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Equivalence-singularity dichotomy for the Wiener measures on path groups and loop groups

By

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1. Introduction

In this paper, we will show a dichotomy between the equivalence and the singularity for the Wiener measure on path groups and loop groups under their group transformations. It is a natural extention of the following well-known fact for the abstract Wiener space: the Wiener measure is equivalent to its shifted measure by an element of Cameron-Martin subspace, and singular by other elements (see an excellent review [6].) As for path groups, the criterion for equivalence is known which we here call the finite energy condition (see Albeverio-Høegh-Krohn [1] and Shigekawa [8]). As for loop groups, the finite energy condition also gives a sufficient condition for the equivalence (Malliavin-Malliavin [7]). We will show that this condition also gives the criterion for equivalence in the case of loop groups. To show the dichotomy, we give a simple proof of the ergodicity for the Wiener measures on path groups and loop groups under the actions of finite energy paths and loops. This was first proved by Gross [3], by using the Ito-Wiener multiple integral expansion and of the support theorem. In this paper, we will give a proof based on the notion of quasi-homeomorphism and quasi-sure analysis.

The organization of this paper is as follows. In section 2, we show ergodicity on path groups. Section 3 is devoted to show the ergodicity on loop groups. In section 4, we show the equivalence-singularity dichotomy.

2. Ergodicity on path groups

In this section, we will prove the ergodicity on path groups under the action of finite energy paths. For the proof, we shall define the gradient operator which was first introduced by Gross [2]. Comparing this with H-derivative on the Wiener space, we will obtain ergodicity.

Let G be a d-dimensional compact connected Lie group and $\mathfrak{g}$ be its Lie algebra. Suppose we are given an $Ad(G)$ invariant inner product $(\cdot,\cdot)_{\mathfrak{g}}$. We shall fix an orthonormal basis $\{A_\alpha\}_{\alpha=1}^{d}$ of $\mathfrak{g}$ with respect to this inner

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product. For $g \in G$, $(Adg)^i_j$ denotes the components of $Adg\colon \mathfrak{g} \to \mathfrak{g}$ with respect to the basis $\{A_\alpha\}_{\alpha=1}^d$. Denote by PG the path group:

$$PG := \{\gamma \in C([0, 1] \to G);\ \gamma(0) = e\} \tag{2.1}$$

where e is the identity of G. We denote by L_k the left multiplication in PG by k. Let I be the unique strong solution to the following stochastic differential equation:

$$\begin{cases} d\gamma_t = \sum_{\alpha=1}^{d} A_\alpha(\gamma_t) \circ dB_t^\alpha \\ \gamma_0 = e, \end{cases} \tag{2.2}$$

where $\circ$ stands for the Fisk-Stratonovich symmetric integral.

Denote by W^d the d-dimensional Wiener space, i.e.

$$W^d := \{B \in C([0, 1] \to \mathbf{R}^d);\ B_0 = 0\} \tag{2.3}$$

with the standard Wiener measure m. Let μ be the image measure of m by I. I is a measure theoretical isomorphism from (W^d, m) to (PG, μ) ([8, Lemma 3.2]).

Let K be the set of paths of finite energy:

$$K := \left\{ k \in PG;\ \begin{matrix} k = (k_t) \text{ is absolutely continuous w.r.t. } t \\ \text{and } \int_0^1 |k_t^{-1} \dot{k}_t|_{\mathfrak{g}}^2\, dt < \infty \end{matrix} \right\} \tag{2.4}$$

We shall frequently use such matrix notation $k_t^{-1}\dot{k}_t$ for ease of reading instead of $(L_{k_t^{-1}})_* \dot{k}_t$. Let H be the set of $\mathfrak{g}$ valued finite energy paths:

$$H := \left\{ h \in C([0, 1] \to \mathfrak{g});\ \begin{matrix} h = (h_t) \text{ is absolutely continuous w.r.t. } t \\ \text{and } \int_0^1 |\dot{h}_t|_{\mathfrak{g}}^2\, dt < \infty,\ h_0 = 0 \end{matrix} \right\} \tag{2.5}$$

H is a Hilbert space with the inner product:

$$(a, b)_H := \int_0^1 (\dot{a}_t, \dot{b}_t)_{\mathfrak{g}}\, dt. \tag{2.6}$$

H is usually called the Cameron-Martin subspace. Let $C_{cyl}^\infty(PG)$ be the set of cylindrical functions:

$$C_{cyl}^\infty(PG) := \left\{ F\colon PG \to \mathbf{R};\ \begin{matrix} \text{There exists } f \in C^\infty(G^n) \text{ such that} \\ F(\gamma) = f(\gamma_{t_1}, \dots, \gamma_{t_n})\ n = 1, 2, \dots \end{matrix} \right\} \tag{2.7}$$

We denote by $\mathscr{D}$ the set of functions F on PG which satisfy following conditions.

