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On the space of schlicht projective structures on compact Riemann surfaces with boundary

By

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§ 1. Introduction

Let Γ be an arbitrary Fuchsian group acting on the upper half plane $\mathbf{H} = \{z \in \mathbf{C}; \operatorname{Im} z > 0\}$. We denote by $S(\Gamma)$ the set consisting of the Schwarzian derivative S_f of all the univalent meromorphic functions f on $\mathbf{H}$ with $f \circ \gamma = \chi(\gamma) \circ f$ on $\mathbf{H}$ for some group homomorphism $\chi: \Gamma \to$ Möb. Then it turns out that $S(\Gamma)$ is a bounded closed subset of the complex Banach space $B_2(\mathbf{H}, \Gamma)$ (see § 2 for its precise definition). It is an interesting matter to investigate how (the Bers model of) the Teichmüller space $T(\Gamma)$ is embedded in $S(\Gamma)$. Generally, $\overline{T(\Gamma)} \subsetneq S(\Gamma)$ holds. In fact, first Gehring has shown that $\overline{T(1)} \subsetneq S(1)$ in [7], and later the author proved in [14] that $\overline{T(\Gamma)} \subsetneq S(\Gamma)$ for any Fuchsian group Γ of the second kind. Moreover, recently K. Matsuzaki showed in [9] the existence of certain infinitely generated Fuchsian groups Γ of the first kind such that $\overline{T(\Gamma)} \subsetneq S(\Gamma)$. But, it is still a difficult problem to decide whether $\overline{T(\Gamma)} = S(\Gamma)$ for a finitely generated Fuchsian group Γ of the first kind. (We remark that this problem is equivalent to the Bers conjecture: any b-group is a boundary group of the Teichmüller spaces.)

On the other hand, Gehring has shown in [6] that $\operatorname{Int} S(1) = T(1)$. Furthermore Žuravlev showed in [17] that $T(\Gamma)$ is the zero component of $\operatorname{Int} S(\Gamma)$ for an arbitrary Fuchsian group Γ. Thus, it is naturally conjectured that $\operatorname{Int} S(\Gamma) = T(\Gamma)$ for any Γ. In this direction, Shiga proved in [13] that the above conjecture holds if Γ is finitely generated Fuchsian group of the first kind, equivalently, if $B_2(\mathbf{H}, \Gamma)$ is finite dimensional.

The main theorem in this article (Theorem 2.1) is the claim that $\operatorname{Int} S(\Gamma) = T(\Gamma)$ for any Fuchsian group Γ uniformizing a compact (bordered) Riemann surface with nonempty boundary, in other words, for finitely generated, purely hyperbolic Fuchsian group Γ of the second kind. In order to prove this theorem, we shall utilize Gehring's method in [6] with several localization techniques for overcoming difficulties caused by the group action. Here we remark that our proof does not depend on Žuravlev's result.

The proof of the main theorem divides into several steps as follows. In § 2, we prepare terminologies and notations for later use, and state the main theorem and some lemmas. Let Γ be an arbitrary Fuchsian group, $\varphi \in \operatorname{Int} S(\Gamma)$ and f

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be a univalent function such that $S_f = \varphi$. To say that $\varphi \in T(\Gamma)$, we have to show essentially that $D = f(\mathbf{H})$ is a quasidisk, or equivalently, D is a locally connected John domain (if D is bounded).

In §3, we will show that D is locally connected, but "locally" in $\Omega(G)$ where $G = f^{-1}\Gamma f$ (Proposition 3.1). Roughly speaking, in the "island" D, there is no very deep bay. In fact, if such a deep bay exists, one can construct a G-equivariant meromorphic map g on D with small Schwarzian which shuts its inlet (thus, is not univalent) by bending D a little, and this will lead to a contradiction. As a corollary of this result, we see that $\partial D = \partial(\hat{\mathbf{C}} \setminus \bar{D})$, in particular, $\hat{\mathbf{C}} \setminus \bar{D} \neq \emptyset$, for any $\varphi \in \operatorname{Int} S(\Gamma)$.

In §4, we also see that D is a John domain, at least "locally" in $\Omega(G)$ (Proposition 4.1). Roughly speaking again, there is no peninsula so much constricted in the island D. In fact, if such a peninsula, one can construct a G-equivariant meromorphic map g on D with small Schwarzian which touches the opposite shore of D by lengthening a narrow part of the peninsula, and this also will lead to a contradiction.

In both steps, we shall accomplish the construction of g as follows: first, we construct a G-equivariant quasi-regular (in fact, quasiconformal locally, but not necessarily injective) map h with small deformation which has the same properties as g except the holomorphy. By an appropriate construction of h, the Beltrami coefficient μ of h^{-1} can be well-defined, so we can choose $w^\mu \circ h$ as g, where w^μ is a μ-qc map of $\hat{\mathbf{C}}$ (here, for example, μ was extended to 0 in $h(D)^c$). For estimation of the norm of the Schwarzian derivative of $w^\mu \circ h$, we shall utilize the "local norm technique" as in [16].

In §5, for a Fuchsian group uniformizing a compact bordered Riemann surface with nonempty boundary, we prove that the boundary of D/G in $\Omega(G)/G$ is a disjoint union of quasi-analytic curves by invoking the annular covering argument. Thus, in particular, the induced conformal map $F: \mathbf{H}/\Gamma \to D/G$ by f can be naturally extended to a homeomorphism $\overline{\mathbf{H}/\Gamma} \to \overline{D/G}$. Furthermore, in §6, F turns out to be extended to a quasiconformal map $\tilde{F}: \Omega(\Gamma)/\Gamma \to \Omega(G)/G$ which can be lifted to a quasiconformal map $\tilde{f}: \Omega(\Gamma) \to \Omega(G)$. This fact follows essentially from the existence of a G-equivariant quasiconformal reflection with respect to ∂D. Since $\tilde{f}$ may be continued to a quasiconformal self-map of $\hat{\mathbf{C}}$, it has shown that $\varphi = S_f \in T(\Gamma)$.

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§2. Preliminaries and the main theorem

In this section, we shall fix the terminologies needed below and state related facts and the main theorem. As a general reference, we refer to the textbook [10] by S. Nag.

Projective structures on a Riemann surface. Let R be a hyperbolic Riemann surface and $p:\mathbf{H}\to R$ be a holomorphic universal covering of R, where $\mathbf{H}=\{z\in\mathbf{C};\ \mathrm{Im}\, z>0\}$ is the upper half plane. A *projective chart* on the Riemann surface R is a complex chart on R such that the transition functions are (locally) restrictions of Möbius transformations. Two projective charts on R are *equivalent* if their union is also a projective chart. Equivalence classes of projective charts on R are called *projective structures* on R.

Let σ be a projective structure on R represented by a chart $\{\psi_\alpha\colon U_\alpha\to V_\alpha;\ \alpha\in A\}$. Set $\tilde U_\alpha := p^{-1}(U_\alpha)$, and write $\tilde\psi_\alpha=\psi_\alpha\circ p$ on $\tilde U_\alpha$ for $\alpha\in A$. Then $(\tilde U_\alpha)_{\alpha\in A}$ is an open covering of $\mathbf{H}$. Define $\psi_{\alpha\beta}=\psi_\alpha\circ\psi_\beta^{-1}$ on $\psi_\beta(U_\alpha\cap U_\beta)$, then $\psi_{\alpha\beta}$ is a restriction of Möbius transformation on each component of $\psi_\beta(U_\alpha\cap U_\beta)$, by the very hypothesis. And we have

$$\tilde\psi_\alpha=\psi_{\alpha\beta}\circ\tilde\psi_\beta \qquad \text{on } \tilde U_\alpha\cap\tilde U_\beta \tag{2.1}$$

for any $\alpha,\beta\in A$.

Here, we recall some of the properties of the Schwarzian derivative. The Schwarzian derivative S_f of a non-constant meromorphic function f on a plane domain is defined by

$$S_f=\left(\frac{f''}{f'}\right)'-\frac{1}{2}\left(\frac{f''}{f'}\right)^2.$$

S_f is holomorphic at a point if and only if f is locally schlicht (= locally univalent) at the point. And, $S_f=0$ on a domain $D\subset\hat{\mathbf{C}}$ if and only if f is a restriction of a Möbius transformation. Further, if f and g are meromorphic functions and if $f\circ g$ is defined, then the following important formula (the Cayley identity) holds:

$$S_{f\circ g}=(S_f)\circ g\cdot(g')^2+S_g. \tag{2.2}$$

By the above properties and (2.1), we obtain that

$$S_{\tilde\psi_\alpha}=S_{\tilde\psi_\beta} \qquad \text{on } \tilde U_\alpha\cap\tilde U_\beta$$

for any $\alpha,\beta\in A$. Thus a holomorphic function $\varphi\colon\mathbf{H}\to\mathbf{C}$ is well-defined by

$$\varphi=S_{\tilde\psi_\alpha} \qquad \text{on } \tilde U_\alpha$$

for any α. Moreover, by the relation (2.2), we can see that the holomorphic function φ satisfies the following functional equations:

$$(\varphi\circ\gamma)\cdot(\gamma')^2=\varphi$$

for all $\gamma\in\Gamma$, where $\Gamma<\text{Möb}$ is the covering transformation group of $p:\mathbf{H}\to R$. The above φ is called a *holomorphic quadratic differential* for Γ on $\mathbf{H}$. We will denote by $Q(\mathbf{H},\Gamma)$ the set of all the holomorphic quadratic differentials for Γ on $\mathbf{H}$.

Conversely, let a holomorphic quadratic differential φ for Γ on $\mathbf{H}$ be given. Consider the following homogeneous linear ordinary differential equation:

$$y'' + \frac{1}{2}\varphi y = 0 \quad \text{on } \mathbf{H}. \tag{2.3}$$

Since $\mathbf{H}$ is simply connected, there exists a pair of fundamental solutions (y_0, y_1) on $\mathbf{H}$ uniquely determined by the initial condition

$$y_0(i) = 0,\ y_0'(i) = 1;\quad y_1(i) = 1,\ y_1'(i) = 0. \tag{2.4}$$

Noting that $y_0'y_1 - y_0y_1' \equiv 1$, we obtain that $f^\varphi := y_0/y_1$ satisfies the following conditions:

$$S_{f^\varphi} = \varphi \quad \text{on } \mathbf{H}, \tag{2.5}$$

$$f^\varphi(z) = (z - i) + O(|z - i|^3) \quad \text{as } z \to i = \sqrt{-1}. \tag{2.6}$$

It should be remarked that $f^\varphi: \mathbf{H} \to \hat{\mathbf{C}}$ is uniquely determined by the above conditions (2.5) and (2.6).

Let $\gamma \in \Gamma$ be represented by $\gamma(z) = \dfrac{az+b}{cz+d}$ for some $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbf{C})$. Conventionally, we write $(\gamma')^{-1/2} = cz + d$, then $\tilde{y}_j = (y_j \circ \gamma)(\gamma')^{-1/2}$ becomes a solution of (2.3) again. ($\tilde{y}_j$ may be considered as the analytic continuation of the solution y_j along with a path from i to $\gamma(i)$.) Therefore $\tilde{y}_0$ and $\tilde{y}_1$ are uniquely represented by linear combinations of y_0 and y_1 as

$$\begin{aligned} \tilde{y}_0 &= Ay_0 + By_1 \\ \tilde{y}_1 &= Cy_0 + Dy_1, \end{aligned} \tag{2.7}$$

where A, B, C and D are constants. Since $\tilde{y}_0'\tilde{y}_1 - \tilde{y}_0\tilde{y}_1' \equiv 1$, $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{SL}(2, \mathbf{C})$.

We denote by $\chi^\varphi(\gamma)$ the Möbius transformation $\dfrac{Az+B}{Cz+D}$, which is independent of the choice of signature of $(\gamma')^{-1/2}$. The group homomorphism $\chi^\varphi: \Gamma \to \text{Möb}$ is called a *holonomy homomorphism* associated with φ. By (2.7), we have the following transformation formula for f^φ:

$$f^\varphi \circ \gamma = \chi^\varphi(\gamma) \circ f^\varphi \quad \text{for all } \gamma \in \Gamma. \tag{2.8}$$

Such a meromorphic map f^φ as $S_{f^\varphi} = \varphi$ is called a *developing map* of φ, and also the pair $(f^\varphi, \chi^\varphi)$ is called a *deformation* of the Fuchsian group Γ.

In this article, we will call φ a *schlicht projective structure* if its developing map f^φ is schlicht (= univalent) in $\mathbf{H}$. Let $S(\Gamma)$ denote the set of totality of schlicht projective structures for Γ on $\mathbf{H}$. The Nehari-Kraus theorem states that if f^φ is schlicht in $\mathbf{H}$ then

$$\|\varphi\|_{\mathbf{H}} = \sup_{z \in \mathbf{H}} |\varphi(z)|(2\operatorname{Im} z)^2 \le 6.$$

So it is natural to consider a complex Banach space $B_2(\mathbf{H}, \Gamma) = \{\varphi \in Q(\mathbf{H}, \Gamma);\ \|\varphi\|_{\mathbf{H}} < \infty\}$. Of course $S(\Gamma) \subset B_2(\mathbf{H}, \Gamma)$.

By Hurwitz's theorem, it turns out that $S(\Gamma)$ is closed in $B_2(\mathbf{H}, \Gamma)$.

On the other hand, $S(\Gamma)$ is closely related to the Teichmüller space $T(\Gamma)$ of Γ, where (the Bers model of) the Teichmüller space $T(\Gamma)$ of Γ is defined by

$$T(\Gamma) = \{\varphi \in Q(\mathbf{H}, \Gamma); f^\varphi \text{ can be extended to a } \Gamma\text{-compatible quasiconformal homeomorphism of } \hat{\mathbf{C}}\},$$

where we say that f is *Γ-compatible* if $f \circ \gamma \circ f^{-1} \in \text{Möb}$ for all $\gamma \in \Gamma$.

It is a well-known fact that $T(\Gamma)$ is a bounded connected open set of $B_2(\mathbf{H}, \Gamma)$. Clearly $T(\Gamma) \subset S(\Gamma)$, and it is conjectured that $T(\Gamma) = \operatorname{Int} S(\Gamma)$, where $\operatorname{Int} S(\Gamma)$ denotes the interior of $S(\Gamma)$ in the Banach space $B_2(\mathbf{H}, \Gamma)$.

Now we state the main theorem, which is a generalization of Gehring's result in [6].

2.1. Main Theorem. *If Γ is a finitely generated, purely hyperbolic Fuchsian group of the second kind, then $T(\Gamma) = \operatorname{Int} S(\Gamma)$.*

2.2. Remark. For a Fuchsian group Γ acting on $\mathbf{H}$, the following conditions are mutually equivalent:

(i) Γ is finitely generated, purely hyperbolic and of the second kind,

(ii) Γ is a Schottky group,

(iii) Γ is a uniformizing group of a compact bordered Riemann surface with nonempty boundary, more presicely, Γ is the covering transformation group of a holomorphic universal covering $p: \mathbf{H} \to R$, where R is a compact Riemann surface of genus g (≥ 0) with mutually closed topological disks $\bar{D}_1, \ldots, \bar{D}_m$ removed $(m \geq 1)$.

In case of (iii), we say that R is of conformal type $(g, 0, m)$, and we should note that Γ is a free group of rank $2g + m - 1$.

In the sequel, we are mainly concerned with the properties of a point in $\operatorname{Int} S(\Gamma)$ for an arbitrary Fuchsian group Γ, more precisely, the shape of the domain $D = D^\varphi = f^\varphi(\mathbf{H})$ for $\varphi \in \operatorname{Int} S(\Gamma)$.

First, for $\varphi \in S(\Gamma)$, the holonomy homomorphism $\chi^\varphi: \Gamma \to \text{Möb}$ is injective and $G = \chi^\varphi(\Gamma) < \text{Möb}$ acts on $D = f^\varphi(\mathbf{H})$ discontinuously, therefore $D \subset \Omega(G)$, in particular, G is a Kleinian group. Furthermore, for $\varphi \in \operatorname{Int} S(\Gamma)$, $\chi^\varphi: \Gamma \to G$ enjoies the following property.

2.3. Lemma (cf. [15]). *For $\varphi \in \operatorname{Int} S(\Gamma)$, the holonomy homomorphism $\chi^\varphi: \Gamma \to G = \chi^\varphi(\Gamma)$ is a type-preserving isomorphism.*

Proof. First, we remark that the mapping $\varphi \mapsto \operatorname{tr}^2 \chi^\varphi(\gamma)$ is holomorphic on $B_2(\mathbf{H}, \Gamma)$, where γ is a fixed element of Γ and $\operatorname{tr}^2 g = (a + d)^2$ if the Möbius transformation g is represented by $g(z) = \dfrac{az + b}{cz + d}$ with $ad - bc = 1$.

If γ is parabolic or elliptic, i.e., $\operatorname{tr}^2 \gamma = 4\cos^2 q\pi$ for some rational number q, then $\operatorname{tr}^2 \chi^\varphi(\gamma) = 4\cos^2 q\pi$ for $\varphi \in T(\Gamma)$ by quasiconformal homogeneity of $T(\Gamma)$.

