J. Math. Kyoto Univ. (JMKYAZ)
35-4 (1995) 631-638

The homotopy type of the space of rational functions

Dedicated Professor Seiya Sasao on his 60th birthday

By

M. A. GUEST, A. KOZLOWSKI, M. MURAYAMA and K. YAMAGUCHI

1. Introduction

For each positive integer d, let Hol_d denote the space of all holomorphic (equivalently, algebraic) maps of degree d from the Riemann sphere $S^2 = \mathbf{C} \cup \infty$ to itself. This space is of interest both from a classical and a modern point of view (see [1], [5]). Let Hol_d^* be the subspace of Hol_d consisting of maps which preserve a basepoint of S^2. It is well known that Hol_1 is the group of fractional linear transformations $\mathrm{PSL}_2(\mathbf{C})$ and that Hol_1^* may be identified with the affine transformation group of $\mathbf{C}$. It is an elementary fact that Hol_d and Hol_d^* are connected spaces. The fundamental groups of these spaces are $\mathbf{Z}/2d$, $\mathbf{Z}$ respectively; these computations are due to Epshtein ([6]) and Jones (see [8]). The following more general result was obtained by Segal:

Theorem 0 ([8]). *Let* Map_d *be the space of all continuous maps of degree* d *from* S^2 *to itself and let* Map_d^* *be the subspace consisting of maps* f *such that* $f(\infty) = 1$. *Then the natural inclusion maps induce the following isomorphisms of homotopy groups*:

(1) *If* $k < d$, *then* $\pi_k(\mathrm{Hol}_d^*) = \pi_k(\mathrm{Map}_d^*) = \pi_{k+2}(S^2)$.

(2) *If* $k < d$, *then* $\pi_k(\mathrm{Hol}_d) = \pi_k(\mathrm{Map}_d)$.

The stable homotopy type of Hol_d^* was studied in [3]. In this note we shall extend the above results by determining some further homotopy groups of the space Hol_d. Our results are as follows:

Theorem 1. (1) *For* $k \geq 2$,

$$\pi_k(\mathrm{Hol}_d) = \begin{cases} \pi_k(S^3) & d = 1 \\ \pi_k(S^3) \oplus \pi_k(S^2) & d = 2 \\ \mathbf{Z}/2 & d \geq 3,\ k = 2 \end{cases}$$

(2) *If* $k \geq 3$ *and* $d \geq 3$, *then* $\pi_k(\mathrm{Hol}_d) = \pi_k(\mathrm{Hol}_d^*) \oplus \pi_k(S^3)$.

(3) *In particular, if* $d > k \geq 3$, *then* $\pi_k(\mathrm{Hol}_d) = \pi_{k+2}(S^2) \oplus \pi_k(S^3)$.

Communicated by Prof. A. Kono, December 6, 1994

Theorem 2. *The space* Hol_2 *may be identified with a homogeneous space of the form* $(SL_2(\mathbf{C}) \times SL_2(\mathbf{C}))/H$, *where* H *is isomorphic to* $\mathbf{C}^* \rtimes \mathbf{Z}/4$. *In this semi-direct product, the action of* $\mathbf{Z}/4 = \langle \sigma : \sigma^4 = 1 \rangle$ *is given by* $\sigma \cdot \alpha = \alpha^{-1}$ *for* $\alpha \in \mathbf{C}^*$. *In particular,* Hol_2 *is homotopy equivalent to* $(S^3 \times S^3)/(S^1 \rtimes \mathbf{Z}/4)$.

Theorem 3. (1) *The universal cover of* Hol_2^* *is homotopy equivalent to* S^2.
(2) *The universal cover of* Hol_2 *may be identified with a homogeneous space of the form* $(SL_2(\mathbf{C}) \times SL_2(\mathbf{C}))/D$, *where* D *is isomorphic to* $\mathbf{C}^*$. *In particular, it is homotopy equivalent to* $S^3 \times S^2$.

In Theorem 1, the case $d = 1$ follows from the fact that Hol_1 may be identified with $\mathrm{PSL}_2(\mathbf{C})$ and hence is homotopy equivalent to $\mathbf{R}P^3$; the case $d = 2$ is direct consequence of (2) of Theorem 3.

