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On derivatives of holomorphic functions on a complex Wiener space

By

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This note is a complement of the joint work with Professor J. REN (cf. [1]). Let us keep the same notations as in [1]. Recall that a holomorphic function $F \in H^p(X, \mu)$ on a complex Wiener space X is defined by the limit in $L^p(X, \mu)$ of holomorphic polynomials on X.

1. H-derivatives

1.1. Proposition. *Let $F \in H^p(X, \mu)$, $h \in H$, then*

$$D_h F(x) = \lim_{\varepsilon \to 0} \frac{F(x+\varepsilon h) - F(x)}{\varepsilon} \text{ exists a.e.}$$

Proof. Let P_n be a sequence of holomorphic polynomials such that

$F = L^p - \lim_{n \to +\infty} P_n$.

Define $G_n(x, \xi) = P_n(x+\xi h)$ and $G(x, \xi) = F(x+\xi h)$, $\xi \in \mathbf{C}$. It is clear that $G_n(x, *)$ are holomorphic functions on $\mathbf{C}$. Let $R>r>0$, by Cauchy formula:

$$G_n(x, \xi) = \frac{1}{2\pi i} \int_{|\eta|=R} \frac{G_n(x, \eta)}{\eta - \xi} d\eta \ , \quad |\xi| < r \ .$$

Therefore:

$$\sup_{|\xi| \leq r} |G_n(x, \xi)| \leq \frac{R}{2\pi(R-r)} \int_0^{2\pi} |G_n(x, Re^{i\theta})| d\theta \ .$$

Taking the expectation relative to x, we get:

$$\begin{aligned} \mathbf{E}(\sup_{|\xi| \leq r} |G_n(x, \xi)|) &\leq \frac{R}{2\pi(R-r)} \mathbf{E}\Big(\int_0^{2\pi} |G_n(x, Re^{i\theta})| d\theta\Big) \\ &= \frac{R}{2\pi(R-r)} \int_0^{2\pi} \mathbf{E}|G_n(x, Re^{i\theta})| d\theta \\ &= \frac{R}{2\pi(R-r)} \int_0^{2\pi} \Big(\int_X P_n(x) e^{\langle x, Re^{i\theta}h\rangle - R^2\|h\|_H^2/4} d\mu(x)\Big) d\theta \\ &\leq \frac{R}{R-r} \|P_n\|_{L^p(X, \mu)} \exp\{(q-1) R^2 \|h\|_H^2 / 4\} \ . \end{aligned}$$

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It follows that there exists a subsequence n_k such that

$G_{n_k}(x, \xi)$ converge to $G(x, \xi)$ uniformly in $|\xi| \leq r$ a.e.

So $F(x+\xi h)$ is holomorphic in $|\xi|<r$. In particular, we have:

$$D_h F(x) = \left\{\frac{d}{d\varepsilon}F(x+\varepsilon h)\right\}_{\varepsilon=0} \text{ exists.}$$

It is natural now to ask if $F \in H^p(X, \mu)$ belongs to some Sobolev space $W_{p,r}(X)$? In infinite dimensional case, this is false, as shown by the following example.

1.2. Example. Let φ_k be a Hilbertian basis of $H^{*(1,0)}$, define:

$$F(x) = \sum_{k\geq 1} \frac{\langle \varphi_k, x\rangle^k}{k\sqrt{k!}} .$$

Then $F \in H^2(X, \mu)$. However let C be the Cauchy operator on X:

$$Cu_k = \sqrt{k}u_k$$

where $u_k(x) = \dfrac{\langle \varphi_k, x\rangle^k}{\sqrt{k!}}$. We have:

$$CF = \sum_{k\geq 1} (1/\sqrt{k}) u_k$$

and $\displaystyle\int_X |CF(x)|^2 d\mu(x) = \sum_k \frac{1}{k} = +\infty$.

2. Malliavin derivatives

2.1. A family of Borelian probability measures. Introduce first by

$$\rho_0 = \mu ,$$

$$\rho_n(A) = (2/\pi)\int_X d\rho_{n-1}(x) \int_{\mathbf{D}} \mathbf{1}_A(\xi, x) \log\frac{1}{|\xi|} d\sigma(\xi), \ A \subset X \text{ Borelian}$$

where $\mathbf{D}$ is unit disc of $\mathbf{C}$ and σ Lebesgue measure on $\mathbf{D}$. As remarked in [1], the measures introduced here ρ_n are singulier to the Wiener measure μ. So given a holomorphic function $F \in H^p(X, \mu)$, we have to extend its definition.

2.2. Redefinition of a holomorphic function. Let P_n be an approximating sequence of holomorphic polynomials of F in $L^p(X, \mu)$. By proposition 5.3 of [1], P_n is a Cauchy sequence in $L^p(X, \rho_k)$ for $0 \leq k \leq n$. So taking $\nu = \dfrac{\rho_0 + \cdots + \rho_n}{n+1}$, P_n is also a Cauchy sequence in $L^p(X, \nu)$. Now let $\tilde{F} = \mathrm{Lim}_{n\to+\infty} P_n$ in $L^p(X, \nu)$. Then $\tilde{F}$ satisfies the following properties

(i) $\tilde{F} = F$ μ-a.e. and $\tilde{F}$ is ρ_k-measurable $(0 \leq k \leq n)$.