Since $T(\Gamma)$ is open in $B_2(\mathbf{H}, \Gamma)$, the identity theorem implies that $\mathrm{tr}^2\, \chi^{\varphi}(\gamma) = 4\cos^2 q\pi$ for all $\varphi \in B_2(\mathbf{H}, \Gamma)$. Thus χ^{φ} preserves types elliptic and parabolic.

Finally, let $\gamma \in \Gamma$ be hyperbolic. Since χ^{φ} is injective for $\varphi \in \mathrm{Int}\, S(\Gamma)$, $\chi^{\varphi}(\gamma)$ should not be elliptic. Suppose that $\chi^{\varphi_0}(\gamma)$ becomes parabolic, i.e., $\mathrm{tr}^2\, \chi^{\varphi_0}(\gamma) = 4$, for some φ_0, the identity theorem produces again that $\mathrm{tr}^2\, \chi^{\varphi}(\gamma)$ is a nonconstant map on $\mathrm{Int}\, S(\Gamma)$, so the image of $\mathrm{Int}\, S(\Gamma)$ under this map is open neighborhood of 4. In particular, for sufficiently large $n \in \mathbf{N}$ there exists a point φ_1 in $\mathrm{Int}\, S(\Gamma)$ such that $\mathrm{tr}^2\, \chi^{\varphi_1}(\gamma) = 4\cos^2 \dfrac{\pi}{n}$, which is a contradiction. Q.E.D.

2.4. Remark. The proof of the above lemma in [15] relies upon the λ-lemma. The author has learned the idea in the above proof from H. Shiga.

Hyperbolic sup norm. For later use, we shall fix several notations in the more general situation. Let D be a hyperbolic simply connected domain with the Poincaré metric $\rho_D(z)|dz|$ of negative constant curvature -4, and let G be a Kleinian group acting on D (, which is not necessarily a component of $\Omega(G)$). The complex Banach space $B_2(D, G)$ is defined as the set

$$\{\varphi\colon D \to \mathbf{C}\colon \text{holomorphic map}; (\varphi \circ L)(L')^2 = \varphi \text{ for all } L \in G,\ \|\varphi\|_D < \infty\},$$

where $\|\varphi\|_D = \sup_{z \in D} |\varphi(z)|\, \rho_D(z)^{-2}$. If G is the trivial group, we write $B_2(D, G)$ simply by $B_2(D)$. Let $f\colon \mathbf{H} \to D$ be a conformal map, then by the conformal invariance of the Poincaré metric, we have

$$\|(\varphi \circ f)(f')^2\|_{\mathbf{H}} = \|\varphi\|_D \qquad \text{for } \varphi \in B_2(D). \tag{2.9}$$

G-Schwarzian domains. Under the above preparations, we shall state a characterization of such a domain D as $f^{\varphi}(\mathbf{H})$ obtained from some $\varphi \in \mathrm{Int}\, S(\Gamma)$.

2.5. Lemma. *Let Γ be an arbitrary Fuchsian group acting on $\mathbf{H}$. For $\varphi \in S(\Gamma)$, φ belongs to $\mathrm{Int}\, S(\Gamma)$ if and only if the domain $D = f^{\varphi}(\mathbf{H})$ has the following property: There exists a positive constant $\varepsilon > 0$ such that any non-constant meromorphic map $g\colon D \to \hat{\mathbf{C}}$ with $S_g \in B_2(D, G)_{\varepsilon}$ must be univalent in $\mathbf{H}$, where $G = \chi^{\varphi}(\Gamma)$ and $B_2(D, G)_{\varepsilon} = \{\psi \in B_2(D, G);\ \|\psi\|_D < \varepsilon\}$.*

When a Kleinian group G acts on a hyperbolic domain D (not necesarily simply connected), D is called a *G-Schwarzian domain with constant* ε if the above property holds.

Proof. By (2.2) and (2.9) we obtain the equality

$$\|S_{g \circ f^{\varphi}} - \varphi\|_{\mathbf{H}} = \|S_g\|_D,$$

which implies what we need here. Q.E.D.

Next we refer to the local quasiconformal homogeneity of $\mathrm{Int}\, S(\Gamma)$, which plays an important role in §4. The proof of the following proposition is deeply indebted to a group equivariant version of the λ-lemma.

2.6. Proposition ([15]). *Let V be a connected component of* $\operatorname{Int} S(\Gamma)$. *For any $\varphi_1, \varphi_2 \in V$, there exists a quasiconformal self-map F of $\hat{\mathbf{C}}$ with the following properties*:

(1) $f^{\varphi_2} = F \circ f^{\varphi_1}$ *on* $\mathbf{H}$,

(2) $\chi^{\varphi_2}(\gamma) = F \circ \chi^{\varphi_1}(\gamma) \circ F^{-1}$ *on* $\hat{\mathbf{C}}$ *for all* $\gamma \in \Gamma$.

An estimate of the hyperbolic sup norm. In what follows, it becomes important to estimate the magnitude of the hyperbolic sup norm $\|\varphi\|_D$, and so we now give a method to controle the norm by the another (relatively) local data easy to treat. Let $A \in [1, \infty)$ be a constant and D a proper subdomain of $\mathbf{C}$. Define $\mathscr{D}_A(D)$ by the collection of all disks $B(z_0, r) = \{z \in \mathbf{C}; |z - z_0| < r\}$ such that $B(z_0, Ar) \subset D$.

In this article, an orientation-preserving homeomorphism (or, non-constant continuous map) $f: D_1 \to D_2$ shall be called, conventionally, a *k-quasiconformal* map (or, *k-quasiregular* map, respectively) where $k \in [0, 1)$ is a constant if f has locally L^2-derivatives such that $|\partial_{\bar{z}} f| \leq k |\partial_z f|$ almost everywhere in D_1. A quasiconformal map is often called a qc map, for short. And, we denote here by $\mu[f]$ the *Beltrami coefficient* $\partial_{\bar{z}} f / \partial_z f$ of quasiconformal map (or, quasi-regular map) f. We remark that, since $\partial_z f \neq 0$ a.e., $\mu[f]$ is well-defined. Thus, a quasiconformal map f is k-qc if and only if $\|\mu[f]\|_\infty \leq k$. We should remark that such an f is ordinarily called K-quasiconformal where $K = \dfrac{1+k}{1-k} \in [1, \infty)$, and this terminology has a advantageous property that the composition map $f_1 \circ f_2$ is $K_1 \cdot K_2$-qc if f_1 is K_1-qc and f_2 is K_2-qc. With these notations, we have the following

2.7. Proposition (cf. [2], [16]). *Let D be a simply connected hyperbolic subdomain of $\mathbf{C}$, $A \geq 1$ and $k \in [0, 1)$ be constants, and f be a non-constant meromorphic function on D. If $f|_\Delta$ can be extended to a k-qc map of $\hat{\mathbf{C}}$ for any $\Delta \in \mathscr{D}_A(D)$, then $\|S_f\|_D \leq 96kA^2$.*

Conversely, if $\|S_f\|_D \leq 2kA^2$ then $f|_\Delta$ can be extended to a k-qc map of $\hat{\mathbf{C}}$ for any $\Delta \in \mathscr{D}_A(D)$.

Bers projection. The measurable Riemann mapping theorem due to Ahlfors-Bers claims that, for $\mu \in L^\infty(\mathbf{C})$ with $\|\mu\|_\infty < 1$, there exists a unique quasiconformal homeomorphism of $\hat{\mathbf{C}}$, denoted by w^μ, such that $\partial_{\bar{z}} w^\mu = \mu \partial_z w^\mu$ a.e. and $w^\mu(0) = 0$, $w^\mu(1) = 1$, $w^\mu(\infty) = \infty$.

Let G be a Kleinian group acting on an open set $D \subset \hat{\mathbf{C}}$. We set $E = \mathbf{C} \setminus D$. $L^\infty(E, G)$ and $M(E, G)$ denote the complex Banach space $\{\mu \in L^\infty(\mathbf{C});$ $\mu = 0$ on $D, (\mu \circ L) \cdot \bar{L}'/L' = \mu$ a.e. for all $L \in G\}$ and its open unit ball, respectively. If $G = 1$, we shall write $L^\infty(E) = L^\infty(E, 1)$ and $M(E) = M(E, 1)$ for simplicity.

For $\mu \in M(E, G)$, by the automorphy of μ, w^μ conjugates G to another Kleinian group, i.e., $w^\mu G (w^\mu)^{-1} <$ Möb, and $w^\mu|_D$ is conformal since $\mu = 0$ on

D. As a result, the Schwarzian derivative of $w^{\mu}|_D$ is well-defined and turns out to be a (bounded) holomorphic quadratic differential for G on D. Particularly, when D is simply connected domain of hyperbolic type, we denote by $\Phi_D(\mu)$ the Schwarzian derivative of $w^{\mu}|_D$, and which is called the (generalized) *Bers projection* of $\mu \in M(E, G)$. As is well-known, Φ_D: $M(E, G) \to B_2(D, G)$ is holomorphic and its differential at the origin is represented as an integral operator (cf. [15]):

$$d_0\Phi_D[\nu](z) = -\frac{6}{\pi}\iint_E \frac{\nu(\zeta)}{(\zeta - z)^4}\, d\xi d\eta \qquad (\zeta = \xi + i\eta)$$

for every $\nu \in L^{\infty}(E, G)$. In the special case that $D = \mathbf{H}$, $\Phi_{\mathbf{H}}$ is the original Bers projection and its image $\Phi_{\mathbf{H}}(M(\mathbf{H}^c, G))$ is the Teichmüller space $T(G)$ of the Fuchsian group G.

As a corollary of the main theorem, we can verify the following:

2.8. Corollary. *Let D be a simply connected subdomain of $\hat{\mathbf{C}}$ of hyperbolic type and E its complement. Suppose that a Schottky group G acts on D. Then the following conditions are equivalent to each other*:

(1) *$d_0\Phi_D$: $L^{\infty}(E, G) \to B_2(D, G)$ is surjective,*

(2) *$d_0\Phi_D$: $L^{\infty}(E) \to B_2(D)$ is surjective, and*

(3) *D is a quasidisk.*

Proof. As the claim (2)$\Leftrightarrow$(3) is a special case $G = 1$ of (1)$\Leftrightarrow$(3), it suffices to prove (1)$\Leftrightarrow$(3). The part (3)$\Rightarrow$(1) is a direct consequence of the submersivity of the generalized Bers projection (cf. Bers [3], Earle-Nag [5]). Thus we have only to prove that (1) implies (3). First observe that if $d_0\Phi_D$: $L^{\infty}(E, G) \to B_2(D, G)$ is surjective then $\Phi_D(M(E, G))$ is a neighborhood of 0 in $B_2(D, G)$ (see, for instance, [1] Proposition 2.5.9), that is, D is a G-Schwarzian domain. Let f: $\mathbf{H} \to D$ be a Riemann mapping function of D and φ its Schwarzian derivative. Then, the above observation shows that $\varphi \in \operatorname{Int} S(\Gamma)$ where Γ denotes the Fuchsian group $f^{-1}Gf$. Here we may assume that $f = f^{\varphi}$. Since χ^{φ}: $\Gamma \to G$ is a type-preserving isomorphism by Lemma 2.3, Γ is also a Schottky group. (Here note that Schottky groups are characterized as the finitely generated, purely loxodromic free Kleinian groups by Maskit's theorem [8].) Therefore Theorem 2.1 produces that $\operatorname{Int} S(\Gamma) = T(\Gamma)$. Thus we have shown that $\varphi \in T(\Gamma)$, in particular, $D = f^{\varphi}(\mathbf{H})$ is a quasidisk. Q.E.D.

Quasidisks. Finally, we shall mention a characterization of the quasidisks, where we recall that the quasidisk is defined as an image of the unit disk (or the upper half plane) under a quasiconformal self-map of $\hat{\mathbf{C}}$. Before stating the result, we shall define a distance ("path diameter distance" w.r.t. the Euclidean metric) δ_D on any open subset D of $\mathbf{C}$. For given two points z_1, z_2 in D, we set

$$\delta_D(z_1, z_2) = \inf_{\alpha \subset D} \operatorname{diam} \alpha,$$

where the infimum is taken over the paths α connecting z_1 and z_2 in D and

$\operatorname{diam}\alpha = \sup_{w_1, w_2 \in \alpha}|w_1 - w_2|$. If z_1 and z_2 do not belong to the same component of D, we define $\delta_D(z_1, z_2) = \infty$. As is easily seen, δ_D satisfies the axiom of distance except that δ_D possibly takes the value ∞. In particular, δ_D is certainly a distance on D if D is a domain, and $\delta_D(z_1, z_2) \geq |z_1 - z_2|$ by definition.

A bounded simply connected domain D is called *linearly connected* if $\delta_D(z_1, z_2) \leq C|z_1 - z_2|$ for any $z_1, z_2 \in D$, or equivalently, for an arbitrary disk Δ, any two points in $D \cap \Delta$ can be connected by a path in $D \cap \Delta_A$, where C and A is constants (≥ 1) depending only on D and Δ_A denotes $\{|z - z_0| < Ar\}$ if $\Delta = \{|z - z_0| < r\}$. It is worthy to know the fact that a linearly connected, bounded, simply connected domain is always a Jordan domain (see Theorem 3.3 below).

A bounded simply connected domain D is called a *John domain* if, for an arbitrary disk Δ, any two points in $D \setminus \Delta_A$ can be connected by a path in $D \setminus \Delta$, where A is a constant (≥ 1) depending only on D.

2.9. Theorem (cf. Gehring [6], Pommerenke [12]). *A bounded simply connected domain D is a quasidisk if and only if D is a linearly connected John domain.*

§3. The first construction of non-univalent meromorphic map with G-invariant small Schwarzian

In this section, we shall proceed in a general situation. Let G be an arbitrary Kleinian group, D be a G-invariant hyperbolic plane domain and $p: \Omega(G) \to R = \Omega(G)/G$ be the natural projection. Here we should remark that $D \subset \Omega(G)$, for $\hat{\mathbf{C}} \setminus D$ is a G-invariant closed set containing at least three points, thus $\hat{\mathbf{C}} \setminus D \supset \Lambda(G)$. In this section, it is our main job to prove the following

3.1. Proposition. *Suppose that a Kleinian group G acts on a simply connected plane domain $D \subset \mathbf{C}$ of hyperbolic type. If D is a G-Schwarzian domain with constant $\varepsilon > 0$, the following is valid for an appropriate constant $B > 1$ depending only on ε: for an arbitrary $\Delta \in \mathscr{D}_B(\Omega(G))$ such that $p|_{\Delta_B}$ is injective, any two points in $\Delta \cap D$ can be joined by a path in $\Delta_B \cap D$.*

Before stepping into the proof of the above proposition, we state a few corollaries.

3.2. Corollary. *If a hyperbolic simply connected plane domain D is G-Schwarzian for some Kleinian group G acting on D, and if $\Lambda(G) \neq \partial D$ then $\partial D = \partial D^*$ where D^* is the exterior of D. In particular, $D^* \neq \emptyset$.*

Proof of Corollary 3.2. Since always $\Lambda(G) \subset \partial D$, the hypothesis implies that there exists a point z_0 in $\partial D \setminus \Lambda(G) = \partial D \cap \Omega(G)$. The limit set $\Lambda(G)$ is contained in the closure of the orbit $G \cdot z_0$ of z_0, and on the other hand, $G \cdot z_0$ is contained in $\partial D \cap \Omega(G)$, thus we obtain that $\Lambda(G) \subset \overline{G \cdot z_0} \subset \overline{\partial D \cap \Omega(G)}$. As a consequence, we have $\overline{\partial D \cap \Omega(G)} = \partial D$. Clearly, $\partial D^* \subset \partial D$, so it is sufficient to prove that

$\partial D \cap \Omega(G) \subset \partial D^*$. If not, since the free regular set ${}^\circ\Omega(G) = \Omega(G) \setminus \{$elliptic fixed points of $G\}$ has at most countable complement in $\Omega(G)$, there exists a point $w_0 \in \partial D \cap {}^\circ\Omega(G) \setminus \partial D^*$. Pick and fix another point w_1 in ∂D. Then, since $w_0 \in \operatorname{Int} \bar{D} \cap {}^\circ\Omega(G)$, there exists an injective disk Δ with center w_0 and radius $r > 0$ such that $\Delta \subset \bar{D}$ and $w_1 \notin \Delta$. Now we take a sufficiently large n so that $\sin \frac{\pi}{n} < \frac{1}{4B}$, and set $e_j = \frac{r}{2} \exp\left(\frac{2\pi i j}{n}\right) + w_0$ for $j = 0, 1, \dots, n$. Then $e_j \in \partial \Delta_{1/2}$ and $|e_{j+1} - e_j| = r \sin \frac{\pi}{n} < \frac{r}{4B}$ for $j = 0, 1, \dots, n-1$. Since $e_j \in \Delta \subset \bar{D}$, we can choose a point $a_j \in D$ such that $|a_j - e_j| < \frac{r}{4B}$ for each $j = 1, \dots, n$, and set $a_0 = a_n$. Then $a_j, a_{j+1} \in B\left(e_j, \frac{r}{2B}\right)$ and the disk $B(e_j, r/2)$ is included in the injective disk Δ, so Proposition 3.1 guarantees the existence of a path $\gamma_j \subset B(e_j, r/2) \cap D$ connecting a_j and a_{j+1} $(j = 0, 1, \dots, n-1)$. Therefore $\gamma = \bigcup_{j=0}^{n-1} \gamma_j$ is a closed path in D separating w_0 from w_1, which contradicts the connectedness of ∂D. Q.E.D.