In section 2, we shall consider the homogeneous structure of Hol_2 based on the action of $\mathrm{Hol}_1 \times \mathrm{Hol}_1$ by pre- and post-composition, and give the proof of Theorem 2 and (2) of Theorem 3. In section 3, we shall investigate the space Hol_2^*, and give the proof of (1) of Theorem 3. In section 4, we shall prove Theorem 1. In section 5 we shall give an application of these results to the C_2-operad structure on $\coprod_{d \geq 0} \mathrm{Hol}_d^*$. In particular, we show that the C_2-structure on $\coprod_{d \geq 0} \mathrm{Hol}_d^*$ is not compatible with that on $\Omega^2 S^2$ up to homotopy.

After completing this paper, we found that results similar to Theorems 2 and 3 are also contained in a preprint "Remarks on quadratic rational maps" of J. Milnor.

2. The Homogeneous Structure of Hol_2

From now on, we identify Hol_d with the space of functions $f = p_1/p_2$, where p_1, p_2 are coprime polynomials such that $\max\{\deg(p_1), \deg(p_2)\} = d$. The group Hol_1 acts on Hol_d by pre- and post-compositions: for $(\mathrm{A}, \mathrm{B}) \in \mathrm{Hol}_1 \times \mathrm{Hol}_1$ and $f \in \mathrm{Hol}_d$ we have

$$(\mathrm{A}, \mathrm{B}) \cdot f(\mathrm{z}) = \mathrm{A}(f(\mathrm{B}^{-1}(\mathrm{z}))).$$

The following proposition is well known, but we shall give a proof for the sake of completeness.

Proposition 2.1. *The group* $\mathrm{Hol}_1 \times \mathrm{Hol}_1$ *acts transitively on* Hol_2.

Proof. Let $f = \mathrm{p}/\mathrm{q} \in \mathrm{Hol}_2$. It suffices to show that $\mathrm{A}(f(\mathrm{B}(\mathrm{z}))) = z^2$ for some $\mathrm{A}, \mathrm{B} \in \mathrm{Hol}_1$. Since Hol_1 acts transitively on S^2, there is a function $\mathrm{A} \in \mathrm{Hol}_1$ such that $\mathrm{A}(f(\infty)) = \infty$. Hence, without loss of generality, we may suppose that $f(\infty) = \infty$, i.e. that $\deg(\mathrm{p}) = 2 > \deg(\mathrm{q})$.

Claim: If $f(\infty) = \infty$, then there is some $(\mathrm{A}, \mathrm{B}) \in \mathrm{Hol}_1^* \times \mathrm{Hol}_1^*$ such that

$$\mathrm{A}(f(\mathrm{B}(\mathrm{z}))) = \mathrm{z}^2 \quad \text{or} \quad (\mathrm{z} + \mathrm{z}^{-1})/2.$$

We shall prove this by considering separately the cases $\deg(q) = 0$, $\deg(q) = 1$.

(i) If $\deg(q) = 0$, we may suppose that $f(z) = p(z) = a(z + b)^2 + c$ for some $a \neq 0$, $b, c \in \mathbf{C}$. If we put $A(z) = a^{-1}(z - c)$, $B(z) = z - b$ then $A(f(B(z))) = z^2$, as required. (ii) If $\deg(q) = 1$, we may suppose that $q(z) = z + a$, and $p(z) = b\{(z + a)^2 + c(z + a) + d^2\}$ with $b \neq 0$, $d \neq 0$. Putting $A(z) = (z - bc)/2bd$, $B(z) = dz - a$ we see that $A(f(B(z))) = (z + z^{-1})/2$. This completes the proof of the claim.

Let $g(z) = (z + z^{-1})/2$. If $A(z) = (z + 1)/(z - 1)$, $B(z) = (z - 1)/(z + 1)$ then $A(g(B(z))) = z^2$. This completes the proof of Proposition 2.1.

