a finite number of simply connected domains or multiply connected domains that are disjoint each other from a simply connected domain Ω surrounded by a rectifiable Jordan curve.

Now, from $Lu = 0$, we have

$$\partial u_{\mathrm{odd}}/\partial t + ia_{\mathrm{odd}}(t, x)\partial u_{\mathrm{odd}}/\partial x = -ia_{\mathrm{even}}(t, x)\partial u_{\mathrm{even}}/\partial x \qquad \text{in } U. \tag{2.1}$$

Hence, it follows that

$$\partial u_{\mathrm{odd}}/\partial t + ia_{\mathrm{odd}}(t, x)\partial u_{\mathrm{odd}}/\partial x = 0 \qquad \text{in } D \cap \Omega^c. \tag{2.2}$$

By our assumption $ta(t, x) > 0$ for $t \neq 0$, we see that $a_{\mathrm{odd}}(t, x) > 0$ for $t > 0$.

Now we note that $u_{\mathrm{odd}}(0, x) \equiv 0$. Therefore, applying uniqueness theorem (Ninomiya [6], Strauss-Treves [11] or Zuily [16]) to (2.2), we see that $u_{\mathrm{odd}}(t, x)$ vanishes identically in $D \cap \Omega^c$.

Now, we have the following

Theorem 2.1 (Ninomiya [7]). *Assume that $b(t, x)$ is realvalued C^2, odd with respect to t and positive for $t > 0$. Then, there exists a C^1 solution $v = v(t, x)$ of*

$$\partial v/\partial t + ib(t, x)\partial v/\partial x = 0 \tag{2.3}$$

in a neighborhood of every point on the x axis such that $\partial v/\partial x \neq 0$.

This proof will be given in the appendix. From Theorem 2.1, the equation

$$\partial v/\partial t + ia_{\text{odd}}(t, x)\partial v/\partial x = 0 \tag{2.4}$$

has a C^1 solution $v = v(t, x)$ in a neighborhood of P such that $\partial v/\partial x \neq 0$. Then, we can assume that, from the beginning, v satisfies (2.4) in U and that both of Re $\partial v/\partial x$ and Im $\partial v/\partial x$ are positive in U. Then, from (2.1), we have

$$(\partial v/\partial x)\{\partial u_{\text{odd}}/\partial t + ia_{\text{odd}}(t, x)\partial_{\text{odd}}/\partial x\} = (\partial v/\partial x)\{-ia_{\text{even}}(t, x)\partial u_{\text{even}}/\partial x\} \tag{2.5}$$

in U. Hence we have

$$\int_\Omega (\partial v/\partial x)\{\partial u_{\text{odd}}/\partial t + ia_{\text{odd}}(t, x)\partial u_{\text{odd}}/\partial x\}\mathrm{d}t\mathrm{d}x \tag{2.6}$$

$$= \int_\Omega (\partial v/\partial x)\{-ia_{\text{even}}(t, x)\partial u_{\text{even}}/\partial x\}\mathrm{d}t\mathrm{d}x\,.$$

One can easily verify that the lefthand side of (2.6) =

$$\int_\Omega \mathrm{d}\{u_{\text{odd}}\mathrm{d}v\} = 0 \qquad \text{because of } u_{\text{odd}} \equiv 0 \text{ on } \partial\Omega\,.$$

Therefore,

$$\int_\Omega (\partial v/\partial x)\{a_{\text{even}}(t, x)\partial u_{\text{even}}/\partial x\}\mathrm{d}t\mathrm{d}x = 0\,.$$

But this contradicts the fact that

$$\operatorname{Im}\,[(\partial v/\partial x)\{a_{\text{even}}\partial u_{\text{even}}/\partial x\}]$$
$$= a_{\text{even}}(t, x)\cdot\{\operatorname{Re}\,\partial u_{\text{even}}/\partial x\cdot\operatorname{Im}\,\partial v/\partial x + \operatorname{Im}\,\partial u_{\text{even}}/\partial x\cdot\operatorname{Re}\,\partial v/\partial x\}$$

is positive in $\omega \subset \Omega$. Q.E.D.