By the next characterization of Jordan domains, we obtain a further information about G-Schwarzian simply connected domains.

3.3. Theorem (Newman [11] Chap. VI, Theorem 14.1 and Theorem 16.2). *A hyperbolic simply connected domain $D \subset \hat{\mathbf{C}}$ is a Jordan domain if and only if D is uniformly locally connected, more precisely, for any positive number ε there exists a positive η such that, for all pairs of points $x, y \in D$, $d(x, y) < \eta$ implies that $\delta_D(x, y) < \varepsilon$, where δ_D denotes the "path diameter distance" with respect to the spherical metric d of $\hat{\mathbf{C}}$.*

3.4. Corollary. *Let D be a hyperbolic, G-Schwarzian, simply connected plane domain for some Kleinian group G and D' a Jordan domain such that $\partial(D \cap D') \subset {}^\circ\Omega(G)$, where ${}^\circ\Omega(G)$ denotes the free regular set of G, i.e., ${}^\circ\Omega(G) = \Omega(G) \setminus \{$elliptic fixed points of $G\}$. Then, each component of $D \cap D'$ is a Jordan domain.*

Proof of Corollary 3.4. Since $D \cap D'$ is simply connected, it suffices to show that each component of $D \cap D'$ is uniformly locally connected, by Theorem 3.3. By Corollary 3.2, we can assume that D and D' are both bounded domains. Let $0 < \varepsilon < \operatorname{diam} \partial D \cap \partial D'$. Since some compact neighborhood of $\partial(D \cap D')$ is contained in free regular set of G, Proposition 3.1 yields that there exists a positive η_1 such that $\delta_D(x, y) < \varepsilon/2$ for any $x, y \in D \cap D'$ with $|x - y| < \eta_1$.

On the other hand, Theorem 3.3 implies that there exists a positive η_2 such that $\delta_{D'}(x, y) < \varepsilon/2$ for any $x, y \in D'$ with $|x - y| < \eta_2$.

Let D_1 be a component of $D \cap D'$ and $x, y \in D_1$ with $|x - y| < \eta_0 = \min\{\eta_1, \eta_2\}$. Then, there exist paths γ_1, γ_2 from x to y such that $\gamma_1 \subset D, \gamma_2 \subset D'$ and that $\operatorname{diam} \gamma_j < \varepsilon/2$. And, since D_1 is connected, there exists a path γ_0 in D_1 from x to y. Since D and D' are both simply connected, γ_1 is homotopic to γ_0 in D and γ_2 is homotopic to γ_0 in D', and so γ_1 is homotopic to γ_2 in $D \cup D'$. Let

γ denote the closed curve $\gamma_1 \cdot \gamma_2^{-1}$, then γ is null homotopic in $(\partial D \cap \partial D')^c$. Since $\operatorname{diam} \gamma < \varepsilon < \operatorname{diam} \partial D \cap \partial D'$, γ bounds no points of $\partial D \cap \partial D'$. And therefore γ is null homotopic in $\Delta \setminus (\partial D \cap \partial D') = (\Delta \setminus \partial D) \cup (\Delta \setminus \partial D')$, where $\Delta = \{z \in \mathbf{C};\, |z - x| < \varepsilon/2\}$. (Remark that $\gamma_j \subset \Delta$.)

Therefore by Alexander's lemma (to be stated below) x and y are known to be connected by a path γ_3 in $(\Delta \setminus \partial D) \cap (\Delta \setminus \partial D') = \Delta \setminus (\partial D \cup \partial D')$. Since $\gamma_3 \subset D_1$ and $\operatorname{diam} \gamma_3 < \operatorname{diam} \Delta = \varepsilon$, it is proved that D_1 is uniformly locally connected.

3.5. Alexander's lemma (cf. Newman [11]). *Let O_1 and O_2 be open sets in $\hat{\mathbf{C}}$. Suppose that $x, y \in O_1 \cap O_2$ are connected by paths γ_i in O_i $(i = 1, 2)$. Then, if $\gamma_1 \cup \gamma_2$ is null homotopic in $O_1 \cup O_2$, x and y are connected by a path in $O_1 \cap O_2$.*

As the final corollary, we state a rather technical lemma which will be used in § 5.

3.6. Lemma. *Under the same hypothesis of Proposition* 3.1, *let w_0 be a point of $\partial D \cap {}^\circ\Omega(G)$ and $\Delta \subset {}^\circ\Omega(G)$ an injective disk centered at w_0. Then there exists a connected open neighborhood V of w_0 in Δ such that $V \cap D$ is connected.*

Proof. Consider the disk $\Delta_{1/B} \in \mathscr{D}_B(\Omega(G))$, where B is the constant which appeared in Proposition 3.1. Let W be a connected component of $\Delta \cap D$ which includes a point in $\Delta_{1/B} \cap D$. Then, by Proposition 3.1, $\Delta_{1/B} \cap D \subset W$. Therefore, we can adopt $\Delta_{1/B} \cup W$ as a neighborhood V. Q.E.D.

Proof of Proposition 3.1. We choose $B \geq C > 1$ so that $B \geq 6C$ and $C \geq 9 + 2^{11} \cdot 3^3/\varepsilon$. For some disk $\Delta = B(z_0, r) \in \mathscr{D}_B(\Omega(G))$, suppose that $p|_{\Delta_B}$ is injective and that two points $z_1, z_2 \in \Delta \cap D$ cannot be connected by any path in $\Delta_C \cap D$ (see Fig. 3.1). Let D_j be the connected component of $\Delta_C \cap D$ containing z_j $(j = 1, 2)$. Noting that $\overline{D_1} \cap D_2 = \emptyset$ by the hypothesis, we can choose a component of $\partial \Delta_C \setminus \overline{D_1}$, say J, containing a point of $\overline{D_2}$. Then the closed interval $I = \partial \Delta_C \setminus J$ has the following properties:

(1) $\bar{D}_1 \cap \partial \Delta_C \subset I$,

(2) $\overline{D_2} \cap (I \setminus \partial I) = \emptyset$, and

(3) $\partial I \subset \partial D$,

where ∂I denotes the set of endpoints of I.

Furthermore, interchanging D_1 and D_2 if necessary, we may assume that

(4) $$|I| \leq \frac{1}{2} |\partial \Delta_C|$$

where $|\cdot|$ denotes the arc length.

We assign the anti-clockwise orientation to I with initial point w_0 and terminal point w_1. In the following, we frequently utilize an auxiliary Möbius transformation $Q(z) = \dfrac{z - w_0}{z - w_1}$, which maps w_0, w_1 to $0, \infty$, respectively. First,

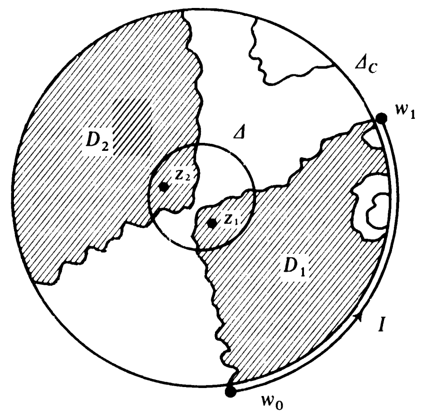

Figure 3.1.

we remark that the quantity $\lambda = Q(z_2)/Q(z_1)$ is near to 1. Precisely, for the principal value σ of $\log \lambda$, we have the next

3.7. Lemma.

$$|\sigma| \le \frac{4|w_0 - w_1|}{(C-1)^2 r}\left(\le \frac{8C}{(C-1)^2}\right), \qquad \textit{and } |\sigma| \le \frac{8}{C-1}.$$

Proof of Lemma 3.7. If we set $u = \dfrac{z_2 - z_1}{z_1 - w_0}$ and $v = \dfrac{z_2 - z_1}{z_1 - w_1}$, then we have $|u|, |v| \le \dfrac{2}{C-1}\left(<\dfrac{1}{2}\right)$, $|u - v| \le \dfrac{2|w_0 - w_1|}{(C-1)^2 r}$ and

$$|\sigma| = \left|\log\frac{1+u}{1+v}\right| = \left|\int_u^v \frac{d\zeta}{1+\zeta}\right| \le 2\int_{[u,v]} |d\zeta| = 2|u-v| \le \frac{4|w_0 - w_1|}{(C-1)^2 r}.$$

Similarly, we have $|\sigma| \le 2(|u| + |v|) \le \dfrac{8}{C-1}$. Q.E.D.

Let $\omega = \operatorname{Im}\sigma = \arg\lambda$ ($|\omega| \le |\sigma| < \pi/4$) and θ_0 be an angle of the ray $Q(I)$ with the positive real line, i.e., $Q(I) = \{re^{i\theta_0}; 0 \le r \le \infty\}$. In order to construct a tame deformation such that its images of z_1 and z_2 coincide, we first define a map $\tilde{T}\colon \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ by

$$\tilde{T}(z) = \begin{cases} z\exp\left(\dfrac{\pi/4 + \theta}{\pi/4 - \omega}\sigma\right) & (-\pi/4 \le \theta < -\omega) \\ \lambda z & (-\omega \le \theta < \pi) \\ z\exp\left(\dfrac{5\pi/4 - \theta}{\pi/4}\sigma\right) & (\pi \le \theta < 5\pi/4) \\ z & (5\pi/4 \le \theta < 7\pi/4) \end{cases}$$

where $z = re^{i(\theta+\theta_0)}$, $\tilde{T}(0) = 0$, and $\tilde{T}(\infty) = \infty$, and further set $T = Q^{-1} \circ \tilde{T} \circ Q$. Then $T(z_1) = z_2$ and we have the following

3.8. Lemma. *$\tilde{T}$ and T are $\dfrac{8}{C-9}$-qc homeomorphism of $\hat{\mathbf{C}}$.*

Proof. If we set

$$t = t(re^{i(\theta+\theta_0)}) = \begin{cases} \dfrac{\sigma}{\pi/4 - \omega} & (-\pi/4 < \theta < -\omega) \\ -4\sigma/\pi & (\pi < \theta < 5\pi/4) \\ 0 & (\text{otherwise}), \end{cases}$$

then $s := ir\dfrac{\partial \tilde{T}}{\partial r} \Big/ \dfrac{\partial \tilde{T}}{\partial \theta} = \dfrac{i}{t+i} = \dfrac{1}{1-it}$ a.e. and we have an estimate $|t| \le \dfrac{|\sigma|}{\pi/4 - |\sigma|}$ $\le 2|\sigma|$ since $|\sigma| \le \dfrac{8}{C-1} \le \dfrac{1}{4} < \pi/4 - \dfrac{1}{2}$. Therefore,

$$|\mu[\tilde{T}]| = \left|\frac{s-1}{s+1}\right| = \left|\frac{it}{2-it}\right| \le \frac{|t|}{2-|t|} \le \frac{|\sigma|}{1-|\sigma|} \le \frac{8}{C-9},$$

i.e., $\|\mu[T]\|_\infty = \|\mu[\tilde{T}]\|_\infty \le \dfrac{8}{C-9}$. Q.E.D.

If we set

$$\tilde{E}_1 = \{z \in \mathbf{C}^*;\ \theta_0 - \pi/4 < \arg z \le \theta_0\},$$
$$\tilde{E}_2 = \{z \in \mathbf{C}^*;\ \theta_0 + \pi \le \arg z < \theta_0 + \pi + \omega\},$$

and $E_j = Q^{-1}(\tilde{E}_j)$ $(j = 1, 2)$, then we obtain the following.

3.9. Lemma. $T(\Delta_C \cup E_1) \subset \Delta_C \cup E_1 \cup E_2 \subset \Delta_{4C} (\subset \Delta_B)$.

Proof. The first inclusion is clear by construction of T. Now we prove the second inclusion. With a suitable normalization, we may assume that $\Delta_C = B(0, 1)$ and $w_1 = \bar{w}_0 = e^{i\varphi_0}$ $(0 < \varphi_0 < \pi)$. The condition (4) implies that $0 < \varphi_0 \le \pi/2$. Thus we observe that E_1 is largest if $\varphi_0 = \pi/2$, in that case

$$E_1 \subset B(1, \sqrt{2}) \subset B(0, 1 + \sqrt{2}) \subset B(0, 4) = \Delta_{4C}. \tag{3.1}$$

Next, we shall consider E_2. We may assume that $\omega > 0$, for otherwise $E_2 = \emptyset$. We need to calculate the radius ρ and the center $c < 0$ of the circle $Q^{-1}\{\arg z = \theta_0 + \omega \bmod \pi\}$ (see Fig. 3.2). The elementary geometry tells us that

$$\rho \sin(\varphi_0 - \omega) = \sin \varphi_0,$$
$$\rho \cos(\varphi_0 - \omega) = \cos \varphi_0 - c,$$

and the latter implies that $-c \le \rho$. By Lemma 3.2,

$$\omega \leq \frac{4|w_1 - w_0|}{(C-1)^2} = \frac{8 \sin \varphi_0}{(C-1)^2} \leq \frac{\varphi_0}{2},$$

since $C \geq 5$. So, $\varphi_0 - \omega \geq \varphi_0/2$, and therefore $\rho = \sin \varphi_0 / \sin(\varphi_0 - \omega) \leq \sin \varphi_0 / \sin(\varphi_0/2) = 2\cos(\varphi_0/2) \leq 2$. The above estimate enables us to deduce that $E_2 \subset B(c, \rho) \subset B(0, \rho - c) \subset B(0, 2\rho) \subset B(0, 4) = \Delta_{4C} \subset \Delta_B$. The proof is now completed. Q.E.D.

Now, we shall go to the next step. We construct a locally injective quasi-regular map $h: D \to \hat{\mathbf{C}}$ as follows. Let $\psi(z) = \xi(z) + i(\eta(z) + \theta_0)$ be the branch of $\log Q(z)$ in D such that $\eta = 0$ on $I \cap \overline{D_1}$, where we note that $\log Q(z)$ has the same imaginary part on $I \cap \overline{D_1}$ by the property (1). Then, clearly $-2\pi < \eta < \pi$, $\eta(z_1) > 0$, $\eta(z_2) < -\pi$, and $|\eta(z_1) - \eta(z_2) - 2\pi| = |\operatorname{Im} \sigma| \leq \dfrac{8}{C-1}$.

At first, we define h on $\Delta_B \cap D$ by

$$h(z) = \begin{cases} z & \text{if } \eta(z) \leq -\pi/4, \\ T(z) & \text{if } -\pi/4 \leq \eta(z). \end{cases}$$

Observe that $\left\{ z \in D;\, \eta(z) \geq -\dfrac{\pi}{4} \right\} \subset \Delta_C \cup E_1$, $h(\Delta_B \cap D) \subset \Delta_B$ and $h = \mathrm{id}$ on $\partial \Delta_B \cap D$ by Lemma 3.9. Since $\bigcup_{L \in G} L(\Delta_B)$ is a disjoint union, we can extend h to a continuous map on D (denoted by the same letter h) as follows:

$$h(z) = \begin{cases} L \circ h \circ L^{-1}(z) & \text{if } z \in L(\Delta_B \cap D) \text{ for some } L \in G, \\ z & \text{otherwise.} \end{cases}$$

By construction, h satisfies the following.

(a) h is $\dfrac{8}{C-9}$-quasi-regular,

(b) $h \circ L = L \circ h$ for all $L \in G$,

(c) $h(z_1) = h(z_2)$,

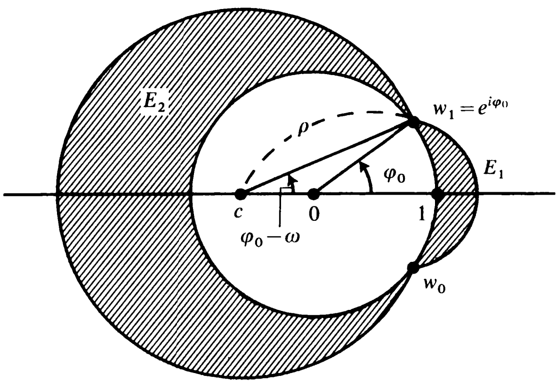

Figure 3.2.

(d) $h(\Delta_B \cap D) \subset \Delta_B$, and $h(D \setminus \Delta_B) \cap \Delta_B = \emptyset$.

And, a crucial point of the above construction is the validity of the following

3.10. Lemma. *The Beltrami coefficient* $\mu[h^{-1}]$ *on* $h(D)$ *of the* (*local*) *inverse of* h *is well-defined, i.e., it is independent of a particular branch of* h^{-1}.

Proof of Lemma 3.10. By equivariance of the definition of h and the property (d), it is sufficient to prove only on Δ_B. Let

$$E_3 = \{z \in \Delta_B \cap D;\ \# h^{-1}(h(z)) > 1\}, \text{ and}$$
$$E_4 = \{z \in \Delta_B \cap D;\ \partial_{\bar{z}} h(z) \neq 0\}.$$

First, $h(E_3) \subset T(\{z \in \Delta_B \cap D;\ -\pi/4 < \eta(z) < \pi\}) \cap \{z \in \Delta_B \cap D;\ -2\pi < \eta(z) < -\pi/4\} \subset Q^{-1}(\{-2\pi < \arg z < -\pi\})$, by definition of h. On the other hand, $h(E_4) \subset T(\{-\pi/4 \leq \eta(z) \leq -\omega\}) \subset Q^{-1}(\{-\pi/4 \leq \arg z \leq 0\})$, and so it follows that $h(E_3) \cap h(E_4) = \emptyset$, this shows that any branch of h^{-1} is holomorphic on $h(E_3)$. Thus, the proof is finished. Q.E.D.