Remark. It follows from the claim that Hol_2^* consists of two $\mathrm{Hol}_1^* \times \mathrm{Hol}_1^*$ orbits. It is well known that the map

$$SL_2(\mathbf{C}) \to \mathrm{Hol}_1, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{az + b}{cz + d},$$

is a double covering and induces an isomorphism $PSL_2(\mathbf{C}) \cong \mathrm{Hol}_1$ of Lie groups. Thus the group $G = SL_2(\mathbf{C}) \times SL_2(\mathbf{C})$ acts (transitively) on Hol_2.

Lemma 2.2. *Let H denote the isotropy subgroup of $SL_2(\mathbf{C}) \times SL_2(\mathbf{C})$ at $z^2 \in \mathrm{Hol}_2$. Then*

$$H = \left\{ \left(\pm \begin{pmatrix} \alpha^2 & 0 \\ 0 & \alpha^{-2} \end{pmatrix}, \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \right), \left(\pm \begin{pmatrix} 0 & i\alpha^2 \\ i\alpha^{-2} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \alpha \\ -a^{-1} & 0 \end{pmatrix} \right) : \alpha \in \mathbf{C}^* \right\}.$$

Proof. This follows by direct calculation.

Next we determine the group structure of H.

Lemma 2.3. *Let $K = \mathbf{C}^* \rtimes \mathbf{Z}/4$ be the group defined by the action of $\mathbf{Z}/4 = \langle \sigma : \sigma^4 = 1 \rangle$ on $\mathbf{C}^*$ by $\sigma \cdot \alpha = \alpha^{-1}$ for $\alpha \in \mathbf{C}^*$. Then H and K are isomorphic Lie groups.*

Proof. We put

$$\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Define $\varphi : K \to H$ by

$$(\alpha, \sigma^m) \mapsto \left(\begin{pmatrix} \alpha^2 & 0 \\ 0 & \alpha^{-2} \end{pmatrix} \sigma_1^m, \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \sigma_2^m \right).$$

It is easy to check that φ is an isomorphism.

Proof of Theorem 2. The first part of Theorem 2 follows from Proposition 2.1, Lemma 2.2 and Lemma 2.3. The inclusion map of the maximal compact subgroup $SU(2) = S^3$ of $SL_2(\mathbf{C})$ induces the homotopy equivalence $(S^3 \times S^3)/(S^1 \rtimes \mathbf{Z}/4) \simeq \mathrm{Hol}_2$.

Next we consider the universal cover of Hol_2. Let D be the subgroup

$$\left\{\left(\begin{pmatrix}\alpha^2 & 0\\ 0 & \alpha^{-2}\end{pmatrix}, \begin{pmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{pmatrix}\right): \alpha\in\mathbf{C}^*\right\}$$

of $SL_2(\mathbf{C}) \times SL_2(\mathbf{C})$, and let D_c be its maximal compact subgroup

$$\left\{\left(\begin{pmatrix}\alpha^2 & 0\\ 0 & \alpha^{-2}\end{pmatrix}, \begin{pmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{pmatrix}\right): |\alpha| = 1,\ \alpha\in\mathbf{C}\right\}.$$

Then D is a normal subgroup of $H \cong \mathbf{C}^* \rtimes \mathbf{Z}/4$ isomorphic to $\mathbf{C}^*$. Similarly D_c is is isomorphic to S^1. Clearly the projection $E = (SL_2(\mathbf{C}) \times SL_2(\mathbf{C}))/D \to (SL_2(\mathbf{C}) \times SL_2(\mathbf{C}))/H = \mathrm{Hol}_2$ is a regular covering. To show that E is simply connected, we shall consider the fibre bundle $SL_2(\mathbf{C}) \to E \to SL_2(\mathbf{C})/D$ whose projection map is induced by the projection onto the second factor.

Lemma 2.4. (1) *E is fibre homotopy equivalent to the fibre bundle* $SU(2) \to Y \to S^2$, *where* $Y = (SU(2) \times SU(2))/D_c$.

(2) *Y can be identified with the unit sphere bundle* $S(\eta^2 \oplus \eta^{-2})$, *where* η *denotes the Hopf complex line bundle over* $S^2 = \mathbf{C}P^1$.