§3. Proof of Theorem B

Assume that the contrary holds. Then we can assume that there exist a neighborhood U of P and C^1 functions u and v such that

$$Lu = 0\,. \tag{3.1}$$

$$\partial v/\partial t + ia_{\text{odd}}(t, x)\partial v/\partial x = 0\,. \tag{3.2}$$

(3.3) Re $\partial u_{\text{even}}/\partial x$, Im $\partial u_{\text{even}}/\partial x$, Re $\partial v/\partial x$ and Im $\partial v/\partial x$ are positive in U.

$$\bigcup_{n=1}^{\infty} V_n \cup V_n' \cup W_n \cup W_n' \subset U . \tag{3.4}$$

By the same way as the previous section, we can conclude that u_{odd} vanishes identically in $\bigcup_{n=1}^{\infty} V_n \cup V_n'$. And by the same way as the previous section, we have

$$\iint_{W_n} d\{u_{\text{odd}} dv\} = \iint_{W_n} (\partial v/\partial x)\{-ia_{\text{even}}\partial u_{\text{even}}/\partial x\} dt dx \tag{3.5}$$

and

$$\iint_{W_n'} d\{u_{\text{odd}} dv\} = \iint_{W_n'} (\partial v/\partial x)\{-ia_{\text{even}}\partial u_{\text{even}}/\partial x\} dt dx \qquad (n = 1, 2, \dots.) . \tag{3.6}$$

Therefore, from $u_{\text{odd}} = 0$ on $\partial W_n \backslash \{t = c\} \cup \partial W_n' \backslash \{t = c\}$ for every $n \in N$, we obtain

$$\left| \int_{a_{n+1}}^{b_n} u_{\text{odd}}(c, x)(\partial v(c, x)/\partial x) dx \right| = \left| \iint_{W_n} (\partial v/\partial x)\{a_{\text{even}}\partial u_{\text{even}}/\partial x\} dt dx \right| \tag{3.7}$$

and

$$\left| \int_{b_n'}^{a_{n+1}'} u_{\text{odd}}(c, x)(\partial v(c, x)/\partial x) dx \right| = \left| \iint_{W_n'} (\partial v/\partial x)\{a_{\text{even}}\partial u_{\text{even}}/\partial x\} dt dx \right| . \tag{3.8}$$

Hence there exist suitable positive constants M and m such that

$$M\alpha_n(b_n - a_{n+1}) \geqq m \iint_{W_n} a_{\text{even}}(t, x) dt dx \tag{3.9}$$

and

$$M\alpha_n'(a_{n+1}' - b_n') \geqq m \iint_{W_n'} a_{\text{even}}(t, x) dt dx \tag{3.10}$$

where

$$\alpha_n \equiv \max_{a_{n+1} \leqq x \leqq b_n} |u_{\text{odd}}(c, x)|$$

and

$$\alpha_n' \equiv \max_{b_n' \leqq x \leqq a_{n+1}'} |u_{\text{odd}}(c, x)| .$$

As the other case can be also shown, we suppose

$$\varliminf_{n \to \infty} \frac{\iint_{W_n} a_{\text{even}}(t, x) dt dx}{b_n - a_{n+1}} \equiv K > 0 .$$

Then, from (3.9) and $\lim_{n\to\infty} \alpha_n = 0$, we obtain:

$$0 \geqq mK > 0 .$$

This is absurd. Q.E.D.

Appendix

We shall prove Theorem 2.1. First, we easily see that the following lemma holds:

Lemma 4.1. *Let $A(t, x)$ be a realvalued C^2 function such that $A(t, x) \geqq 0$ for $t \geqq 0$. Then, there exist a neighborhood $U(\mathrm{P})$ of* P *and a positive constant C such that*

(i) $|A_x(t, x)| \leqq C\sqrt{A(t, x)}$;

(ii) $\left|\int_{t'}^{t} A_x(s, x)ds\right| \leqq C\left|\int_{t'}^{t} A(s, x)ds\right|^{1/2}$ in $\mathrm{U(P)}_+ \equiv \mathrm{U(P)} \cap \{t \geqq 0\}$.

