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A note on homogeneous Kähler manifolds of semi-simple Lie groups

By

Yūta HAYASHI and Kazufumi NAKAJIMA

1. By a homogeneous Kähler manifold G/K we mean a homogeneous space of a connected Lie goup G by a closed subgroup K equipped with a G-invariant Kähler structure. Denote by $\mathrm{Aut}(G/K)$ the group of all holomorphic isometries of G/K and by $\mathrm{Aut}(G/K)_0$ its identity component. We now assume that G is semi-simple. It is well-known that if G/K is a hermitian symmetric space of the non-compact type, then G coincides with $\mathrm{Aut}(G/K)_0$ under the effectiveness assumption on G. This does not hold in general for the compact case. Actually, the unitary symplectic group $Sp(n)$ acts transitively on $2n-1$ dimensional complex projective space. The purpose of the the present note is to show the following

Theorem. *Let G/K be a homogeneous Kähler manifold of a semi-simple Lie group G. Then* $\mathrm{Aut}(G/K)$ *is semi-simple. Assume further that G acts effectively on G/K and that every simple factor of G is of non-compact type. Then G coincides with* $\mathrm{Aut}(G/K)_0$.

2. Proof of Theorem. Let G/K be a homogeneous Kähler manifold of a connected semi-simple Lie group G by a closed subgroup K. We assume that the action of G is effective. Let B be a maximal compact subgroup of G containing K. By a result of Borel [1] (see also Shima [3]), G/B is a hermitian symmetric space of the non-compact type, the projection $\pi: G/K \to G/B$ is holomorphic and B/K is a Kähler C-space, i.e, a compact simply connected homogeneous Kähler manifold, equipped with the induced Kähler structure. We set $G' = \mathrm{Aut}(G/K)_0$. From our assumption, G is a subgroup of G'. Let K' denote the isotropy subgroup of G' at the origin of G/K. We then have $G/K = G'/K'$. For every element $g \in G'$, $\pi \circ g$ sends B/K to a single point, because G/B is holomorphically equivalent to a bounded domain. Thus g induces a holomorphic transformation $\hat{g}$ of G/B and the correspondence $\tau: g \to \hat{g}$ gives a homomorphism of G' to $\mathrm{Aut}(G/B)$. (Note that $\mathrm{Aut}(G/B)$ coincides with the group of all holomorphic transformations of G/B.) Through the map τ, G' acts on G/B transitively and we can write as $G/B = G'/B'$, where B' is the isotropy subgroup of G' at the origin of G/B.

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Let $\mathfrak{g}$, $\mathfrak{g}'$, $\mathfrak{k}$, $\mathfrak{k}'$, $\mathfrak{b}$, $\mathfrak{b}'$ be the Lie algebras of G, G', K, K', B, B' respectively. We denote by the same letter τ the homomorphism of $\mathfrak{g}'$ to the Lie algebra of Aut(G'/B') corresponding to the group homomorphism τ. We also denote by ρ the natural homomorphism of $\mathfrak{b}'$ to the Lie algebra of Aut(B'/K'). Let $\mathfrak{n}'$ be the kernel of τ in $\mathfrak{g}'$. Then $\mathfrak{n}'$ is an ideal of $\mathfrak{g}'$ contained in $\mathfrak{b}'$. We assert that ρ is injective on $\mathfrak{n}'$. Indeed, suppose that $\rho(x)=0$ for $x\in\mathfrak{n}'$. Then $x\in\mathfrak{k}'$, and ad x maps $\mathfrak{g}'$ to $\mathfrak{b}'$ and $\mathfrak{b}'$ to $\mathfrak{k}'$. Since ad x is semi-simple, we know that ad x sends $\mathfrak{g}'$ to $\mathfrak{k}'$ and hence $x=0$, proving the assertion.

We can now show the semi-simplicity of Aut(G/K). Let $\mathfrak{r}'$ denote the radical of $\mathfrak{g}'$. From the fact stated in the introduction, $\tau(\mathfrak{g})$ coincides with the Lie algebra of Aut(G/B). Therefore, $\tau(\mathfrak{g}')=\tau(\mathfrak{g})$ and hence $\tau(\mathfrak{g}')$ is semi-simple, showing that $\mathfrak{r}'\subset\mathfrak{n}'$. Since B'/K' is a Käler C-space, every subgroup of Aut(B'/K') acting transitively on B'/K' is semi-simple (c.f. [2]). Therefore the solvable ideal $\rho(\mathfrak{r}')$ of $\rho(\mathfrak{b}')$ must be trivial. Since ρ is injective on $\mathfrak{r}'$, we get $\mathfrak{r}'=0$, proving that Aut(G/K) is semi-simple.

Next we show the latter part of our theorem. We already know that $\mathfrak{g}'$ is semi-simple. Therefore there exists a semi-simple ideal $\mathfrak{g}''$ of $\mathfrak{g}'$ such that $\mathfrak{g}'=\mathfrak{n}'\oplus\mathfrak{g}''$. Let α denote the projection of $\mathfrak{g}'$ to $\mathfrak{n}'$ according to this decomposition. Since $\rho(\mathfrak{n}')$ is an ideal of the compact semi-simple Lie algebra $\rho(\mathfrak{b}')$, $\mathfrak{n}'$ is of compact type. We now assume that every simple factor of $\mathfrak{g}$ is non-compact. Then $\mathfrak{n}'\cap\mathfrak{g}=0$ and $\alpha(\mathfrak{g})=0$. The first equality means that $\mathfrak{g}$ is isomorphic to $\mathfrak{g}''$ and the second one implies that $\mathfrak{g}$ is containd in $\mathfrak{g}''$. Therefore $\mathfrak{g}=\mathfrak{g}''$. Recall that K' is the centralizer of a toral subgroup of G' ([1]]. Then $\mathfrak{k}'=(\mathfrak{k}'\cap\mathfrak{n}')\oplus(\mathfrak{k}'\cap\mathfrak{g})$. This yields $\mathfrak{n}'\subset\mathfrak{k}'$, because $\mathfrak{k}=\mathfrak{k}'\cap\mathfrak{g}$ and $\dim\mathfrak{g}/\mathfrak{k}=\dim\mathfrak{g}'/\mathfrak{k}'$. Now from the effectiveness of $\mathfrak{g}'$, we conclude that $\mathfrak{n}'=0$, completing the proof of our theorem.

DEPARTMENT OF MATHEMATICS
RITSUMEIKAN UNIVERSITY
KYOTO 603, JAPAN
DEPARTMENT OF MATHEMATICS
RITSUMEIKAN UNIVERSITY
KYOTO 603, JAPAN

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