Proof. (1) The inclusion map $SU(2) \times SU(2) \to SL_2(\mathbf{C}) \times SL_2(\mathbf{C})$ induces the desired fibre homotopy equivalence $Y \to E$.
(2) The second factor

$$S = \left\{\begin{pmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{pmatrix}: |\alpha| = 1,\ \alpha\in\mathbf{C}\right\}$$

of D_c is the standard embedding of S^1 into $SU(2)$, and $S^1 \to SU(2) \to SU(2)/S \approx S^2$ is the Hopf bundle. We may use the identifications $SU(2) = Sp(1) = S^3 \subset \mathbf{H}$, and extend the action of D_c naturally to $SU(2) \times \mathbf{H}$. By considering the transition functions of the vector bundle $(SU(2) \times \mathbf{H})/D_c$, it is not difficult to see that Y is equivalent to $S(\eta^2 \oplus \eta^{-2})$.

Proof of (2) *of Theorem* 3. It follows from the homotopy exact sequence that Y (and hence E) is simply connected. Hence E is the universal covering of Hol_2. Since $\pi_2(BU(2)) = \mathbf{Z}$, a 2-dimensional complex vector bundle ξ over S^2 is determined by its first Chern class $c_1(\xi)$. As $c_1(\eta^2 \oplus \eta^{-2}) = 0$, it follows that $\eta^2 \oplus \eta^{-2}$ is trivial. Hence $E \simeq Y \simeq S^3 \times S^2$.

3. The Space Hol_2^*

In this section we shall use the actions of Hol_1 (and Hol_1^*) on Hol_d by post-composition: $A \cdot f(z) = A(f(z))$ for $(A, f)\in \mathrm{Hol}_1 \times \mathrm{Hol}_d$. First, we have two easy lemmas:

Lemma 3.1. *Let $d \geq 1$. The group* Hol_1 *acts freely on* Hol_d *by post-composition. Similarly,* Hol_1^* *acts freely on* Hol_d^* *by post-composition.*

Proof. This follows immediately from the fact that any map of non-zero degree is surjective.

Lemma 3.2. *Let $d \geq 1$. Then the natural inclusion map* $\mathrm{j}_d\colon \mathrm{Hol}_d^* \to \mathrm{Hol}_d$ *induces a homeomorphism* $\tilde{\mathrm{j}}_d\colon \mathrm{Hol}_1^* \backslash \mathrm{Hol}_d^* \approx \mathrm{Hol}_1 \backslash \mathrm{Hol}_d$.

Proof. Since Hol_1 acts transitively on S^2, the induced map $\tilde{\mathrm{j}}_d\colon \mathrm{Hol}_1^* \backslash \mathrm{Hol}_d^* \to \mathrm{Hol}_1 \backslash \mathrm{Hol}_d$ is surjective. Since $(\mathrm{Hol}_1 \cdot f) \cap \mathrm{Hol}_d^* = \mathrm{Hol}_1^* \cdot f$ for any $f \in \mathrm{Hol}_d^*$, $\tilde{\mathrm{j}}_d$ is injective. If we identify these spaces by $\tilde{\mathrm{j}}_d$, it is easy to see that the topologies coincide.

Proposition 3.3 ([4]). *There is a fibration* $S^1 \to \mathrm{Hol}_2^* \to \mathbf{R}P^2$.

Remark. Cohen and Shimamoto ([4]) deduce this from results of Donaldson ([5]) and Atiyah and Hitchin ([1]) on monopoles. We shall give a direct and elementary proof.

Proof. By Lemma 3.1 we have a principal bundle

$$\mathrm{Hol}_1^* \to \mathrm{Hol}_2^* \to \mathrm{Hol}_1^* \backslash \mathrm{Hol}_2^*.$$

By Theorem 2 and Lemma 3.2, $\mathrm{Hol}_1^* \backslash \mathrm{Hol}_2^* \approx \mathrm{Hol}_1 \backslash \mathrm{Hol}_2 \simeq (\mathrm{S}^3/\mathrm{S}^1)/\{\pm 1\} \approx \mathrm{S}^2/\{\pm 1\} = \mathbf{R}P^2$. Since $\mathrm{Hol}_1^* \simeq \mathrm{S}^1$, we have the required fibration $\mathrm{S}^1 \to \mathrm{Hol}_2^* \to \mathbf{R}P^2$.