The above lemma says that the next definition:

$$\mu = \begin{cases} \mu[h^{-1}] & \text{on } h(D) \\ 0 & \text{on } \mathbf{C} \setminus h(D) \end{cases}$$

is well-defined and the properties (a) and (b) imply that $\mu \in M(\mathbf{C}, G)$ and $\|\mu\|_\infty \leq \dfrac{8}{C-9} (< 1)$. Now, we define a quasi-regular map $g: D \to \hat{\mathbf{C}}$ by $g = w^\mu \circ h$, then by the chain rule for quasi-regular mappings, we can see that $\mu[g] = 0$ a.e. on D. By virtue of Weyl's lemma for quasi-regular mappings, $g: D \to \hat{\mathbf{C}}$ is known to be meromorphic, moreover property (c) of h implies that $g(z_1) = g(z_2)$, that is, g is *not* univalent. By the fact that $\mu \in M(\mathbf{C}, G)$, w^μ transforms the Kleinian group G to another one by conjugation, thus g does so, in other words, $S_g(z)dz^2$ is G-invariant.

Finally, we shall give an estimate of the hyperbolic sup norm of S_g which will lead to a contradiction.

Before into the final step, we prepare some lemmas. The proof of the first is quite elementary (see [16] Proof of Proposition 2.4).

3.11. Lemma. *For any constant* $A \geq 1$, *the following is valid. If* $\Delta' \in \mathscr{D}_A(D)$ *and if* $L \in$ Möb *satisfies that* $L(D) \subset \mathbf{C}$ *then* $L(\Delta') \in \mathscr{D}_{A'}(L(D))$ *where* $A' = \dfrac{A + A^{-1}}{2}$.

3.12. Lemma. *Let* $E = \{z \in D;\ -\pi/4 < \eta(z) < -\omega\}$ *and* $A \geq 3$ *be a constant. Then, for* $\Delta' \in \mathscr{D}_A(D)$ *such that* $\Delta' \cap E \neq \emptyset$, h *coincides with* T *on* Δ'.

Proof of Lemma 3.12. We may assume that $\Delta_C = B(0, 1)$. Suppose that $\Delta' = B(c, \rho) \in \mathscr{D}_A(D)$ satisfies $\Delta' \cap E \neq \emptyset$. Then clearly $-5\pi/4 < \eta < \pi - \omega$ on Δ'. And we note that

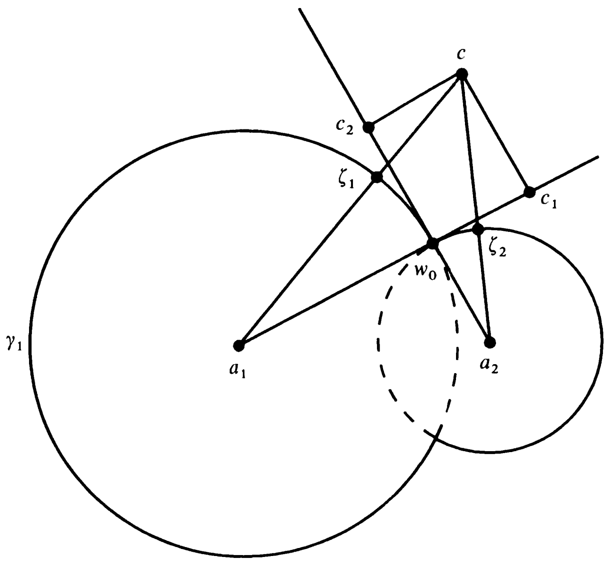

Fig. 3.3.: Case 1.

$$A\rho \leq |c - w_i| \qquad (i = 0, 1) \tag{3.2}$$

from the assumption and the fact that $w_i \in \partial D$ (the property (3)). Pick a point ζ from $\Delta' \cap E$, then $|c - \zeta| < \rho$, $|\zeta - w_1| \leq \operatorname{diam} E \leq \operatorname{diam} E_1 \leq 2\sqrt{2}$ and $|\zeta| < 1 + \sqrt{2}$ by (3.1). Hence, $A\rho \leq |c - w_1| \leq |c - \zeta| + |\zeta - w_1| < \rho + 2\sqrt{2}$, thus we have $\rho \leq \dfrac{2\sqrt{2}}{A - 1} \leq \sqrt{2}$. Since $|c| \leq |c - \zeta| + |\zeta| < \rho + (1 + \sqrt{2})$, we have that $\Delta' \subset B(0, |c| + \rho) \subset B(0, 1 + \sqrt{2} + 2\rho) \subset B(0, 6) \subset \Delta_B$. Thus we need only to prove that $\Delta' \cap E' = \emptyset$ where $E' = Q^{-1}(\{z \in \mathbf{C}^*;\ -5\pi/4 < \arg z < -3\pi/4\}) = Q^{-1}(\{z \in \mathbf{C}^*;\ 3\pi/4 < \arg z < 5\pi/4\})$.

Suppose that $\Delta' \cap E' \neq \emptyset$. We may assume that $|w_0 - c| \leq |w_1 - c|$. Let C_1 and C_2 denote the circles (or lines) including circular arcs (or segments) $\gamma_1 = Q^{-1}(\{\zeta;\ \arg \zeta - \theta_0 = -\pi/4\})$ and $\gamma_2 = Q^{-1}(\{\zeta;\ \arg \zeta - \theta_0 = -3\pi/4\})$, respectively. Here we note that C_1 is necessarily a circle, say $C_1 = \{z;\ |z - a_1| = r_1\}$, by (3.1), whereas C_2 is possibly a line. Further remark that, by assumption, γ_1 and γ_2 perpendicularly intersect at the two points w_0 and w_1 and that $\gamma_j \cap \partial\Delta' \neq \emptyset$ for $j = 1, 2$.

Let ζ_j denote the nearest point of C_j to c for $j = 1, 2$. If $\zeta_j \in C_j \setminus \gamma_j$, then $A\rho \leq |c - w_0| = \operatorname{dist}(c, \gamma_j) < \rho$, this is impossible. Thus we conclude that $\zeta_j \in \gamma_j$, and hence we have

$$|c - \zeta_j| = \operatorname{dist}(c, \gamma_j) < \rho \qquad \text{for } j = 1, 2. \tag{3.3}$$

Now let c_j be the orthogonal projection of the point c to the normal line of C_j at w_0 $(j = 1, 2)$.

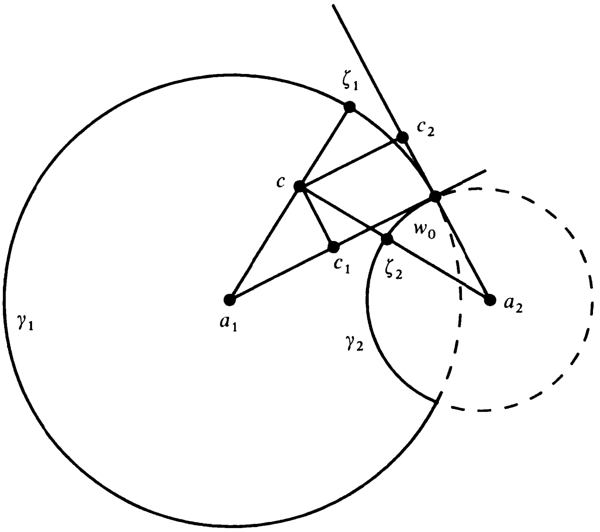

Fig. 3.4.: Case 2.

Case 1: $W = Q^{-1}(\{\zeta;\ -3\pi/4 < \arg\zeta - \theta_0 < -\pi/4\})$ is unbounded (see Fig. 3.3).

In this case, we have $|c_j - w_0| \leq |c - \zeta_j| < \rho$ for $j = 1, 2$. Thus we get an estimate that

$$|c - w_0| = \sqrt{|c_1 - w_0|^2 + |c_2 - w_0|^2} < \sqrt{2}\rho.$$

By (3.2), we have $A\rho < \sqrt{2}\rho$, which is a contradiction.

Case 2: W is bounded.

We may assume that c is in the inside of C_1 and in the outside of C_2 (see Fig. 3.4).

Then, as in Case 1, we know that

$$|c_2 - w_0| < \rho. \tag{3.4}$$

Next, because $|c - a_1| \leq |c - c_1| + |c_1 - a_1| = |c_2 - w_0| + (r_1 - |c_1 - w_0|)$, we obtain that

$$|c_1 - w_0| \leq |c_2 - w_0| + r_1 - |c - a_1| = |c_2 - w_0| + |c - \zeta_1| < 2\rho \tag{3.5}$$

by (3.3) and (3.4). So, by (3.4) and (3.5), we have

$$|c - w_0| = \sqrt{|c_1 - w_0|^2 + |c_2 - w_0|^2} < \sqrt{5}\rho.$$

Combined with (3.2), we can deduce that $A\rho < \sqrt{5}\rho$, which contradicts the hypothesis that $A \geq 3$.

Proof of Propositon 3.1 (*continued*). Here, we shall examine the local qc extendability of h. Let $A \geq 6$ and $\Delta \in \mathscr{D}_A(D)$. If $\Delta \cap L(E) \neq \emptyset$ for some $L \in G$, then

$L^{-1}(\Delta) \cap E \neq \emptyset$ and, by Lemma 3.11, $L^{-1}(\Delta) \in \mathscr{D}_{A'}(D)$ where $A' = \dfrac{A + A^{-1}}{2} \geq 3$. So, Lemma 3.12 implies that $h = T$ on $L^{-1}(\Delta)$, i.e., $h = L \circ h \circ L^{-1} = L \circ T \circ L^{-1}$ on Δ. From Lemma 3.8, we know that $h|_\Delta$ can be extended to a k-qc homeomorphism $L \circ T \circ L^{-1}$ of $\hat{\mathbf{C}}$, where $k = \dfrac{8}{C - 9}$.

Otherwise, $\Delta \cap (\bigcup_{L \in G} L(E)) = \emptyset$, so h is a restriction of a Möbius transformation on Δ. Thus, in any case, $h|_\Delta$ can be extended to a k-qc homeomorphism of $\hat{\mathbf{C}}$. On the other hand, w^μ is originally a global k-qc map, and hence $g|_\Delta = w^\mu \circ h|_\Delta$ can be extended to a k'-qc homeomorphism of $\hat{\mathbf{C}}$, where $\dfrac{1 + k'}{1 - k'} = \left(\dfrac{1 + k}{1 - k}\right)^2$. By quite easy calculations, we have $k' = \dfrac{2k}{1 + k^2} \leq \dfrac{16(C - 9)}{(C - 9)^2 + 64} \leq \dfrac{16}{C - 9}$, therefore combined with Lemma 2.7, we obtain an estimate

$$\|S_g\|_D \leq 96 A^2 k' \leq 96 \cdot 6^2 \cdot \frac{16}{C - 9} < \varepsilon.$$

From the first hypothesis, g should be univalent on D, which contradicts the fact that $g(z_1) = g(z_2)$. Thus, z_1 and $z_2 \in \Delta \cap D$ must be connected by a path in $\Delta_C \cap D$, therefore, in $\Delta_B \cap D$.

§4. The second construction of non-univalent meromorphic map with G-invariant small Schwarzian

In this section, we shall make another construction of non-univalent meromorphic map, which is, in a sense, a dual of the one in §3. At first, we prove a rather technical proposition, which holds for general Kleinian groups.

4.1. Proposition. *Suppose that a Kleinian group G acts on a simply connected plane domain $D \subset \mathbf{C}$ of hyperbolic type and let $p: \Omega(G) \to \Omega(G)/G$ be the natural projection. If D is a G-Schwarzian domain with constant $\varepsilon > 0$, the following is valid for an appropriate constant $C > 1$ depending only on ε: for each disk $\Delta \in \mathscr{D}_C(\Omega(G))$ such that $\bar{E} \subset \Omega(G)$ and that $p|_{\bar{E}}$ is injective where $E = D \setminus D_0$ and D_0 is some connected component of $D \setminus \bar{\Delta}$, any two points $z_0, z_1 \in D \setminus \bar{\Delta}_C$ can be joined by a path in $D \setminus \bar{\Delta}$.*

We remark that D_0 is ordinarily the "main body" of $D \setminus \bar{\Delta}$, precisely speaking, the unique component of $D \setminus \bar{\Delta}$ containing G-equivalent points.

Proof. Let $C > 5 + 2^{10} \cdot 3 \cdot 5^2/\varepsilon$ and assume that $\Delta \in \mathscr{D}_C(\Omega(G))$, D_0 and E satisfy the above hypothesis. Suppose that some pair of points $z_1, z_2 \in D \setminus \bar{\Delta}_C$ cannot be joined by any path in $D \setminus \bar{\Delta}$. We denote by D_j the connected component of $D \setminus \bar{\Delta}$ which contains z_j for $j = 1, 2$. By the assumption, $D_1 \neq D_2$.

Let I be the closure of connected component of $\partial\Delta\setminus\bar{D}_1$ containing points of $\partial\Delta\cap\bar{D}_2$. Obviously $\partial\Delta\cap\bar{D}_2\subset I$, and $\partial\Delta\cap\bar{D}_0\subset I$ or $\partial\Delta\cap\bar{D}_0\subset\partial\Delta\setminus I^\circ$, where $I^\circ=I\setminus\partial I$. Replacing I by $\partial\Delta\setminus I^\circ$ and interchanging D_1 and D_2 if necessary, we may assume that

$$\partial\Delta\cap\bar{D}_1\subset\partial\Delta\setminus I^\circ,\ \partial\Delta\cap\bar{D}_2\subset I,\ \text{ and }\ \partial\Delta\cap\bar{D}_0\subset I. \tag{4.1}$$

We assign the anti-clockwise orientation to I, and let w_0, w_1 be the initial and terminal point of I, respectively. Note here that $w_i\in\partial D$ by construction. Now we introduce a Möbius transformation $Q(z)=\dfrac{z-w_0}{z-w_1}$. Let σ be the principal value of $\log\lambda$, where $\lambda=Q(z_2)/Q(z_1)$. By Lemma 3.2, we know that $|\sigma|\le\dfrac{8}{C-1}$. We define a family of quasiconformal maps $\tilde{T}_t\colon\hat{\mathbf{C}}\to\hat{\mathbf{C}}$ for $0\le t\le 1$ by

$$\tilde{T}_t(z)=\begin{cases} z\exp\left(\dfrac{\theta-\pi/3}{\pi/3}t\sigma\right) & (\pi/3\le\theta<2\pi/3)\\ z\exp(t\sigma) & (2\pi/3\le\theta<4\theta/3)\\ z\exp\left(\dfrac{5\pi/3-\theta}{\pi/3}t\sigma\right) & (4\pi/3\le\theta<5\pi/3)\\ z & (0\le\theta<\pi/3 \text{ or } 5\pi/3\le\theta<2\pi)\end{cases}$$

where $z=re^{i(\theta+\theta_0)}$ and θ_0 is an angle of the ray $Q(I)$ with the positive real line. Next, let $T_t=Q^{-1}\circ\tilde{T}_t\circ Q$ for $0\le t\le 1$. By the same way as in the proof of Lemma 3.8, we have the following

4.2. Lemma. *$\tilde{T}_t$ and T_t are $\dfrac{4t}{C-5}$-qc homeomorphism of $\hat{\mathbf{C}}$ for $0\le t\le 1$.*

Let $\psi(z)=\xi(z)+i(\eta(z)+\theta_0)$ be a branch of $\log Q(z)$ in D such that $\eta=0$ on $\partial D_0\cap D(\subset I^\circ)$. Then it is clear that $-\pi<\eta<2\pi$ on D, $\eta<0$ on $D_0\cup D_2$ and that $\eta>\pi$ on D_1. First, define $h_t(z)$ for $z\in E=D\setminus D_0$ by the rule

$$h_t(z)=\begin{cases} z & \text{if } \eta(z)<0,\\ T_t(z) & \text{if } 0\le\eta(z)<\pi,\\ Q^{-1}(e^{t\sigma}Q(z)) & \text{if } \pi\le\eta(z).\end{cases}$$

Noting that $\bigcup_{L\in G}L(E)$ is a disjoint union by assumption, next we extend h_t to a mapping on $\bigcup_{L\in G}L(E)$ (still written by the same notation h_t) as follows: $h_t=L\circ h_t\circ L^{-1}$ on $L(E)$. Since $h(z)=z$ for any $z\in\partial E\cap D=\partial D_0\cap D$ by the choice of η, we can continuously extend h_t by difining as the identity map on $D\setminus\bigcup_{L\in G}L(E)$. By (4.1), these quasi-regular mappings $h_t\colon D\to\hat{\mathbf{C}}$ $(0\le t\le 1)$ satisfy the following conditions.

(a) h_t is $\dfrac{4t}{C-5}$-quasi-regular,

(b) $h_t\circ L=L\circ h_t$ for any $L\in G$,

(c) $h_1(z_1)=h_1(z_2)=z_2$, and

(d) $h_t(z)$ continuously depends on $(t, z) \in [0, 1] \times D$.