Next, let us consider a mapping $(t, x) \overset{F}{\to} (t_1, x_1)$ defined by

$$t_1 = \int_0^t b(s, x)ds \qquad \text{and} \qquad x_1 = x - x_0 , \qquad \text{provided } t \geqq 0 .$$

F gives a homeomorphism from $U(\mathrm{P})_+$ onto $F(U(\mathrm{P})_+)$; t is expressed as $t = t(t_1, x_1)$. Let a function $c(t_1, x_1)$ defined in $F(U(\mathrm{P})_+)$ be

$$\begin{aligned} c(t_1, x_1) &= \int_0^t ib_x(s, x)ds + 1 \\ &= i\int_0^{t(t_1, x_1)} b_x(s, x_1 + x_0)ds + 1 . \end{aligned}$$

Next we set $C(t_1, x_1) = c(|t_1|, x_1)$; $C(t_1, x_1)$ is defined in a neighborhood of the origin. Then we have the following

Lemma 4.2 (Ninomiya [6]). *There exists a neighborhood* V *of the origin such that $C(t_1, x_1) \in C^{1/2}(\mathrm{V})$.*

Proof of Lemma 4.2 ([6]). It holds that

$$\int_0^t b_x(s, x)ds - \int_0^{t'} b_x(s, x')ds = \int_{t'}^t b_x(s, x)ds + \int_0^t \{b_x(s, x) - b_x(s, x')\}ds .$$

Let t and t' be nonnegative. By virtue of Lemma 4.1, taking a smaller neighborhood $U(\mathrm{P})$ of P if necessary, we have

$$\left|\int_0^t b_x(s, x)ds - \int_0^{t'} b_x(s, x')ds\right| \leqq C_1\left|\int_{t'}^t b(s, x)ds\right|^{1/2} + \int_0^{t'} |b_x(s, x) - b_x(s, x')|ds$$

$$\leqq C_1\left\{\left|\int_0^t b(s, x)ds - \int_0^{t'} b(s, x')ds\right| + C_2t'|x - x'|\right\}^{1/2} + C_3t'|x - x'|$$

$$\leqq C_4\left\{\left|\int_0^t b(s, x)ds - \int_0^{t'} b(s, x')ds\right| + |x - x'|\right\}^{1/2}$$

in $U(\mathrm{P})_+$ where C_i $(i = 1, 2, 3, 4)$ denote positive constants. Let (t_1, x_1) and (t_1', x_1') be in $F(U(\mathrm{P})_+)$. Then, we have

$$|C(t_1, x_1) - C(t_1', x_1')| \leqq C_4\{|t_1 - t_1'| + |x_1 - x_1'|\}^{1/2}$$

in $F(U(\mathrm{P})_+)$. From this, it clearly follows that

$$C(t_1, x_1) \in C^{1/2}(\mathrm{V})$$

where $V = F(\mathrm{U}(\mathrm{P})_+) \cup \{(t_1, x_1); (-t_1, x_1) \in F(\mathrm{U}(\mathrm{P})_+)\}$.

By virtue of Lemma 4.2, we obtain the following

Theorem 4.3. *There exists a* $C^{1+1/2}(V_0)$ *solution* $z = z(t_1, x_1)$ *of*

$$i\partial z/\partial x_1 + C(t_1, x_1)\partial z/\partial t_1 = 0$$

with $dz \neq 0$ *in a neighborhood* V_0 *of the origin.*

Theorem 4.3 follows from a classical result on the Beltrami equation. Now, let us define a function $h = h(t, x)$ by

$$h(t, x) = z\left(\int_0^t b(s, x)ds, x - x_0\right).$$

Let (t, x) be in $F^{-1}(\mathrm{V}_{0_+})$ where $\mathrm{V}_{0_+} = V_0 \cap \{t_1 \geqq 0\}$. Then,