2.3. Theorem. Given $F \in H^2(X, \mu)$, then

(i) $\tilde{\mathscr{L}}F = \mathrm{Lim}_{t\to 0} \dfrac{\tilde{T}_t F - \tilde{F}}{t}$ *exists in* $L^2(X, \rho_2)$;

(ii) $\int_X |\widetilde{\mathscr{L}}F|^2 d\rho_2 \leq \int_X |F|^2 d\mu$

where T_t is Ornstein-Uhlenbeck semi-group on X.

Proof. Let P be a holomorphic polynomial, we have

$$T_tP(x) = \int_X P(e^{-t}x + (1-e^{-2t})^{1/2}y) d\mu(y) = P(e^{-t}x) .$$

Therefore

2.3.1. $$\mathscr{L}P(x) = \mathrm{Lim}_{t\to 0}\frac{T_tP(x) - P(x)}{t} = \mathrm{Lim}_{t\to 0}\frac{P(e^{-t}x) - P(x)}{t}$$
$$= \mathrm{Lim}_{\varepsilon\to 0, \varepsilon>0}\frac{P((1-\varepsilon)x) - P(x)}{\varepsilon} .$$

It follows from proposition 5.4 of [1] that

$$\int_X |\mathscr{L}P|^2 d\rho_1 \leq \int_X |P|^2 d\mu .$$

Now by definition of ρ_2 and proposition 5.3 of [1], we get

2.3.2. $$\int_X |\mathscr{L}P|^2 d\rho_2 \leq \int_X |P|^2 d\mu .$$

Replacing P by $\mathscr{L}P$ and integrating the two sides of (5.9) of [1] with respect to ρ_1, we obtain

$$\int_X |\mathscr{L}^2P|^2 d\rho_2 \leq \int_X |\mathscr{L}P|^2 d\rho_1 .$$

So

2.3.3. $$\int_X |\mathscr{L}^2P|^2 d\rho_2 \leq \int_X |P|^2 d\mu .$$

Put $G(t, x) = T_tP(x)$, we have $G'(t, x) = \frac{d}{dt}G(t, x) = T_t\mathscr{L}P(x)$. We have:

$$T_tP(x) = P(x) + \int_0^t \mathscr{L}T_sP(x) ds ,$$

$$\sup_{0\leq t\leq 1}|T_tP(x)|^2 \leq 2(|P(x)|^2 + \int_0^1 |\mathscr{L}T_sP(x)|^2 ds) .$$

2.3.4. $$\int_X \sup_{0\leq t\leq 1}|T_tP(x)|^2 d\rho_2(x)$$
$$\leq 2\left(\int_X |P(x)|^2 d\rho_2(x) + \int_0^1 ds \int_X |\mathscr{L}T_sP(x)|^2 d\rho_2(x)\right) .$$

Now taking $\mathscr{L}P$ as P in 2.3.4. and using 2.3.2 and 2.3.3, we obtain

2.3.5. $$\int_X \sup_{0\leq t\leq 1}|G'(t, x)|^2 d\rho_2(x) \leq 4\int_X |P|^2 d\mu .$$

Take P_n as a approximating sequence of holomorphic polynomials of F in $L^2(X, \mu)$ and $G_n(t, x) = T_tP_n(x)$. By 2.3.4. and 2.3.5, there exist a subset $A\subset X$ such that $\rho_2(A) = 1$ and a subsequence n_k such that for $x\in A$ $G'_{n_k}(t, x)$ converge uniformly in $t\in[0, 1]$ and $G_n(t, x)$ converge to $\widetilde{T}_tF(x)$.

Therefore for $x \in A$, $\frac{d}{dt}\widetilde{T}_t F(x)$ exists and

$$\int_X \sup_{0 \leq t \leq 1} \left|\frac{d}{dt}\widetilde{T}_t F(x)\right|^2 d\rho_2(x) \leq 4\int_X |P|^2 d\mu \ .$$

As $\left|\frac{\widetilde{T}_t F - \widetilde{F}}{t}\right| \leq \sup_{0 \leq t \leq 1} \left|\frac{d}{dt}\widetilde{T}_t F(x)\right|$ for $0 \leq t \leq 1$, by Lebesgue dominated theorem, we get (i). (ii) follows from 2.3.2.

2.4. Higher order Malliavin derivatives.

2.4.1. Theorem. We have:

i) *$\widetilde{\mathscr{L}}^n F$ exists in $L^2(X, \rho_{2n})$;*

ii) $\int_X |\widetilde{\mathscr{L}}^n F|^2 d\rho_{2n} \leq \int_X |F|^2 d\mu$.

Proof. By induction.

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