Now we let $J = \{t \in [0, 1]; \exists L \in G \setminus \{1\}$ such that $h_t(E^\circ) \cap L(h_t(E^\circ)) \neq \emptyset\}$, where $E^\circ = \operatorname{Int} E = D \setminus \bar{D}_0$.

4.3. Lemma. *For any* $t \in [0, 1] \setminus J$, *the following Beltrami coefficient* μ_t *is well-defined*:

$$\mu_t = \begin{cases} \mu[h_t^{-1}] & \text{on } h_t(D), \\ 0 & \text{elsewhere.} \end{cases}$$

Proof of Lemma 4.3. Set $E_1 = \{z \in D; \pi/3 \leq \eta(z) \leq 2\pi/3\}$ $(\subset \Delta)$. Then, by definition, h_t is holomorphic off $\bigcup_{L \in G} L(E_1)$. Therefore, it is sufficient to show that

$$h_t(E_1) \cap h_t(D \setminus E_1) = \emptyset. \tag{4.2}$$

Since $h_t = T_t$ on $\Delta \cap E$, h_t is injective in $\Delta \cap E$, so we have

$$h_t(E_1) \cap h_t(\Delta \cap E \setminus E_1) = \emptyset. \tag{4.3}$$

Noting here that $|\arg(\tilde{T}_t(z)/z)| \leq |\arg(e^{t\sigma})| = t|\operatorname{Im} \sigma| \leq \dfrac{8t}{C-1} < \pi/6$, we have $|\arg(Q(h_t(z))/Q(z)| < \pi/6$ for all $z \in E$. Hence, we obtain

$$Q(h_t(E_1)) \subset \{\zeta; \pi/6 < \arg \zeta - \theta_0 < 5\pi/6\}, \text{ and} \tag{4.4}$$

$$Q(h_t(E \setminus \Delta)) \subset \{\zeta; 5\pi/6 < \arg \zeta - \theta_0 < 13\pi/6\}. \tag{4.5}$$

In particular, we have

$$h_t(E_1) \cap h_t(E \setminus \Delta) = \emptyset. \tag{4.6}$$

By (4.3) and (4.6), we can see that

$$h_t(E_1) \cap h_t(E \setminus E_1) = \emptyset. \tag{4.7}$$

Further (4.4) implies that

$$h_t(E_1) \cap D_0 = \emptyset \tag{4.8}$$

since $Q(D_0) \subset \{\zeta; -\pi < \arg \zeta < 0\}$. Noting that $h_t = \mathrm{id}$ on $D_0' := D_0 \setminus \bigcup_{L \in G \setminus \{1\}} L(E)$, we have the following equality

$$h_t(D \setminus E_1) = h_t(E \setminus E_1) \cup h_t(D_0) = h_t(E \setminus E_1) \cup D_0' \cup (\textstyle\bigcup_{L \in G \setminus \{1\}} L(h_t(E))).$$

Combining $h_t(E_1) \cap (\bigcup_{L \in G \setminus \{1\}} L(h_t(E))) = \emptyset$ with (4.7) and (4.8), we obtain (4.2), thus we finish the proof. Q.E.D.

Let $t \in [0, 1] \setminus J$. By properties (a) and (b), one can see that $\mu_t \in M(\mathbf{C}, G)$ with $\|\mu_t\|_\infty \leq \dfrac{4t}{C-5}$. We define a quasi-regular map $g_t \colon D \to \hat{\mathbf{C}}$ by $g_t = w^{\mu_t} \circ h_t$. Then, it follows that g_t is meromorphic, for $\mu[g_t] = 0$ a.e. Since $g_t \circ L = \chi_t(L) \circ g_t$, where $\chi_t(L) = w^{\mu_t} \circ L \circ (w^{\mu_t})^{-1} \in \text{Möb}$ for all $L \in G$, the Schwarzian derivative of g_t

is G-automorphic. Now we shall estimate $\|S_{g_t}\|_D$.

4.4. Lemma. *Let $A \geq 5$, then $h_t = T_t$ on Δ' for any $\Delta' \in \mathcal{D}_A(D)$ with $\Delta' \cap E_1 \neq \emptyset$.*

Proof of Lemma 4.4. Let $\Delta' = B(c, r) \in \mathcal{D}_A(D)$ such that $\Delta' \cap E_1 \neq \emptyset$. Then it suffices to show that $\Delta' \subset \Delta$. Suppose that $\Delta' \not\subset \Delta$, then one and only one of the following happens:

(1) $I^\circ \cap \partial\Delta' \neq \emptyset$ and $Q^{-1}(\{\zeta : \arg\zeta - \theta_0 = \pi/3\}) \cap \partial\Delta' \neq \emptyset$.

(2) $(\partial\Delta \setminus I) \cap \partial\Delta' \neq \emptyset$ and $Q^{-1}(\{\zeta : \arg\zeta - \theta_0 = 2\pi/3\}) \cap \partial\Delta' \neq \emptyset$.

In both cases, there are two circular arcs γ_1 and γ_2 with the following properties.

(i) γ_j is a subarc of a circle C_j centered at a_j with endpoints w_0 and w_1,

(ii) $\gamma_j \cap \partial\Delta' \neq \emptyset$ for $j = 1, 2$, and

(iii) γ_1 intersects γ_2 at w_i with angle $\pi/3$ for $i = 0, 1$.

Without loss of generarity, we may assume that $|w_0 - c| \leq |w_1 - c|$. Since $\Delta' \in \mathcal{D}_A(D)$ and $w_0 \in \partial D$ we have

$$|w_0 - c| \geq A\rho. \tag{4.9}$$

Now let ζ_j be the intersection point of the circle C_j and the ray starting from a_j and passing through c, where we should note that $c \neq a_j$ by (4.9). Moreover by (ii) and (4.9), we can see that $\zeta_j \in \gamma_j$ and $|c - \zeta_j| = \operatorname{dist}(c, \gamma_j) < \rho$, so we have that

$$|\zeta_j - w_0| \geq |w_0 - c| - |\zeta_j - c| > (A - 1)\rho. \tag{4.10}$$

Let m_j be the midpoint of γ_j, that is, $m_j \in \gamma_j$ such that $|m_j - w_0| = |m_j - w_1|$. By

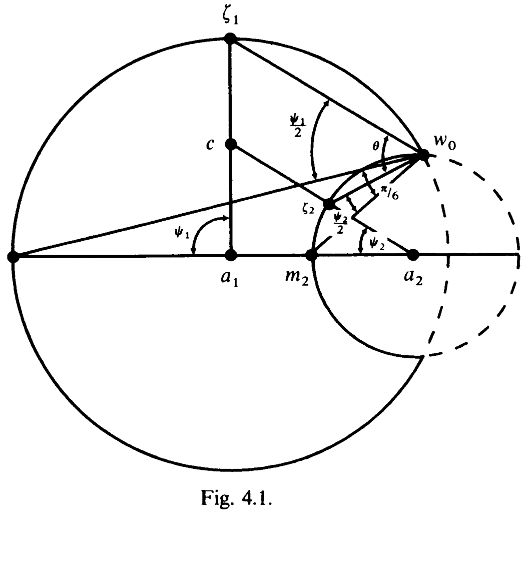

Fig. 4.1.

the property (iii), the elementary geometry tells us that $\angle m_1 w_0 m_2 = \pi/6$. Set $\theta = \angle \zeta_1 w_0 \zeta_2 \in (0, \pi)$, then we can verify that

$$\theta \geq \pi/6. \tag{4.11}$$

Indeed, this is evident if the region W bounded by $\gamma_1 \cup \gamma_2$ is covex. Next, we consider the case that W is not convex. We may assume that m_2 is contained in the inner domain of the circle C_1 (see Fig. 4.1).

Let $\psi_j = \angle m_j a_j \zeta_j \in (0, \pi)$ for $j = 1, 2$. Then $\psi_1 = \angle m_1 a_1 c \leq \angle m_1 a_2 c = \psi_2$. Noting that $\angle m_j w_0 \zeta_j = \psi_j/2$, we have that $\theta = \pi/6 + (\psi_1 - \psi_2)/2 \geq \pi/6$.

Now the cosine formula says that

$$\begin{aligned} |\zeta_1 - \zeta_2|^2 &= |\zeta_1 - w_0|^2 + |\zeta_2 - w_0|^2 - 2|\zeta_1 - w_0||\zeta_2 - w_0| \cos\theta \\ &\geq |\zeta_1 - w_0|^2 + |\zeta_2 - w_0|^2 - 2|\zeta_1 - w_0||\zeta_2 - w_0| \cos\pi/6 \qquad \text{(by (4.11))} \\ &\geq (A-1)^2 \rho^2 (2 - \sqrt{3}) \qquad \text{(by (4.10))}. \end{aligned}$$

On the other hand, $|\zeta_1 - \zeta_2| \leq |\zeta_1 - c| + |\zeta_2 - c| < 2\rho$, so we obtain that $\sqrt{2-\sqrt{3}}(A-1)\rho < 2\rho$, that is, $A < 1 + 2/\sqrt{2-\sqrt{3}} < 5$. This contradicts the assumption that $A \geq 5$. Q.E.D.

Proof of Proposition 4.1 (*continued*). Take a number $A \geq 1$ such that $A' = \dfrac{A + A^{-1}}{2} \geq 5$. Let $\Delta' \in \mathscr{D}_A(D)$. If $\Delta' \cap (\bigcup_{L \in G} L(E_1)) = \emptyset$, then $h_t|_{\Delta'}$ is a restriction of a Möbius transformation by constrution. If $\Delta' \cap (\bigcup_{L \in G} L(E_1)) \neq \emptyset$, then

$$\Delta' \cap L^{-1}(E_1) \neq \emptyset \text{ for some } L \in G.$$

Since $L(\Delta') \cap E_1 \neq \emptyset$, Lemma 3.11 and Lemma 4.4 yield that $h_t = T_t$ on $L(\Delta')$, hence $h_t = L^{-1} \circ h_t \circ L = L^{-1} \circ T_t \circ L$ on Δ'. Consequently, $h_t|_{\Delta'}$ can be extended to a $\dfrac{4t}{C-5}$-qc map by Lemma 4.2. In any case, $h_t|_{\Delta'}$ can be extended to a global k-qc map, where $k = \dfrac{4t}{C-5}$. Because w^{μ_t} is originally a k-qc map, $g|_{\Delta'}$ can be extended to a k'-qc map, where $\dfrac{1+k'}{1-k'} = \left(\dfrac{1+k}{1-k}\right)^2$. Since $k' = \dfrac{2k}{1+k^2} \leq 2k = \dfrac{8t}{C-5}$, by Proposition 2.7, we obtain an estimate

$$\|S_{g_t}\|_D \leq 96A^2 k' \leq 96A^2 \cdot \frac{8t}{C-5}.$$

By assumption on C, we have $\|S_{g_t}\|_D < \varepsilon$ if we take $A = 10$, for example. Let $f: \mathbf{H} \to D$ be a Riemann mapping of D, then the above implies that $S_{g_t \circ f}$ belongs to the ball centered at S_f of radius ε in $B_2(\mathbf{H}, \Gamma)$ where $\Gamma = f^{-1}Gf$. Remarking here the ball above is contained in $S(\Gamma)$ by the assumption on D, we can deduce from Proposition 2.6 that there exists a global qc extension $\tilde{g}_t: \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ of g_t such that $\tilde{g}_t \circ L = \chi_t(L) \circ \tilde{g}_t$ on $\hat{\mathbf{C}}$ for all $L \in G$. (Or, by utilizing a group equivariant version of the ultimate λ-lemma due to Slodkowski (cf. Earle-Kra-Krushkal [4]), we have directly this result.) Therefore h_t also has a global qc extension $(w^{\mu_t})^{-1} \circ \tilde{g}_t$, in particular, the next lemma follows.

4.5. Lemma. *For any* $t \in [0, 1] \setminus J$, $h_t : D \to \hat{\mathbf{C}}$ *can be extended to a homeomorphism* $\tilde{h}_t$ *of* $\hat{\mathbf{C}}$ *commuting with* G.

Now suppose that $J = \emptyset$, then the above lemma implies that $h_1 : D \to \hat{\mathbf{C}}$ is injective, which contradicts the property (c). So, J must be nonempty. Let t_0 be the infimum of J. Since J is open in $(0, 1]$, we remark that $t_0 \in [0, 1] \setminus J$, in particular Lemma 4.5 is applicable to t_0. Let t_n $(n = 1, 2, \dots)$ be a sequence in J converging to t_0. As $t_n \in J$,

$$(4.12) \qquad h_{t_n}(E^\circ) \cap h_{t_n}(L_n(E^\circ)) \neq \emptyset \text{ for some } L_n \in G \setminus \{1\}.$$

Suppose that there exists an $L \in G \setminus \{1\}$ such that $L = L_n$ for infinitely many n's. Then, by the property (d) and the fact that $\bar{E} \subset \Omega(G)$, (4.12) forces that $\tilde{h}_{t_0}(\bar{E}) \cap \tilde{h}_{t_0}(L(\bar{E})) \neq \emptyset$, which is impossible because $\bar{E} \cap L(\bar{E}) = \emptyset$ and $\tilde{h}_{t_0} : \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ is injective. Thus, we may assume that L_n is a distinct sequence in $G \setminus \{1\}$. In this case, as is easily seen from (4.12), $\tilde{h}_{t_0}(\bar{E}) \cap \Lambda(G) \neq \emptyset$, which is contradictory to the fact that $\bar{E} \cap \Lambda(G) = \emptyset$ and that $\tilde{h}_{t_0}(\Lambda(G)) = \Lambda(G)$.

In any case, contradictions are deduced, which implies the falsity of the assumption that $z_1, z_2 \in D \setminus \bar{\Delta}_C$ cannot be joined by any path in $D \setminus \bar{\Delta}$. Q.E.D.

§5. The boundary of $R_0 = D/G$ is a disjoint union of quasi-analytic curves

Let Γ be an arbitrary Fuchsian group and $\varphi \in \operatorname{Int} S(\Gamma)$. We denote by R the (possibly disconnected) Riemann surface $\Omega(G)/G$ where $G = \chi^{\varphi}(\Gamma)$.

In this sectoin, we shall study the relative boundary ∂R_0 of the subdomain $R_0 = D/G = f^{\varphi}(\mathbf{H})/G$ in R. Our main aim here is to prove that ∂R_0 is a disjoint union of quasi-analytic curves under the suitable hypothesis, where the *quasi-analytic curve* means the quasiconformal homeomorphic image of the circle. To this end, we recall a notion of the annular covering. Let α be a homotopically nontrivial simple closed curve in a hyperbolic Riemann surface R. Let $\pi : \mathbf{H} \to R$ be a holomorphic universal covering of R, and γ an element of the covering transformation group $\Gamma_0 <$ Möb of π which covers α, i.e., the terminal point of a lifting curve $\tilde{\alpha}$ of α with respect to π equals to $\gamma(z_0)$ where z_0 is the initial point of $\tilde{\alpha}$.

As is easily seen, the quotient Riemann surface $\mathbf{H}/\langle\gamma\rangle$ is conformally equivalent to an annulus $A = \{z \in \mathbf{C};\, c < |z| < 1\}$, where $0 \leq c < 1$ satisfies the relation $\cosh(\pi^2/\log c) = |\operatorname{tr}\gamma|/2$, which is not so significant below.

Let $\pi_1 : \mathbf{H} \to A$ be a holomorphic covering with the covering transformation group $\langle\gamma\rangle$. The induced holomorphic covering map $q = q_\alpha : A \to R$ such that $\pi = q \circ \pi_1$ is called an *annular covering* with respect to α. By construction, $\hat{\alpha} = \pi_1(\tilde{\alpha})$ is the unique closed lift of α, in other words, any other lift of α than $\hat{\alpha}$ is *not* closed. Using the tool above, we shall prove the following result.

5.1. Theorem. *Let Γ be an arbitrary Fuchsian group of the second kind acting on the upper half plane $\boldsymbol{H}$. For $\varphi \in \operatorname{Int} S(\Gamma)$, let $G = \chi^{\varphi}(\Gamma)$, $D = f^{\varphi}(\mathbf{H})$, $R = \Omega(G)/G$, and $R_0 = D/G$. If a connected component α of ∂R_0 is compact and contains no branch points of the natural projection $p: \Omega(G) \to R$, then α is a quasi-analytic curve and a one-sided boundary component.*

5.2. Corollary. *In particular, when Γ is a finitely generated, purely hyperbolic Fuchsian group of the second kind, the relative boundary ∂R_0 of R_0 is a disjoint union of finitely many quasi-analytic curves, and thus the conformal map F^{φ}: $S_0 \to R_0$ induced by $f^{\varphi}: \mathbf{H} \to D$, $\varphi \in \operatorname{Int} S(\Gamma)$, naturally extends to a homeomorphism $F^{\varphi}: \overline{S_0} \to \overline{R_0}$, where $S_0 = \boldsymbol{H}/\Gamma$ and $\overline{S_0}$ is its closure in $S = \Omega(\Gamma)/\Gamma$, in other words, $f^{\varphi}: \mathbf{H} \to D$ naturally extends to a homeomorphism $f^{\varphi}: \bar{\mathbf{H}} \setminus \Lambda(\Gamma) \to \bar{D} \setminus \Lambda(G)$.*

Proof of Corollary 5.2. By the hypothesis, R is compact, thus so is ∂R_0. Therefore the former part of the above assertion directly follows from Theorem 5.1. In particular, ∂S_0 and ∂R_0 consist of mutually disjoint simple closed curves, therefore the latter part can be deduced from the general version of the famous Carathéodory theorem. Q.E.D.