$$h_t(t, x) = b(t, x)z_{t_1}(t_1, x_1)$$

and

$$h_x(t, x) = z_{t_1}(t_1, x_1)\int_0^t b_x(s, x)ds + z_{x_1}(t_1, x_1).$$

Hence it follows that

$$\begin{aligned} Ah(t, x) &\equiv \partial h/\partial t + ib(t, x)\partial h/\partial x \\ &= b(t, x)\left[iz_{x_1}(t_1, x_1) + \left\{\int_0^t ib_x(s, x)ds + 1\right\}z_{t_1}(t_1, x_1)\right] \\ &= b(t, x)[iz_{x_1}(t_1, x_1) + c(t_1, x_1)z_{t_1}(t_1, x_1)] \\ &= b(t, x)[i\partial z/\partial x_1 + C(t_1, x_1)\partial z/\partial t_1] = 0. \end{aligned}$$

Finally, let us define a function $u = u(t, x)$ by $u = h(|t|, x)$. We can easily verify that $Au(t, x) = 0$ and $du \neq 0$ in a neighborhood $F^{-1}(\mathrm{V}_{0_+}) \cup F^{-1}(\mathrm{V}_{0_+})_-$ of P where $F^{-1}(\mathrm{V}_{0_+})_- = \{(t, x); (-t, x) \in F^{-1}(\mathrm{V}_{0_+})\}$. Q.E.D.

Remark 1. From the proof above the following is easily known: for every point P on the x axis, there are a neighborhood $U(\mathrm{P})$ of P and a function $u \in C^1(U(\mathrm{P})_+)$ such that $Lu = 0$ and $du \neq 0$ in $U(\mathrm{P})_+$.

Remark 2. From Ahlfors [1], we see that the following theorem holds:

Theorem. *There exists a* C^1 *solution* $v = v(t_1, x_1)$ *satisfying the Beltrami equation*

$$\partial \bar{v}/\partial z = \mu \partial v/\partial z \quad (z = t_1 + ix_1)$$

in R^2 *such that* $|\partial v/\partial \bar{z}|^2 - |\partial v/\partial z|^2 > 0$ *under the following assumptions*:
(i) μ *is a measurable function with* $\|\mu\|_\infty \leqq k < 1$.
(ii) p *is a fixed exponent such that* $2 < p$ *and* $kC_p < 1$ *where* C_p *is a constant stated in Theorem D.*
(iii) μ *has a distributional derivative* μ_z *such that* $\mu_z \in \mathrm{L}_p$.

Using this theorem, Theorem D is proved as follows: the assumptions of Theorem D admit an application of the theorem above to conclude that the equation

$$\partial v/\partial \bar{z} - \tilde{\beta}(t_1, x_1)\partial v/\partial z = 0$$

has a C^1 solution v in a neighborhood $U_1(\subset U_0)$ of the origin such that

$$|\partial v/\partial \bar{z}|^2 - |\partial v/\partial z|^2 > 0 .$$

Then we shall define $u = u(t, x)$ by

$$u(t, x) = v\left(\int_0^t a(s, x)ds, x - x_0\right).$$

Then it holds that, in a neighborhood of P,

$$Lu(t, x) = a(t, x)(1 + \alpha)(\partial v/\partial \bar{z} - \tilde{\beta}\partial v/\partial z) = 0 \quad \text{with } \partial u/\partial x \neq 0 .$$

Theorem D is thus proved.

Remark 3. As is already stated, one can verify that the assumption of Theorem D is satisfied when $a_{\text{even}} \equiv 0$ and $a_{\text{odd}}(t, x)$ vanishes of finite order on $t = 0$. Naturally Nirenberg's example does not satisfy the assumption of Theorem D; in more details, we can verify that the condition that $\beta(t, x)$ is extended as a continuous function of t_1 and of x_1 in a neighborhood of the origin is violated.

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Added in proof.

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