(1) F is in $L^p(PG, \mu)$ for any $p > 1$.

(2) For any $h \in H$, $\varepsilon \mapsto F(e^{\varepsilon h}\gamma)$ is an $L^p(PG, \mu)$ valued differentiable function.

(3) The derivative $\frac{d}{d\varepsilon} F(e^{\varepsilon h}\gamma)|_{\varepsilon=0}$ is continuous in h. In other words, there exists $\nabla F \in \bigcap_{p>1} L^p(PG, \mu; H)$ such that $(\nabla F, h)_H = \frac{d}{d\varepsilon} F(e^{\varepsilon h}\gamma)|_{\varepsilon=0}$.

For later use, we note the following fact.

Proposition 2.1. *Let k be a path of finite energy. Then, the measures $\mu \circ L_k^{-1}$ and μ are equivalent and its Radon-Nikodym derivative*

$$J_k(\gamma) := \frac{d\mu \circ L_k^{-1}}{d\mu}(\gamma) \tag{2.8}$$
$$= \exp\left\{ \sum_{i,j=1}^{d} \int_0^1 (Ad\gamma_t^{-1})_j^i (k_t^{-1}\dot{k}_t)_i \, dB_t^j - \frac{1}{2}\int_0^1 |k_t^{-1}\dot{k}_t|_{\mathfrak{g}}^2 \, dt \right\}$$

is in $\mathscr{D}$, where $\gamma = I(B)$ and $k_t^{-1}\dot{k}_t = \sum_{i=1}^{d} (k_t^{-1}\dot{k}_t)_i A_\alpha^i$. Moreover, for any $h \in H$, the $L^p(PG, \mu)$ valued function $\varepsilon \mapsto J_{e^{\varepsilon h}}$ is differentiable and its derivative j_h is

$$j_h := \frac{d}{d\varepsilon} J_{e^{\varepsilon h}}|_{\varepsilon=0} = \sum_{i,j=1}^{d} \int_0^1 (Ad\gamma_t^{-1})_j^i (\dot{h}_t)_i \, dB_t^j.$$

Proof. See [7, Theorem 2.3.1] and [3, Corollary 3.6].

Following [3], we shall define the gradient operator. For this, we need the following.

Proposition 2.2. *The operator ∇ which sends $F \in \mathscr{D}$ to ∇F is a closable operator from $L^2(PG, \mu)$ to $L^2(PG, \mu; H)$.*

Proof. It is sufficient to show that the adjoint operator is densely defined. Let F, $\varphi \in \mathscr{D}$ and $h \in H$. Then,

$$(\nabla F, \varphi \cdot h)_{L^2(PG,\mu;H)} = \int (\nabla F(\gamma), h)_H \varphi(\gamma) \, d\mu \tag{2.9}$$
$$= \int \left(\frac{d}{d\varepsilon} F(e^{\varepsilon h}\gamma) \right)\Big|_{\varepsilon=0} \varphi(\gamma) \, d\mu$$
$$= \frac{d}{d\varepsilon} \int F(e^{\varepsilon h}\gamma)\varphi(\gamma) \, d\mu \Big|_{\varepsilon=0}$$
$$= \frac{d}{d\varepsilon} \int F(\gamma)\varphi(e^{-\varepsilon h}\gamma) J_{e^{\varepsilon h}}(\gamma) \, d\mu \Big|_{\varepsilon=0}$$
$$= -\int F(\gamma)(\nabla\varphi(\gamma), h)_H \, d\mu + \int F(\gamma)\varphi(\gamma) j_h(\gamma) \, d\mu.$$

This shows that

$$Dom(\nabla^*) \supset \mathscr{D} \otimes H, \quad \nabla^*(\phi \otimes h) = -(\nabla\phi, h)_H + j_h \phi. \tag{2.10}$$

The proof is complete.