In order to prove Theorem 5.1, first we need the next

5.3. Lemma. *In addition to the hypothesis in Theorem* 5.1, *further assume that Γ has infinitely many elements. Then, there exists a certain exhausting sequence $(R_n)_{n=1}^{\infty}$ of R_0 with the following properties*:

(0) *each R_n is a subdomain of R_0,*
(i) *$R_1 \subset R_2 \subset \ldots, \bigcup_{n=1}^{\infty} R_n = R_0$,*
(ii) *each $\alpha_n = \partial R_n \cap R_0$ is a homotopically non-trivial smooth simple closed curve which is freely homotopic to any other α_m, and moreover if R is not biholomorphic to the punctured plane $\mathbf{C}^*$ then α_n is not homotopic to any puncture of R,*
(iii) *every limit point of $(\alpha_n)_{n=1}^{\infty}$ is contained in α, and*
(iv) *$R_0 \setminus R_1$ contains no branch points of p.*

Proof of Lemma 5.3. First we shall show that α is open in ∂R_0, in other words, $V \cap \partial R_0 = \alpha$ for some open neighbourhood V of α in R. In fact, let T be a relatively compact neighborhood of α with smooth boundary such that $\bar{T} \subset {}^{\circ}R := {}^{\circ}\Omega(G)/G$ and that $\partial T \cap \partial R_0 = \emptyset$, then it suffices to show that $T_0 := T \cap R_0$ consists of finitely many components each of which has at most finitely many boundary components, i.e., T_0 is a finite union of surfaces of finite topological type. Here, we should note that a Riemann surface X is of infinite topological type if and only if there is an infinite family of mutually disjoint, homotopically independent simple closed curves γ_j ($j = 1, 2, \ldots$) in X. (We call $\gamma_1, \gamma_2, \ldots$ is homotopically independent when each γ_j is not freely homotopic to any γ_k for $k \neq j$ nor null-homotopic.) Let $\mathscr{C} = \{\gamma_1, \ldots, \gamma_l\}$ be a finite family of mutually disjoint, homotopically independent simple closed curves in T_0. Then each γ_j is not null-homotopic in T, too. Otherwise, γ_j bounds a topological disk

Δ in T, and furthermore, $\beta = \Delta \cap \partial R_0 \neq \emptyset$ for γ_j is not null-homotopic in T_0. Since Δ is simply connected domain and since $\Delta \subset {}^\circ R$, $p|_{\tilde{\Delta}} : \tilde{\Delta} \to \Delta$ is biholomorphic, where $\tilde{\Delta}$ is a component of $p^{-1}(\Delta)$. Thus, $\partial\tilde{\Delta} \subset \Omega(G)$ separates $\tilde{\beta} := (p|_{\tilde{\Delta}})^{-1}(\beta) \subset \partial D$ from $\partial D \setminus \tilde{\beta}$, this contradicts the connectedness of ∂D. So we have seen that γ_j is not null-homotopic in T.

Next suppose that γ_j is freely homotopic to γ_k for some $k \neq j$ in T. Then γ_j and γ_k bound a topological annulus (ring domain) A in T. Since γ_j is not freely homotopic to γ_k in T_0, $\beta := A \cap \partial R_0 \neq \emptyset$. Moreover, β separates γ_j from γ_k in A. Indeed, if not, there exists an arc δ in $A \setminus \beta$ connecting γ_j and γ_k, then $A \setminus \delta$ is simply connected and contains a nonempty subset β of ∂R_0, which leads to a contradiction in the same way as in the above.

Further suppose that γ_j is freely homotopic to $\gamma_{k'}$ for $k' \neq j, k$ in T. Then similarly γ_j and $\gamma_{k'}$ bound a topological annulus A' in T, and $\beta' := A' \cap \partial R_0$ separates γ_j from $\gamma_{k'}$ in A'. Because $\partial A \cap \partial A' = \gamma_j$, we have $A \subset A'$, $A' \subset A$ or $A \cap A' = \emptyset$. In any case, $\beta \cup \beta'$ devides R_0 into two pieces, which contradicts the connectedness of R_0.

Thus we conclude that each γ_j is freely homotopic in T to at most one other curve. So we can renumber $\mathscr{C} = \{\gamma_1, \ldots, \gamma_l\}$ so that γ_j is freely homotopic to γ_{s+t+j} in T for $j = 1, \ldots, s$ and that γ_{s+j} is not freely homotopic in T to any other curve γ_k for $j = 1, \ldots, t$, where integers $s, t \geq 0$ satisfy $2s + t = l$. In particular, $\{\gamma_1, \ldots, \gamma_{s+t}\}$ is a family of mutually disjoint homotopically independent simple closed curves in T.

On the other hand, as is well-known, a Riemann surface X of finite topological type $(g, 0, m)$ has at most $3g - 3 + 2m$ mutually disjoint homotopically independent simple closed curves if $(g, m) \neq (0, 0), (0, 1), (1, 0)$. Therefore, $l = 2s + t \leq 2(s + t) \leq 2(3g - 3 + 2m)$ if T is of topological type $(g, 0, m)$, hence l is bounded. This means that T_0 should be a finite union of surfaces of finite topological type. Thus, we have proved that compact component α of ∂R_0 is always isolated in ∂R_0.

Let $p_0 : \Omega(\Gamma) \to S = \Omega(\Gamma)/\Gamma$ be the natural projection, and $F : S_0 = p_0(\mathbf{H}) \to R_0$ the conformal mapping induced by $f = f^{\varphi} : \mathbf{H} \to D$. From the above observation, we can take a curve $\tilde{\beta} : [0, 1] \to \Omega(G)$ such that $\tilde{\beta}([0, 1)) \subset D$ and $\tilde{\beta}(1) \in p^{-1}(\alpha)$. Let $\tilde{\beta}_0 := f^{-1} \circ \tilde{\beta} : [0, 1) \to \mathbf{H}$, then we know that $\tilde{\beta}_0(t)$ has a limit, say $\tilde{\beta}_0(1)$, as $t \to 1$ (see, for example, Pommerenke [12] Proposition 2.14).

Clearly $\tilde{\beta}_0(1) \in \hat{\mathbf{R}} \cap \Omega(\Gamma)$, and let I be the connected component of $\hat{\mathbf{R}} \cap \Omega(\Gamma)$ containing $\tilde{\beta}_0(1)$. Now we shall show that the stabilizer H of I in Γ is generated by a hyperbolic or parabolic element. (Here we remark that the latter case happens only when Γ is generated by a single parabolic element.) Otherwise, H must be trivial, and so p_0 is injective on a neighborhood V of I. Thus we can select a sequence $(I_n)_{n=1}^{\infty}$ of simple arcs in $V \cap \mathbf{H}$ with the same end points as I such that $E_n \supset E_{n+1}$ $(n = 1, 2, \ldots)$ and $\bigcap_{n=1}^{\infty} \bar{E}_n = \bar{I}$, where E_n is the region bounded by $I_n \cup \bar{I}$. First, we remark that the cluster set $C = \bigcap_{n=1}^{\infty} \overline{F(p_0(E_n))}$ is contained in ∂R_0. Now, we can choose a compact neighborhod W of α such that $\partial W \cap \partial R_0 = \emptyset$. Here, $F(p_0(I_n))$ is not relatively compact in R and $\emptyset \neq \tilde{\beta}_0([0, 1))$

$\cap I_n \subset p_0^{-1}(F^{-1}(W \cap R_0))$ for sufficiently large n, therefore we may pick a point $Q_n \in F(p_0(I_n)) \cap \partial W$ for large n. Because ∂W is compact, we may assume that Q_n converges to a point $Q \in \partial W$. On the other hand, $Q \in C \subset \partial R_0$, which contradicts the fact $\partial W \cap \partial R_0 = \emptyset$. Thus, we have proved that $H = \mathrm{Stab}_\Gamma(I)$ is generated by a hyperbolic or parabolic element γ_0. Let $(I_n)_{n=1}^\infty$ be a sequence of circular arcs (or horocycles) in $\mathbf{H}$ with the same end points as I such that $E_n \supset E_{n+1}$ $(n = 1, 2, \ldots)$, and $\bigcap_{n=1}^\infty E_n = \emptyset$, where E_n is the region bounded by $I_n \cup \bar{I}$. Then $R_n := R_0 \setminus F(p_0(E_n \cup I_n))$ is a desired exhaustion of R_0 for sufficiently large n.

Q.E.D.

Proof of Theorem 5.1. We shall prove in the case that Γ is an infinite group. (When Γ is finite, $R = \hat{\mathbf{C}}/\Gamma = \hat{\mathbf{C}}$ by the Riemann-Hurwitz formula, thus the proof is much easier.) Moreover, we may assume that R is hyperbolic. In fact, even if not, taking a sufficiently small closed disk $E \subset R$ such that $E \cap \overline{R_0} = \emptyset$ (see Corollary 3.2), we have only to replace R by the hyperbolic surface $R' = R \setminus E$. Therefore we assume that R is hyperbolic and Γ is an infinite group in the sequel.

Let R_n, α_n $(n = 1, 2, \ldots)$ be as in Lemma 5.2 and $\pi\colon A = \{c < |z| < 1\} \to R$ be an annular covering with respect to α_n. (Remark that for freely homotopic curves, we can take the same annular covering.)

Denote by $\hat{\alpha}_n$ the unique closed lift of α_n via π and let $W_n \subset U = \{z; |z| < 1\}$ be a Jordan domain bounded by $\hat{\alpha}_n$. Compositing the map $z \mapsto c/z$ to π if necessary, we may assume that $W_n \subset W_{n+1}$ for all $n \geq 1$, here we should note that $c > 0$ by Lemma 5.3 (ii). Set $W = \bigcup_{n=1}^\infty W_n$ and $\hat{\alpha} = A \cap \partial W$. Clearly, $\pi(\hat{\alpha}) \subset \alpha$. We shall show that the restricted map $\pi|_{\hat{\alpha}}\colon \hat{\alpha} \to \alpha$ is a homeomorphism.

Let w_0 be an arbitrary point of α. By Lemma 3.6, there exists a connected open neighborhood V of w_0 such that $V \cap R_0$ is connected, $V \cap R_1 = \emptyset$, $V \cap (\partial R_0 \setminus \alpha) = \emptyset$ and that V is contained in a topological disk $\tilde{V}$ in ${}^\circ R$. Now we claim that there exists a (unique) component V_0 of $\pi^{-1}(V)$ with the following conditions:

(1) $V_0 \cap \hat{\alpha} \neq \emptyset$,

(2) $(\pi^{-1}(V) \setminus V_0) \cap \hat{\alpha} = \emptyset$,

(3) $V_0 \cap \pi^{-1}(\alpha) = V_0 \cap \hat{\alpha}$.

To prove the above claim, we first remark that $\pi_1 := \pi|_{W \setminus \bar{W}_1}\colon W \setminus \bar{W}_1 \to R_0 \setminus \bar{R}_1$ is biholomorphic. Let V_0 be a component of $\pi^{-1}(V)$ containing a point of $\pi_1^{-1}(V \cap R_0)$. Then $V_0 \cap (W \setminus \bar{W}_1) = \pi_1^{-1}(V \cap R_0)$ because $V \cap R_0$ is connected. In particular, $(\pi^{-1}(V) \setminus V_0) \cap (W \setminus \bar{W}_1) = \emptyset$, thus the condition (2) follows. Since $\pi_0 := \pi|_{V_0}\colon V_0 \to V$ is biholomorphic, we have

$$\begin{aligned}\pi^{-1}(\alpha) \cap V_0 &= \pi_0^{-1}(\alpha \cap V) = \pi_0^{-1}(\partial R_0 \cap V) = \partial \pi_0^{-1}(R_0 \cap V) \cap V_0 \\ &= \partial \pi_1^{-1}(R_0 \cap V) \cap V_0 = \partial(V_0 \cap (W \setminus \bar{W}_1)) \cap V_0 \\ &= \partial W \cap V_0 = \hat{\alpha} \cap V_0,\end{aligned}$$

where we use the fact that $V_0 \cap W_1 = \emptyset$ thus (3) is proved. By the condition (3),

$\zeta_0 := \pi_0^{-1}(w_0) \in V_0 \cap \pi^{-1}(\alpha) \subset \hat{\alpha}$, which implies (1), and as a by-product we obtain that $\alpha = \pi(\hat{\alpha})$. Moreover, condition (3) yields the injectivity of $\pi|_{\hat{\alpha}}: \hat{\alpha} \to \alpha$, therefore we have proved that $\pi|_{\hat{\alpha}}: \hat{\alpha} \to \alpha$ is a homeomorphism.

We shall continue the proof of Theorem 5.1. Since π is a conformal map in a neighborhood of $\hat{\alpha}$, it is sufficient to prove that $\hat{\alpha}$ is a quasi-circle.

Here we mention a lemma which is a direct conclusion of the Koebe distorsion therem: $\dfrac{a}{(1+a)^2} \leq |f(z)| \leq \dfrac{a}{(1-a)^2}$ for $|z| = a < 1$ if f is univalent in the unit disk and $f(0) = f'(0) - 1 = 0$.

5.4. Lemma. *Let f be a conformal mapping from a disk Δ into $\mathbf{C}$. For $K > 1$, let $a \in (0, 1)$ satisfy the equation $K = \dfrac{(1-a)^2}{4a}$. Then there exists a disk $\tilde{\Delta}$ such that $f(\Delta_a) \subset \tilde{\Delta}$ and $\tilde{\Delta}_K \subset f(\Delta)$.*

First, we shall show the following lemma.

5.5. Lemma. *The domain W constructed above is linearly connected.*

Proof of Lemma 5.5. Since $\hat{\alpha} = \partial W \subset A$ is compact, there exists a positive constant $\delta < \operatorname{diam} W$ with the following property: if a disk Δ with $\operatorname{diam} \Delta \leq \delta$ had a nonempty intersection of $\hat{\alpha}$, then $\Delta \subset A \setminus (\pi^{-1}(\partial R_0) \setminus \hat{\alpha})$ and π is injective in Δ.

Take $0 < a < 1$ satisfing the equation $B = \dfrac{(1-a)^2}{4a}$ where B is the constant which appeared in Propostion 3.1. Let Δ be an arbitrary disk. In order to prove linear connectedness, we shall consider several cases.

Case 1. $\Delta \cap \hat{\alpha} = \emptyset$.

In this case $\Delta \subset W$ or $\Delta \cap W = \emptyset$, so any two points in $\Delta \cap W$ can be always joined by a path in $\Delta \cap W$.

Case 2. $\Delta \cap \hat{\alpha} \neq \emptyset$ and $\operatorname{diam} \Delta < a\delta$.

Then $\Delta_{1/a} \subset A$, $\Delta_{1/a} \cap (\pi^{-1}(\partial R_0) \setminus \hat{\alpha}) = \emptyset$, and π is injective in $\Delta_{1/a}$ by the choice of δ.

Let V be a connected component of $p^{-1}(\pi(\Delta_{1/a}))$. Remark that p is injective in V because p is a covering map and $\pi(\Delta_{1/a})$ is simply connected. Now we apply Lemma 5.4 to the conformal mapping $f = (p|_V)^{-1} \circ \pi: \Delta_{1/a} \to V$. Thus we know that $f(\Delta) \subset \tilde{\Delta}$ and $\tilde{\Delta}_B \subset f(\Delta_{1/a}) = V \subset \Omega(G)$ for some disk $\tilde{\Delta}$. Then, any two points in $f(\Delta) \cap D$ can be joined by a path in $\tilde{\Delta}_B \cap D$, so in $V \cap D$ by Proposition 3.1. Since $f(\Delta_{1/a} \cap W) = f(\Delta_{1/a}) \cap D = V \cap D$ by the condition $\Delta_{1/a} \cap (\pi^{-1}(\partial R_0) \setminus \hat{\alpha}) = \emptyset$, any two points in $\Delta \cap W = f^{-1}(f(\Delta) \cap D)$ can be joined by a path in $\Delta_{1/a} \cap W$.

Case 3. $\Delta \cap \hat{\alpha} \neq \emptyset$ and $\operatorname{diam} \Delta \geq a\delta$. If we set $M = 1 + \dfrac{2 \operatorname{diam} W}{a\delta} (> 1/a)$, then $\Delta_M \supset W$. Thus any two points in $\Delta \cap W$ can be joined by a path in $\Delta_M \cap W = W$.

Hence, in any case, arbitrary two points in $\Delta \cap W$ can be joind by a path in $\Delta_M \cap W$, thus W is linearly connected. Q.E.D.

By Lemma 5.5, in particular, W is a bounded Jordan domain. We should remark that we have proved Lemma 5.5 without results in §4. As for Jordan domains, we mention the next elementary fact, which follows from the uniform continuity of a homeomorphic parametrization $S^1 \to \partial W$ and its inverse map.

5.6. Lemma. *Let W be a bounded Jordan domain in* $\mathbf{C}$ *and η any positive number. Then there exists a positive constant δ with the property that, for any cross cut γ of W with* $\operatorname{diam} \gamma \leq \delta$, *it holds that*

$$\min \{\operatorname{diam} W_1, \operatorname{diam} W_2\} \leq \eta,$$

where W_1 and W_2 are two components of $W \setminus \gamma$.

By the help of Lemma 5.6, secondly we shall show the following lemma.

5.7. Lemma. *W is a John domain.*

Proof of Lemma 5.7. Let $0 < a < 1$ such that $C = \dfrac{(1-a)^2}{4a}$, where C is the constant in the statement of Proposition 4.1. Since $\alpha \subset A$ is compact, we can choose a positive $\eta > 0$ so small that any disk Δ with $\bar{\Delta} \cap \hat{\alpha} \neq \emptyset$ and $\operatorname{diam} \Delta \leq 4\eta$ should satisfy $\Delta \subset A \setminus \bar{W}_1$, $\pi|_\Delta$ is injective, and $\pi(\Delta) \subset p({}^\circ\Omega(G))$.

By Lemma 5.6, for sufficiently small $\delta (0 < \delta \leq \eta)$ the following holds: if a disk Δ with diameter $< \delta$ has nonempty intersection with $\hat{\alpha} = \partial W$, then $\operatorname{diam}(W \setminus W_0) < \eta$ where W_0 is the connnected component of $W \setminus \bar{\Delta}$ containing W_1.

Now let Δ be an arbitrary disk centered at z_0.

Case 1. $\Delta \cap \hat{\alpha} = \emptyset$.

In this case, arbitrary two points in $W \setminus \bar{\Delta}$ can be joined by a path in $W \setminus \bar{\Delta}$.

Case 2. $\Delta \cap \hat{\alpha} \neq \emptyset$ and $\operatorname{diam} \Delta < a\delta$.

Since $\bar{\Delta}_{1/a} \cap \hat{\alpha} \neq \emptyset$ and $\operatorname{diam} \Delta_{1/a} < \delta (\leq \eta)$, $\Delta_{1/a} \cap W_1 = \emptyset$ and $\operatorname{diam}(W \setminus W_0) < \eta$ where W_0 is the component of $W \setminus \bar{\Delta}$ containing W_1. Set $\Delta' = \{z \in \mathbf{C}; |z - z_0| < 2\eta\}$, then clearly $\bar{W} \setminus W_0 \subset \Delta'$ and $\bar{\Delta}_{1/a} \subset \Delta'$. Since $\operatorname{diam} \Delta' = 4\eta$, Δ' must satisfy that $\Delta' \subset A \setminus \bar{W}_1$, $\pi|_{\Delta'}$ is injective, and that $\pi(\Delta') \subset p({}^\circ\Omega(G))$.

Let V be a connected component of $p^{-1}(\pi(\Delta'))$, then $p|_V : V \to \pi(\Delta')$ is a biholomorphic map for $\pi(\Delta')$ is simply connected and $\pi(\Delta') \subset p({}^\circ\Omega(G))$. Since π is injective in Δ', $f := (p|_V)^{-1} \circ \pi : \Delta' \to V$ is a conformal homeomorphism. Now we apply Lemma 5.4 for $f|_{\Delta_{1/a}}$, then we obtain that there exists a disk $\tilde{\Delta}$ such that $f(\Delta) \subset \tilde{\Delta}$ and $\tilde{\Delta}_C \subset f(\Delta_{1/a})$.

Fix a point $z_1 \in W_0 \cap (\Delta' \setminus \overline{\Delta_{1/a}})$. Suppose that there exists a point z_2 in $(W \setminus W_0) \setminus \overline{\Delta_{1/a}} (\subset \Delta')$.

We denote by T the component of $W \setminus \bar{\Delta}$ containing z_2, then clearly $T \cap W_0 = \emptyset$ and $\bar{T} \subset \Delta'$. Since $\bar{T} \subset \Delta'$, $\partial f(T) \subset V \cap \partial f(\Delta' \cap W \setminus \bar{\Delta}) = V \cap \partial(D \setminus f(\bar{\Delta}))$, so we obtain that $\partial f(T) \subset \partial(D \setminus f(\bar{\Delta}))$, which implies that $f(T)$ is a connected component of $D \setminus f(\bar{\Delta})$.

On the other hand, $w_j = f(z_j) \in f(\Delta' \cap W \setminus \overline{\Delta_{1/a}}) = V \cap D \setminus f(\overline{\Delta_{1/a}}) \subset D \setminus \overline{\tilde{\Delta}_C}$ $(j = 1, 2)$, thus Proposition 4.1 guarantees that w_1 and w_2 are connected by a path in $D\setminus\bar{\bar{\Delta}} \subset D\setminus f(\bar{\Delta})$. Since $f(T)$ is a component of $D\setminus f(\bar{\Delta})$ and $w_2 \in f(T)$, w_1 must be in $f(T)$ too, i.e., $z_1 \in T$, which is a contradiction. Therefore we conclude that $(W\setminus W_0)\setminus \overline{\Delta_{1/a}} = \emptyset$, i.e., $W\setminus \overline{\Delta_{1/a}} \subset W_0$, which implies that any two points in $W\setminus \overline{\Delta_{1/a}}$ are joined by a path in $W_0 \subset W\setminus \bar{\Delta}$.

Case 3. $\Delta \cap \hat{\alpha} \neq \emptyset$ and $\operatorname{diam} \Delta \geq a\delta$.

Let $M = 1 + \dfrac{2\operatorname{diam} W}{a\delta} (> 1/a)$, then $\Delta_M \supset W$. Thus, trivially it holds that any two points in $W\setminus \bar{\Delta}_M$ can be joined by a path in $W\setminus \bar{\Delta}$.

In any cases, we have proved that any two points in $W\setminus \overline{\Delta_M}$ can be joined by a path in $W\setminus \bar{\Delta}$. Now the proof is completed. Q.E.D.

Combining Lemma 5.5 and Lemma 5.7 with Theorem 2.9, we can immediately obtain Thereom 5.1.

§6. Existence of a topological involution of R w.r.t. ∂R_0

In this section, chiefly we shall be concernd with the following result, which is a crucial part of the proof of our main theorem.

6.1. Theorem. *Let G be a Schottky group of rank $N(\geq 0)$ and $p: \Omega(G) \to R := \Omega(G)/G$ the natural projection. Suppose that R_0 is a proper subdomain of R such that $D = p^{-1}(R_0)$ is a simply connected domain and that ∂R_0 consists of mutually disjoint simple closed curves. Let $f: \mathbf{H} \to D$ be a Riemann mapping of D, Γ the Fuchsian group defined by $\Gamma = f^{-1}Gf$ and $\chi: \Gamma \to G$ the isomorphism defined by $\chi(\gamma) \circ f = f \circ \gamma$ for all $\gamma \in \Gamma$.*

Then f can be extended to a homeomorphism $\tilde{f}: \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ satisfying that $\chi(\gamma) \circ \tilde{f} = \tilde{f} \circ \gamma$. In particular, $D = \tilde{f}(\mathbf{H})$ is a Jordan domain.

To prove Thorem 6.1, the following proposition comprises the key step.

6.2. Proposition. *Under the same hypothesis of Theorem* 6.1, *there exists a topological involution J of R with respect to ∂R_0, which can be lifted. More precisely, $J: R \to R$ is an orientation-reversing homeomorphism such that $J(R_0) \cap R_0 = \emptyset$, $J|_{\partial R_0} = \mathrm{id}_{\partial R_0}$, $J \circ J = \mathrm{id}_R$ and there exists a homeomorphism $j: \Omega(G) \to \Omega(G)$ which satisfies that $p \circ j = J \circ p$, $j|_{\partial D\setminus \Lambda(G)} = \mathrm{id}_{\partial D\setminus \Lambda(G)}$, $j \circ j = \mathrm{id}_{\Omega(G)}$ and that $j \circ L = L \circ j$ for all $L \in G$.*

6.3. Remark. By Proposition 6.4 below, the above lift $j: \Omega(G) \to \Omega(G)$ naturally extends to a self-homeomorphism of $\hat{\mathbf{C}}$, and then $j|_{\partial D} = \mathrm{id}_{\partial D}$.

Proof of Theorem 6.1. First remark that the conformal map $f: \mathbf{H} \to D$ naturally extends to a homeomorphism $f: \bar{\mathbf{H}}\setminus \Lambda(\Gamma) \to \bar{D}\setminus \Lambda(G)$ by the proof of Corollary 5.2.

Let $j: \Omega(G) \to \Omega(G)$ be the involution satisfying the statement in Proposition 6.2, then we define $\tilde{f}: \Omega(\Gamma) \to \Omega(G)$ by the rule

$$\tilde{f}=\begin{cases} f & \text{on } \bar{\mathbf{H}}\setminus\Lambda(\Gamma) \\ j\circ f\circ j_0 & \text{on } \hat{\mathbf{C}}\setminus\bar{\mathbf{H}}, \end{cases}$$

where j_0 denotes the conjugation map $z\mapsto\bar{z}$.

Since the limit sets of Schottky groups are totally disconnected, it suffices to prove the following purely topological proposition, which essentially follows from the fact that for any plane domain Ω, the Kerékjártó-Stoïlow compactification of Ω is homeomorphic to the quotient space $\bar{\Omega}/\sim$ obtained by collapsing each boundary component of Ω in $\hat{\mathbf{C}}$ to one point (with the quotient topology).

6.4. Proposition. *Let E_1, E_2 be totally disconnected compact subsets of $\hat{\mathbf{C}}$, and set $\Omega_i=\hat{\mathbf{C}}\setminus E_i$ for $i=1, 2$. Then, any homeomorphism $f\colon \Omega_1\to\Omega_2$ (if exists) uniquely extends to a homeomorphism $\tilde{f}\colon\hat{\mathbf{C}}\to\hat{\mathbf{C}}$.*

Sketch of the proof of Proposition 6.4. Let $z\in E_1$. By the Zoretti theorem (cf [N: p. 109]), we can take a nesting sequence α_n, $n=1, 2,\dots$ of Jordan curves in Ω_1 shrinking to the one point z. Then $\{f(\alpha_n)\}$ is a nesting sequence of the Jordan curves shrinking to the exactly one point, say, w. Thus we can assign $\tilde{f}(z)$ as the limit point w of $f(\alpha_n)$ for $z\in E_1$. Defining $\tilde{f}=f$ on $\Omega_1=\hat{\mathbf{C}}\setminus E_1$ we have a homeomorphic extension $\tilde{f}\colon\hat{\mathbf{C}}\to\hat{\mathbf{C}}$ of f. Q.E.D.

Now, our only task is to prove Proosition 6.2! As a preparation, we now state general results about relations between geometric properties of covering spaces and algebraic ones of the fundamental groups. The proof of these results is straightforward, so we shall omit it. Suppose that $p\colon\Omega\to R$ is a normal (= Galois) covering between Riemann surfaces (or, more generally, manifolds). Let R_0 be a subdomain of R and $\iota\colon R_0\to R$ denote the inclusion map. Pick a point a_0 from R_0 and z_0 from Ω with $p(z_0)=a_0$. The inclusion map $\iota\colon R_0\to R$ naturally induces a homomorphism $\iota_*\colon\pi_1(R_0, a_0)\to\pi_1(R, a_0)$. Let $\lambda\colon\pi_1(R, a_0)\to G$ be the monodromy homomorphism with respect to z_0, where G is a covering transformation group of $p\colon\Omega\to R$. Namely, $g=\lambda([\alpha])$ for $g\in G$ and $[\alpha]\in\pi_1(R, a_0)$ if and only if the final point of the lift $\tilde{\alpha}$ of α with initial point z_0 coincides with $g(z_0)$.

6.5. Proposition. (1) *Any one of the following implies the others.*

(1a) *Each component of $p^{-1}(R_0)$ is simply connected,*

(1b) *$\lambda\circ\iota_*$ is injective,*

(1c) *ι_* is injective and $\iota_*(\pi_1(R_0, a_0))\cap\ker\lambda=1$.*

(2) *Any one of the following implies the others.*

(2a) *$p^{-1}(R_0)$ is connected,*

(2b) *$\lambda\circ\iota_*$ is surjective,*

(2c) *$\pi_1(R, a_0)=\ker\lambda\cdot\iota_*(\pi_1(R_0, a_0))$.*

6.6. Corollary. *The following conditions are equivalent to each other.*

(a) *$p^{-1}(R_0)$ is a simply connected domain,*

(b) *$\lambda\circ\iota_*\colon\pi_1(R_0, a_0)\to G$ is an isomorphism,*

(c) $\iota_*: \pi_1(R_0, a_0) \hookrightarrow \pi_1(R, a_0)$ *is an embedding and* $\pi_1(R, a_0) = \ker \lambda \rtimes \pi_1(R_0, a_0)$ *(semi-direct product).*

We now return to the case we have considered, i.e., $\Omega = \Omega(G)$ and $p: \Omega \to R = \Omega(G)/G$ is a Schottky covering. Since $p^{-1}(R_0) = D$ is a simply connected domain, by the Corollary 6.6, it turns out that the homomorphism $\iota_*: \pi_1(R_0, a_0) \to \pi_1(R, a_0)$ has a cross-section $s: \pi_1(R, a_0) \to \pi_1(R_0, a_0)$, e.g., $s = (\lambda \circ \iota_*)^{-1} \circ \lambda$. Through the natural homomorphisms $h: \pi_1(R_0, a_0) \to H_1(R, \mathbf{Z})$ and $h_0: \pi_1(R_0, a_0) \to H_1(R_0, \mathbf{Z})$ (Hurewicz homomorphisms), ι_* and s induce homomorphisms $\iota_\#: H_1(R_0, \mathbf{Z}) \to H_1(R, \mathbf{Z})$, $s_\#: H_1(R, \mathbf{Z}) \to H_1(R_0, \mathbf{Z})$ of the first homology groups, which make the following diagram (6.1) commute. Here we should remark that the kernel of the Hurewicz homomorphism is the commutator subgroup of the fundamental group.

(6.1)
$$\begin{array}{ccccc} \pi_1(R_0, a_0) & \xrightarrow{\iota_*} & \pi_1(R, a_0) & \xrightarrow{s} & \pi_1(R_0, a_0) \\ \downarrow{\scriptstyle h_0} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h_0} \\ H_1(R_0, \mathbf{Z}) & \xrightarrow{\iota_\#} & H_1(R, \mathbf{Z}) & \xrightarrow{s_\#} & H_1(R_0, \mathbf{Z}) \end{array}$$

Since $s_\# \circ \iota_\# = (s \circ \iota_*)_\# = \mathrm{id}_{H_1(R_0, \mathbf{Z})}$, we obtain the following

6.7. Proposition. *The homomorphism* $\iota_\#: H_1(R_0, \mathbf{Z}) \to H_1(R, \mathbf{Z})$ *induced by the inclusion map* $\iota: R_0 \to R$ *is injective.*

By use of the proposition above, we can show the following preliminary

6.8. Lemma. *The exterior* $R_1 = \overline{R_0}$ *of* R_0 *is homeomorphic to* R_0.

Proof. First, we recall that the Schottky group G is a free group of finite rank N. Next, let $(g, 0, m)$ be the topological type of R_0, i.e., R_0 is a genus g compact surface (without punctures) with m mutually disjoint closed topological disks removed. As is well-known, the fundamental group $\pi_1(R_0, *)$ is a free group of rank $2g + m - 1$. Now, Corollary 6.6 yields that $\pi_1(R_0, *)$ is isomorphic to G, thus $N = 2g + m - 1$.

For a while, suppose that R_1 is connected. Let $(g', 0, m')$ be the topological type of R_1, then clearly $m = m'$ and $g + g' + m - 1 = N$ since R is of genus N compact surface and $R = \overline{R_0} \cup \overline{R_1}$, thus $g' = g$ which asserts that R_0 and R_1 are of same topological type $(g, 0, m)$. So, we have only to prove the connectedness of R_0.

Let $c_1, \ldots, c_m$ be boundary loops of R_0 which are consistently oriented. Denote by $C_1, \ldots, C_m$ the homology classes of $c_1, \ldots, c_m$ in R_0, respectively. Here, notice that $H_1(R_0, \mathbf{Z})$ has $A_1, B_1, \ldots, A_g, B_g, C_1, \ldots, C_m$ as a generator over $\mathbf{Z}$ with the sole relation $C_1 + \cdots + C_m = 0$, where $A_1, B_1, \ldots, A_g, B_g$ denote fundamental cycles cutting handles of R_0. Suppose that R_1 was disconnected. Let R_1' be a connected component of R_1, then R_1' had some boundary components, say, $c_1, \ldots, c_l$ with $1 \leq l < m$. Let $C_i' = \iota_\#(C_i)$ be the homology class of c_i in R, then

$C_1' + \cdots + C_l' = 0$ since $C_1' + \cdots + C_l' = [c_1 + \cdots + c_l] = [-\partial R_1'] = 0$ in $H_1(R, \mathbf{Z})$. By Proposition 6.4, $\iota_\#: H_1(R_0, \mathbf{Z}) \to H_1(R, \mathbf{Z})$ is injective, so we obtain an extra relation $C_1 + \cdots + C_l = 0$ which is a contradiction. Q.E.D.