We denote the closure of ∇ also by ∇ and call it the gradient. To show the ergodicity, we shall characterize K-invariant functions.

Definition 2.3. (1) *Let F be a bounded Borel measurable function on PG, and ν be a probability measure on PG. We call F a K-invariant function (with respect to ν) if for every $k \in K$,*

$$F(k\gamma) = F(\gamma) \quad \nu\text{-a.e.} \tag{2.11}$$

(2) *A probability measure ν on PG is called ergodic under the action of K if every K-invariant function is a constant function* ν-a.e.

Clearly, the K-invariant function with respect to μ is in $\mathscr{D}$, and

$$\nabla F = 0. \tag{2.12}$$

Let D be the H-derivative on W^d. Following [3, Theorem 3.14], the relation between D and ∇ is

$$F \in Dom(\nabla) \text{ if and only if } F \circ I \in Dom(D) \text{ and} \tag{2.13}$$

$$(\nabla F, \nabla F)_H = (D(F \circ I), D(F \circ I))_H. \tag{2.14}$$

(2.14) allows us to use some known facts on the Wiener space.

Theorem 2.4. *μ is ergodic under the action of K.*

Proof. Combining (2.12), (2.13) with (2.14), we obtain that, if F is K-invariant,

$$D(F \circ I) = 0. \tag{2.15}$$

It is well-known that this implies

$$F \circ I = \text{const. } m\text{-a.e.} \tag{2.16}$$

So, the proof is completed.

3. Ergodicity on Loop groups

In the previous section, we have shown ergodicity on path groups. In this section, we shall turn to loop groups. First, we review the quasi-sure analysis (see, e.g. [5]). Let X be a Polish space, and m be a Borel probability measure on X. Suppose that a strongly continuous contraction semigroup $(T_t)_{t>0}$ on $L^2(X, m)$ is given. We assume further that the semigroup is symmetric and Markovian. Then, by the interpolation theorem, $(T_t)_{t>0}$ can be defined on $L^p(X, m)$ as a strongly continuous contraction semigroup for $p \geq 1$. For $r > 0$, $p \geq 1$, set

$$V_r := \frac{1}{\Gamma\left(\frac{r}{2}\right)} \int_0^\infty t^{\frac{r}{2}-1} e^{-t} T_t dt$$

and define a Banach space $(\mathscr{F}_{r,p}, \|\cdot\|_{r,p})$ by

$$\mathscr{F}_{r,p} := V_r(L^p(X, m)) \text{ and}$$
$$\|u\|_{r,p} = \|f\|_p \quad \text{for} \quad u = V_r f,\ f \in L^p(X, m)$$

where $\|f\|_p$ denotes the L^p-norm of f. Then, the (r, p)-capacity $C_{r,p}$ is defined as follows: For an open set $G \subset X$,

$$C_{r,p}(G) := \inf\{\|u\|_{r,p}^p;\ u \in \mathscr{F}_{r,p},\ u \geq 1\ m\text{-a.e. on } G\}$$

and for an arbitrary set $B \subset X$,

$$C_{r,p}(B) := \inf\{C_{r,p}(G);\ G \text{ is open and } G \supset B\}$$

We note that if $\mathscr{F}_{r,p} \cap C_b(X)$ is dense in $\mathscr{F}_{r,p}$, we can take a quasi-continuous modification with respect to $C_{r,p}$ for any $f \in \mathscr{F}_{r,p}$. For $f \in \bigcap_{r,p} \mathscr{F}_{r,p}$, we denote by $\tilde{f}$ one of its quasi-continuous modification with respect to all $C_{r,p}$.

In the case of the path group, we take $L := -\nabla^* \nabla$ as the generator of $(T_t)_{t>0}$. (2.14) shows that L can be identified with the Ornstein-Uhlenbeck operator on W^d. In particular, the (r, p)-capacity associated with L is tight ([10]).

Let π be the map from PG to G defined by $\pi(\gamma) = \gamma(1)$. We denote by ΩG the based loop group $\pi^{-1}(e)$. Following [4, Chapter 5.9], we know that π is non-degenerate in the sense of Malliavin (see also [7], 3.1). So, we can take a family of measures $(\mu_g)_{g \in G}$ on PG supported on $(\pi^{-1}(g))_{g \in G}$ which satisfies, for any $f \in \bigcap_{r,p} \mathscr{F}_{r,p}$,

$$\int f(\gamma)\,\mathrm{d}\mu = \int p(g)\,\mathrm{d}g \int \tilde{f}(\gamma)\,\mathrm{d}\mu_g \tag{3.1}$$

$$\text{the function } g \mapsto \int \tilde{f}(\gamma)\,\mathrm{d}\mu_g \text{ is continuous} \tag{3.2}$$

where $\mathrm{d}g$ is the Haar measure on G and $p(g)\,\mathrm{d}g$ is the distribution of π. Now, we begin with a localization of quasi-invariance. This was shown in [7], but our statement is slightly different.

Lemma 3.1. *Let k be a path of finite energy. Then, μ_g and $\mu_{k(1)^{-1}g} \circ L_k^{-1}$ are equivalent.*

Proof. Let f be in $C_{cyl}^\infty(PG)$ and φ be in $C^\infty(G)$. Noting that $J_k \in \bigcap_{r,p} \mathscr{F}_{r,p}$, we obtain

$$\begin{aligned} \int f(k\gamma)\varphi \circ \pi(k\gamma)\mu(\mathrm{d}\gamma) &= \int f(\gamma)\varphi \circ \pi(\gamma) J_k(\gamma)\mu(\mathrm{d}\gamma) \\ &= \int \varphi(g)p(g)\,\mathrm{d}g \int f(\gamma)\tilde{J}_k(\gamma)\mu_g(\mathrm{d}\gamma). \end{aligned} \tag{3.3}$$

On the other hand,

$$\int f(k\gamma)\varphi \circ \pi(k\gamma)\mu(\mathrm{d}\gamma) = \int \varphi(k(1)g)p(g)\,\mathrm{d}g \int f(k\gamma)\mu_g(\mathrm{d}\gamma) \tag{3.4}$$

$$= \int \varphi(g) p(g) \frac{p(k(1)^{-1}g)}{p(g)} \mathrm{d}g \int f(k\gamma) \mu_{k(1)^{-1}g}(\mathrm{d}\gamma).$$

Since (3.3) equals to (3.4) for any $\varphi \in C^\infty(G)$, we obtain

$$\int f(\gamma) \tilde{J}_k(\gamma) \mu_g(\mathrm{d}\gamma) = \frac{p(k(1)^{-1}g)}{p(g)} \int f(k\gamma) \mu_{k(1)^{-1}g}(\mathrm{d}\gamma) \text{ for a.a. } g \in G. \tag{3.5}$$

By the continuity condition (3.2), (3.5) holds for all $g \in G$. This shows that

$$\mu_g \sim \mu_{k(1)^{-1}g} \circ L_k^{-1}. \tag{3.6}$$

The proof is complete.

To show ergodicity, we take a measurable map s from G to PG which satisfies

$$\pi \circ s = id_G \text{ and} \tag{3.7}$$

$$s(G) \subset K. \tag{3.8}$$

The existence of such s is clear from the local triviality of the fibre bundle (PG, π, G). For instance, in the neighborhood U of e, we can take as s the unique geodesic from e fo the point of U. We shall fix such s. Define the map $f: PG \to \Omega G$ by

$$f(\gamma) := s(\gamma(1))^{-1}\gamma.$$

Let A be a subset of ΩG, and we define $\tilde{A}$ as

$$\tilde{A} := \bigcup_{g \in G} s(g) A.$$

For ease of reading, we set

$$A_g := s(g) A$$
$$X_g := \pi^{-1}(g).$$

Lemma 3.2. *Let k be in PG and A and $\tilde{A}$ be as above. Then*

$$(k\tilde{A} \triangle \tilde{A}) \cap X_g = kA_{k(1)^{-1}g} \triangle A_g$$

where $\triangle$ denotes the symmetric difference of sets.