6.9. Proposition. *Under the same hypothesis of Theorem* 6.1, *we have a system of disjoint simple closed curves* $\{\ell_1, \ldots, \ell_N\}$ *on* R, *which satisfies the following conditions*:

(a) $p: \Omega(G) \to R$ *is the highest covering which lifts* ℓ_i *to a closed loop for* $i = 1, \ldots, N$,
(b) *each* ℓ_i *transversely intersects* ∂R_0 *at exactly two points, and*
(c) $R_0' = R_0 \setminus \bigcup_{i=1}^N \ell_i$ *and* $\underline{R_1'} = R_1 \setminus \bigcup_{i=1}^N \ell_i$ *are both simply connected domains, where* $R_1 = R \setminus \overline{R_0}$.

Proof. Let $(g, 0, m)$ be the topological type of R_0, then $N = 2g + m - 1$ as we have seen before. Let $c_1, \ldots, c_m$ be boundary comopnents of R_0, or equivalently, of R_1. Then there exist mutually disjoint simple closed arcs $u_1, \ldots, u_N$ on $\overline{R_1}$ as follows (see Fig. 6.1):

(i) whole u_i is contained in R_1 except its endpoints,
(ii) for $i = 1, \ldots, m-1$, u_i connects c_m with c_i,
(iii) for $i = m, \ldots, m + 2g - 1 = N$, u_i starts from c_m and returns to c_m, and
(iv) $R_1' := R_1 \setminus \bigcup_{i=1}^N u_i$ is connected.

In order to advance the proof, we require several lemmas as the following.

6.10 Lemma. R_1' *is simply connected.*

Proof of Lemma 6.10. Let Σ_i denote the compact sufrace (with boundary) which is obtained by cutting $\overline{R_1}$ along $u_1 \cup \cdots \cup u_i$. (We set $\Sigma_0 = \overline{R_1}$.) Let $\chi(\Sigma)$ denote the Euler characteristic of a compact surface Σ with boundary, that is

$$\chi(\Sigma) = \#\{\text{vertices}\} - \#\{\text{edges}\} + \#\{\text{faces}\}$$

for an arbitrary triangulation of Σ. Further remark that $\chi(\Sigma_{g,m}) = 2 - 2g - m$, where $\Sigma_{g,m}$ represents a compact orientable surface of genus g with m

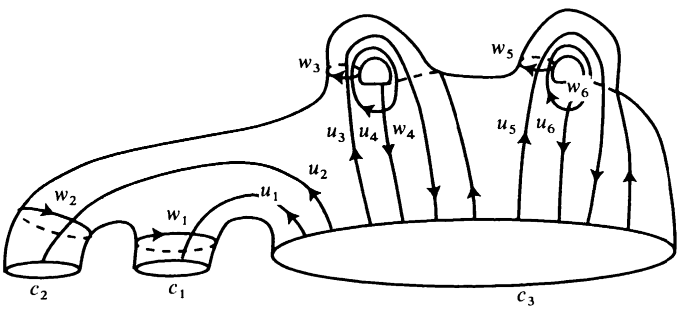

Fig. 6.1. case $(g, 0, m) = (2, 0, 3)$

boundaries. Because Σ_i is obtained by cutting Σ_{i-1} along u_i, we have $\chi(\Sigma_i) = \chi(\Sigma_{i-1}) + 1$ for $i = 1,\ldots, N$. Summarizing these equalities, we obtain that

$$\chi(\Sigma_N) = \chi(\Sigma_0) + N = 2 - 2g - m + N = 1.$$

Therefore R_1' must be homeomorphic to the unit disk. Q.E.D.

6.11. Lemma. *$p^{-1}(R_1)$ is simply connected. In particular, the restriction of p to any component of $p^{-1}(R_1)$ is a universal covering of R_1.*

Proof of Lemma 6.11. $p^{-1}(R_1)$ is the complement of the connected set $\bar{D}$, therefore the above statement is clear. Q.E.D.

Proof of Proposition 6.9 (*continued*). Let $\widetilde{R_1'}$ be connected component of $p^{-1}(R_1')$. By Lemmas 6.10 and 6.11, the restriction map $p|_{\widetilde{R_1'}}: \widetilde{R_1'} \to R_1'$ is bijective. Let $\tilde{\Sigma}$ denote the closure of $\widetilde{R_1'}$.

Now, we can take a simple closed curve w_i: $[0, 1] \to R_1$ which starts from one side of u_i and ends to another side of u_i, and which satisfies that $w_i((0, 1)) \subset R_1'$. Here, we should remark that for $i = 1,\ldots,m-1$, w_i is freely homotopic to the boundary curve c_i. Let $\tilde{w}_i$: $[0, 1] \to \Omega(G)$ be the unique lift in $\tilde{\Sigma}$ of w_i and L_i be the unique element of G with $\tilde{w}_i(1) = L_i(\tilde{w}_i(0))$. Clearly w_i is a homotopically nontrivial loop in R_1, so that $L_i \neq 1$ by virture of Proposition 6.5 (1) and Lemma 6.11. Let u_i^+ be the unique lift of u_i which passes through $\tilde{w}_i(0)$, and set $u_i^- = L_i(u_i^+)$. Then, $p^{-1}(u_i) \cap \tilde{\Sigma} = u_i^+ \cup u_i^-$ and $u_i^+ \cap u_i^- = \emptyset$ (see Fig. 6.2).

Let, for $i = 1,\ldots, N$, $\hat{v}_i^+$ (resp. $\hat{v}_i^-$) be the geodesic curve in $\mathbf{H}$ which connects images a_i^+, b_i^+ (resp. a_i^-, b_i^-) of endpoints of u_i^+ (resp. u_i^-) under f^{-1}; that is, $\hat{v}_i^\pm$ is the semi-circle in $\mathbf{H}$ perpendicularly intersecting $\partial\mathbf{H} = \hat{\mathbf{R}}$ at $a_i^\pm$ and $b_i^\pm$. Set $v_i^\pm = f(\hat{v}_i^\pm)$ and $\ell_i^\pm = u_i^\pm \cup v_i^\pm$ for $i = 1,\ldots, N$. Then it is obvious that $\ell_i^\pm$ are Jordan curves in $\Omega(G)$ and $L_i(\ell_i^+) = \ell_i^-$ by construction. Furthermore, $\ell_1^+, \ell_1^-,\ldots,$

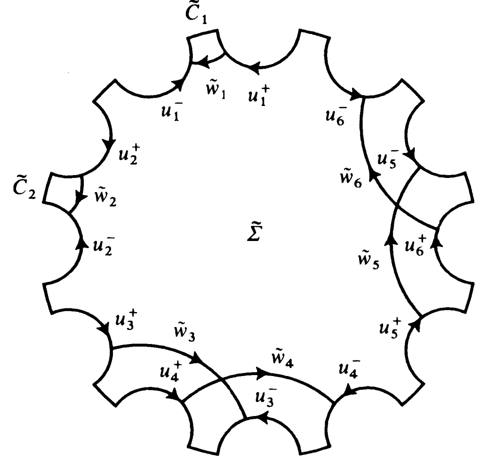

Fig. 6.2.

ℓ_N^- are mutually disjoint. Indeed, since $v_i^\pm$ transversally intersects $v_k^\pm$ at most one point for other $u_k^\pm$ while $u_i^\pm \cap u_k^\pm = \emptyset$ (here, possibly with different signature), if some $\ell_i^\pm$ intersects other $\ell_k^\pm$, $\ell_i^\pm$ transversally intersects $\ell_k^\pm$ at exactly one point, this is a contradiction.

We shall denote by $\operatorname{Ext}\ell_i^\pm$ the component of $\hat{\mathbf{C}} \setminus \ell_i^\pm$ which contains R_1'. By definition, $L_i(\operatorname{Ext}\ell_i^+) \cap \operatorname{Ext}\ell_i^- = \emptyset$ for $i = 1, \dots, N$, and so the subgroup $G_0 := \langle L_1, \dots, L_N \rangle$ of G generated by $L_1, \dots, L_N$ is a Schottky group of the same rank N and $W := \bigcap_{i=1}^N (\operatorname{Ext}\ell_i^+ \cap \operatorname{Ext}\ell_i^-)$ is a fundamental domain of G_0. Here we should note that $\bar{W} \cap \partial D = \partial\tilde{\Sigma} \setminus (u_1^+ \cup u_1^- \cup \cdots \cup u_N^-)$ consists of finite number of lifts of some parts of boundary curves $c_1, \dots, c_m$, therefore, $\bar{W} \cap \partial D \subset \Omega(G)$. As clearly $\bar{W} \setminus \partial D \subset \Omega(G)$, we have $\bar{W} \subset \Omega(G)$, therefore $\Omega(G_0) \subset \Omega(G)$. On the other hand, trivially $\Omega(G_0) \supset \Omega(G)$, thus we conclude that $\Omega(G_0) = \Omega(G)$. Since $\Omega(G_0)/G_0$ and $R = \Omega(G)/G$ are of the same genus N, if $N \neq 1$ the Riemann-Hurwitz theorem implies that the induced covering map $\Omega(G_0)/G_0 = \Omega(G)/G_0 \to \Omega(G)/G$ must be univalent, in other words, $G = G_0$. In case $N = 1$, there exists an element $A \in G$ and a natural number n such that $G = \langle A \rangle$ and $L_1 = A^n$. Since L_1 covers a simple closed curve w_1, n should be 1, therefore $G = G_0$. In any case, $\ell_i := p(\ell_i^\pm)$ for $i = 1, \dots, N$ have all the disired properties, by the construction above. For example, R_0' is known to be simply connected domain by the proof of Lemma 6.10. The proof of Proposition 6.9 is now completed. Q.E.D.

Proof of Proposition 6.2. Let $\ell_1, \dots, \ell_N$ be a system of mutually disjoint simple closed curves on R as in Proposition 6.9. Set $R_0' = R_0 \setminus \bigcup_{i=1}^N \ell_i$ and $R_1' = R_1 \setminus \bigcup_{i=1}^N \ell_i$ are both simply connected domains, where $R_1 = R \setminus \overline{R_0}$. Let W be a component of $p^{-1}(R \setminus \bigcup_{i=1}^N \ell_i)$, then W is a $2N$-ply connected domain with boundary curves $\ell_1^+, \ell_1^-, \dots, \ell_N^-$, where $\ell_i^\pm$ is a closed lift of ℓ_i. We denote by L_i the unique element of G which maps ℓ_i^+ to ℓ_i^-. Let R_k' be the connected component of $p^{-1}(R_k')$ which is contained in W and let Σ_k be the closure of R_k' for $k = 0, 1$. Then, Σ_k seems as a $4N$-gon with $2N$ sides $\ell_i^\pm \cap \Sigma_k$ $(i = 1, \dots, N)$ and $2N$ sides which are lifts of some parts of ∂R_0. Since the order of $\ell_i^\pm \cap \Sigma_0$ in $\partial\Sigma_0$ well corresponds to the one of $\ell_i^\pm \cap \Sigma_1$ in $\partial\Sigma_1$, we can obtain an orientation-reversing homeomorphism $\tilde{J}_0 : \partial\Sigma_0 \to \partial\Sigma_1$ with the following two properties:

(1) $\tilde{J}_0 = \mathrm{id}$ on $\Sigma_0 \cap \partial D = \Sigma_1 \cap \partial D$,
(2) $\tilde{J}_0 \circ L_i = L_i \circ \tilde{J}_0$ on $\Sigma_0 \cap \ell_i^+$ for $i = 1, \dots, N$.

As Σ_0 and Σ_1 are Jordan domains, we can extend $\tilde{J}_0$ to a homeomorphism $\tilde{J}_1 : \Sigma_0 \to \Sigma_1$ with $\tilde{J}_1|_{\partial\Sigma_0} = \tilde{J}_0$. By Property (1), we can further extend $\tilde{J}_1$ to an orientation-reversing homeomorphism $\tilde{J}_2 : \bar{W} \to \bar{W}$ by the rule:

$$\tilde{J}_2 = \begin{cases} \tilde{J}_1 & \text{on } \Sigma_0 \\ \tilde{J}_1^{-1} & \text{on } \Sigma_1. \end{cases}$$

Noting that $\tilde{J}_2 \circ L_i = L_i \circ \tilde{J}_2$ on ℓ_i^+ for $i = 1, \dots, N$, we extend $\tilde{J}_2$ to a homeomorphism $\tilde{J} : \Omega(G) \to \Omega(G)$ as the following:

$$\tilde{J} = L \circ \tilde{J}_2 \circ L^{-1} \qquad \text{on } L(\bar{W}) \text{ for all } L \in G.$$

By construction, $\tilde{J}$ satisfies the following conditions:

(a) $\tilde{J}(D) \cap D = \emptyset$ and $\tilde{J} = \mathrm{id}$ on $\partial D \cap \Omega(G)$,

(b) $\tilde{J} \circ \tilde{J} = \mathrm{id}_{\Omega(G)}$,

(c) $\tilde{J} \circ L = L \circ \tilde{J}$ for any $L \in G$.

Because of (c), $\tilde{J}$ descends to a homeomorphism $J: R \to R$ with $J \circ p = p \circ \tilde{J}$, which has the desired properties. Q.E.D.

In order to complete the proof of Theorem 2.1, we have only to show the following

6.12. Theorem. *Let G be a Schottky group of rank $N (\geq 0)$ and $p: \Omega(G) \to R := \Omega(G)/G$ the natural projection. Suppose that R_0 is a proper subdomain of R such that $D = p^{-1}(R_0)$ is a simply connected domain and that ∂R_0 consists of mutually disjoint quasi-analytic curves. Let $f: \mathbf{H} \to D$ be a Riemann mapping of D, Γ the Fuchsian group defined by $\Gamma = f^{-1}Gf$ and $\chi: \Gamma \to G$ the isomorphism defined by $\chi(\gamma) \circ f = f \circ \gamma$ for all $\gamma \in \Gamma$.*

Then f can be extended to a quasiconformal homeomorphism $\tilde{f}: \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ satisfying that $\chi(\gamma) \circ \tilde{f} = \tilde{f} \circ \gamma$. In particular, $D = \tilde{f}(\mathbf{H})$ is a quasidisk.

Proof. By Theorem 6.1, f can be extended to a homeomorphism $\tilde{f}_0: \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ satisfying that $\chi(\gamma) \circ \tilde{f}_0 = \tilde{f}_0 \circ \gamma$. Let F_0 denote a homeomorphism from $S = \Omega(\Gamma)/\Gamma$ onto R induced by $\tilde{f}_0: \Omega(\Gamma) \to \Omega(G)$. Since $F_0|_{S_0}: S_0 \to R_0$ is conformal, where $S_0 = \mathbf{H}/\Gamma$, and ∂R_0 consists of mutually disjoint quasi-analytic curves, $F_0|_{S_0}$ can be extended to a quasiconformal mapping F_1 on a neighborhood U of $\bar{S}_0$ such that F_1 is diffeomorphic in $U \setminus \bar{S}_0$. Here, we used the well-known fact that a quasiconformal map from the unit disk Δ onto a quasidisk can be extended to a quasiconformal self-map of the whole plane $\hat{\mathbf{C}}$ whose restriction to $\hat{\mathbf{C}} \setminus \bar{\Delta}$ is real-analytic (for example, by the Ahlfors-Weill extension). Then, it is easily seen that there exists a diffeomorphism $F: S \setminus \bar{S}_0 \to R \setminus \bar{R}_0$ which coincides with F_1 on some neighborhood of ∂S_0 and which is homotopic to $F_0|_{S \setminus \bar{S}_0}$ by a homotopy that fixes ∂S_0 pointwise.

We extend F to a homeomorphism from S onto R by defining $F = F_0$ on $\bar{S}_0$, then F becomes quasiconformal and $F \simeq F_0$ in S. Since F_0 can be lifted to $\tilde{f}_0$, F also can be lifted to a homeomorphism $\tilde{f}: \Omega(\Gamma) \to \Omega(G)$ such that $\chi(\gamma) \circ \tilde{f} = \tilde{f} \circ \gamma$ for all $\gamma \in \Gamma$. By Proposition 6.4, $\tilde{f}$ naturally extends to a homeomorphism of $\hat{\mathbf{C}}$, which is denoted also by $\tilde{f}$. Since $F: S \to R$ is quasiconformal, $\tilde{f}$ is also quasiconformal on $\Omega(\Gamma) \subset \hat{\mathbf{C}} \setminus \hat{\mathbf{R}}$. On the other hand, $\hat{\mathbf{R}}$ is a quasiconformally removable set, thus $\tilde{f}$ must be quasiconformal on the whole plane. The proof is finished. Q.E.D.

Proof of Theorem 2.1. Let $\varphi \in \mathrm{Int}\, S(\Gamma)$. Then $G = \chi^{\varphi}(\Gamma)$ is a Schottky group by Lemma 2.3 and Maskit's characterization theorem, and $R_0 := f^{\varphi}(\mathbf{H})/G$ is a proper subdomain of $R := \Omega(G)/G$ with quasi-analytic boundary by Corollary

5.2. Now Theorem 6.12 implies that f^{φ} can be extended to a Γ-compatible quasiconformal homeomorphism $\tilde{f}$ of $\hat{\mathbf{C}}$, which means that $\varphi \in T(\Gamma)$. Thus we have proved now that $\operatorname{Int} S(\Gamma) \subset T(\Gamma)$. Q.E.D.

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Added in proof:
The author recently learned from Professors T. Soma and K. Oshika that Lemma 6.8 directly follows from the theory of I-bundles (see J. Hempel, 3-manifolds, Ann. of Math. Stud., Princeton Univ. Press (1976)).