Proof. We first show that

$$(k\tilde{A})^c \cap X_g = (kA_{k(1)^{-1}g})^c \cap X_g. \tag{3.9}$$

In fact, the complement of left hand side is

$$((k\tilde{A})^c \cap X_g)^c = (k\tilde{A}) \cup (X_g)^c \tag{3.10}$$
$$= (\bigcup_h kA_h) \cup (\bigcup_{l \in G, l \neq g} X_l)$$

$$= (kA_{k(1)^{-1}g}) \cup (\bigcup_{l \in G, l \neq g} X_l)$$
$$= (kA_{k(1)^{-1}g}) \cup X_g^c.$$

In the same way, we obtain

$$\tilde{A}^c \cap X_g = A_g^c \cap X_g. \tag{3.11}$$

Combining (3.10) with (3.11), we have

$$\begin{aligned}(k\tilde{A} \cap \tilde{A})^c \cap X_g &= ((k\tilde{A})^c \cup \tilde{A}^c) \cap X_g \\ &= ((k\tilde{A})^c \cap X_g) \cup (\tilde{A}^c \cap X_g) \\ &= ((kA_{k(1)^{-1}g})^c \cap X_g) \cup (A_g^c \cap X_g) \\ &= ((kA_{k(1)^{-1}g})^c \cup A_g^c)) \cap X_g.\end{aligned} \tag{3.12}$$

Then, we have

$$\begin{aligned}(k\tilde{A} \triangle \tilde{A}) \cap X_g &= \{(k\tilde{A} \cup \tilde{A}) \cap (k\tilde{A} \cap \tilde{A})^c\} \cup X_g \\ &= \{(k\tilde{A} \cup \tilde{A}) \cap X_g\} \cap \{(k\tilde{A} \cap \tilde{A})^c \cap X_g\} \\ &= \{(k\tilde{A} \cup \tilde{A}) \cap X_g\} \cap \{(kA_{k(1)^{-1}g} \cap A_g)^c \cap X_g\} \\ &= \{(kA_{k(1)^{-1}g}) \cup A_g\} \cap \{(kA_{k(1)^{-1}g} \cap A_g)^c \cap X_g\} \\ &= (kA_{k(1)^{-1}g} \cup A_g) \cap (kA_{k(1)^{-1}g} \cap A_g)^c \\ &= kA_{k(1)^{-1}g} \triangle A_g.\end{aligned} \tag{3.13}$$

The proof is complete.

Now, we will show the ergodicity. We set $K_0 := K \cap \Omega G$.

Theorem 3.3. *μ_e is ergodic under the action of K_0.*

Proof. First, we note that, with the same notation in Lemma 3.3, if A is a K_0-invariant set with respect to μ_e, then $\tilde{A}$ is K-invariant with respect to μ. In fact, since μ_g is supported on X_g, we have for any $k \in K$,

$$\begin{aligned}\mu(k\tilde{A} \triangle \tilde{A}) &= \int \mu_g(k\tilde{A} \triangle \tilde{A}) p(g)\,\mathrm{d}g \\ &= \int \mu_g((k\tilde{A} \triangle \tilde{A}) \cap X_g) p(g)\,\mathrm{d}g \\ &= \int \mu_g(kA_{k(1)^{-1}g} \triangle A_g) p(g)\,\mathrm{d}g \\ &= \int \mu_g(ks(k(1)^{-1}g)A \triangle s(g)A) p(g)\,\mathrm{d}g \\ &= \int \mu_g(s(g)(s(g)^{-1}ks(k(1)^{-1}g)A \triangle A))\, p(g)\,\mathrm{d}g.\end{aligned} \tag{3.14}$$

Since $s(g) \in K$, by Lemma 3.1 and the asumption on A, we have

$$\mu(k\tilde{A} \triangle \tilde{A}) = 0. \tag{3.15}$$

Then by Theorem 2.4, we have

$$\mu(\tilde{A}) = 0 \text{ or } 1. \tag{3.16}$$

If we assume $\mu(\tilde{A}) = 0$,

$$\begin{aligned} 0 &= \mu(\tilde{A}) \\ &= \int \mu_g(\tilde{A}) p(g)\,\mathrm{d}g \\ &= \int \mu_g(\tilde{A} \cap X_g) p(g)\,\mathrm{d}g \\ &= \int \mu_g(A_g) p(g)\,\mathrm{d}g. \end{aligned} \tag{3.17}$$

So we obtain

$$\mu_g(A_g) = 0 \text{ for a.a. } g. \tag{3.18}$$

Again by Lemma 3.1, we have

$$\mu_e(A) = 0. \tag{3.19}$$

By the same argument, we obtain $\mu_e(A) = 1$ assuming $\mu(\tilde{A}) = 1$. The proof is complete.

4. Equivalence-Singularity dichotomy

In this section, we will show the equivalence-singularity dichotomy.

As for the equivalence on path groups, we know Proposition 2.1 in one direction, and the inverse was shown in [8]. We shall restate it as follows:

Theorem 4.1. *Let k be in PG. Then, $\mu \circ L_k^{-1}$ is equivalent to μ if and only if k is in K.*

As for loop groups, Lemma 3.1 corresponds to Proposition 2.1. So, we shall show the inverse of Lemma 3.1. For $0 \leq t \leq 1$, we denote by $\mathscr{F}_t$ the σ-field $\sigma[\gamma_s; s \leq t]$.

Theorem 4.2. *Let k be in ΩG. Then, $\mu_e \circ L_k^{-1}$ is equivalent to μ_e if and only if k is in K_0.*

Proof. We only have to show only if part.

Suppose that $\mu_e \circ L_k^{-1}$ and μ_e are equivalent. Since $\mu_e|_{\mathscr{F}_{\frac{1}{2}}} \sim \mu|_{\mathscr{F}_{\frac{1}{2}}}$, we obtain $\mu \circ L_k^{-1}|_{\mathscr{F}_{\frac{1}{2}}} \sim \mu|_{\mathscr{F}_{\frac{1}{2}}}$. By Theorem 4.1, we have that k is absolutely continuous on

$[0, \frac{1}{2}]$ and

$$\int_0^{\frac{1}{2}} |k_t^{-1} \dot{k}_t|_{\mathfrak{g}}^2 \, dt < \infty. \tag{4.1}$$

Since μ_e is invariant under the transformation $(T\gamma)_t = \gamma_{1-t}$, we obtain $\mu_e \sim \mu_e \circ L_k^{-1} \circ T^{-1}$. Noting that $TL_k = L_{T(k)}T$, we have

$$\mu_e \sim \mu_e \circ L_{T(k)}^{-1}. \tag{4.2}$$

So, we obtain by the same argument for k, k is absolutely continuous on $[\frac{1}{2}, 1]$ and

$$\int_{\frac{1}{2}}^1 |k_t^{-1} \dot{k}_t|_{\mathfrak{g}}^2 \, dt < \infty. \tag{4.3}$$

Combining (4.1) and (4.3), we have

$$\int_0^1 |k_t^{-1} \dot{k}_t|_{\mathfrak{g}}^2 \, dt < \infty. \tag{4.4}$$

Remark 4.3. *So far we have considered the left action. By the symmetry of the measures μ and μ_e under the transformation $\gamma \mapsto \gamma^{-1}$, we can replace left by right.*

Now, we recall a wel-known fact about ergodic measures. Let $(\Omega, \mathscr{B})$ denote a measurable space, and $(T_\alpha)_{\alpha \in A}$ denote a family of measurable isomorphisms on $(\Omega, \mathscr{B})$. Let P_1, P_2 be two measures on $(\Omega, \mathscr{B})$ which are quasi-invariant under $(T_\alpha)_{\alpha \in A}$.

Fact. *If P_1 and P_2 are both ergodic under $(T_\alpha)_{\alpha \in A}$, then P_1 and P_2 are either equivalent or singular.*

For the case that Ω is a vector space and $(T_\alpha)_{\alpha \in A}$ are shifts, the proof is given in [9]. One can show the general case with minor modifications. We shall apply this fact to μ and μ_e. Here, we restate our results.

Theorem 4.4. *Let μ be the Wiener measure on the path group PG. Then, for $k \in PG$,*

(1) *$\mu \circ L_k^{-1}$ is equivalent to μ if and only if $k \in K$.*

(2) *$\mu \circ L_k^{-1}$ is singular to μ if and only if $k \in K^c$.*

Proof. By Theorem 2.5 and Remark 4.3, we can apply the fact above to the case $P_1 = \mu$, $P_2 = \mu \circ L_k^{-1}$ and $(R_k)_{k \in K}$. So, $\mu \circ L_k^{-1}$ is either equivalent or singular to μ. We have already shown the criterion for equivalence in Theorem 4.1. The proof is complete.

For the loop group, we have, by the same proof, the following.

Theorem 4.5. *Let μ_e be the pinned Wiener measure on the based loop group*

ΩG. *Then, for* $k \in \Omega G$,

(1) $\mu_e \circ L_k^{-1}$ *is equivalent to* μ_e *if and only if* $k \in K_0$.

(2) $\mu_e \circ L_k^{-1}$ *is singular to* μ_e *if and only if* $k \in K_0^c$.

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