Proof of (1) *of Theorem* 3. Consider the above fibration $\mathrm{S}^1 \to \mathrm{Hol}_2^* \xrightarrow{\pi} \mathbf{R}P^2$. Let $p\colon \mathrm{S}^2 \to \mathbf{R}P^2$ and $q\colon \mathrm{X} \to \mathrm{Hol}_2^*$ be the universal coverings. Since X is simply connected, there is a lift $\theta\colon \mathrm{X} \to \mathrm{S}^2$ such that $p \circ \theta = \pi \circ q$. It follows by diagram chasing that $\theta_*\colon \pi_k(\mathrm{X}) \to \pi_k(\mathrm{S}^2)$ is an isomorphism for all k. Hence θ is a homotopy equivalence.

4. The Proof of Theorem 1

Let $\iota_n \in \pi_n(\mathrm{S}^n)$ be the oriented generator and $\eta_2 \in \pi_3(\mathrm{S}^2)$ be the class of the Hopf map. We put $\eta_n = \sum^{n-2} \eta_2 \in \pi_{n+1}(\mathrm{S}^n)$ for $n > 2$. The following three results are well known and we omit the proofs.

Lemma 4.1 ([9]). (1) $\pi_n(S^n) = \mathbf{Z}\{\iota_n\}$.
(2) $\pi_3(S^2) = \mathbf{Z}\{\eta_2\}$, $\pi_{n+1}(S^n) = \mathbf{Z}/2\{\eta_n\}$ *for* $n > 2$.
(3) $\pi_{n+2}(S^n) = \mathbf{Z}/2\{\eta_n^2\}$ *for* $n > 1$. *Here we put* $\eta_n^2 = \eta_n \circ \eta_{n+1}$.

Lemma 4.2 ([7], [9]). (1) $[\iota_2, \iota_2] = 2\eta_2$.
(2) *If* $k > 2$, $[\iota_2, \alpha] = 0$ *for any* $\alpha \in \pi_k(S^2)$.
Here [,] *denotes the Whitehead product.*

Let $\mathrm{Map}(\mathrm{S}^n, \mathrm{X})$ be the space of all continuous maps from S^n to X, and let

$\mathrm{Map}^*(\mathrm{S}^n, \mathrm{X})$ be the subspace consisting of based maps. For a map f we denote by $\mathrm{Map}_f(\mathrm{S}^n, \mathrm{X})$ or $\mathrm{Map}_f^*(\mathrm{S}^n, \mathrm{X})$ the path-component containing f.

Lemma 4.3 ([10]). *Let $f \in \mathrm{Map}^*(S^n, X)$ and let $\mathrm{Map}_f^*(S^n, X) \to \mathrm{Map}_f(S^n, X) \xrightarrow{ev} X$ be the evaluation fibration. If we use the identification $\pi_k(\mathrm{Map}_f^*(S^n, X)) = \pi_{k+n}(X)$, then the boundary operator $\partial: \pi_{k+n}(X) \to \pi_{k-1}(X)$ of the homotopy exact sequence associated with the evaluation fibration is given (up to sign) by the Whitehead product: $\partial(\alpha) = [\alpha, f]$.*

Proof of Theorem 1. (1) It suffices to consider the case $d \geq 3$. Let $I: \mathrm{Hol}_d^* \to \mathrm{Map}_d^*$ and $J: \mathrm{Hol}_d \to \mathrm{Map}_d$ be the inclusion maps. By Theorem 0, I and J are homotopy equivalences up to dimension d. Consider the following commutative diagram:

$$\begin{array}{ccccc}
\mathrm{Hol}_d^* & \longrightarrow & \mathrm{Hol}_d & \xrightarrow{ev} & S^2 \\
{\scriptstyle I}\downarrow & & {\scriptstyle J}\downarrow & & \downarrow{\scriptstyle =} \\
\mathrm{Map}_d^* & \longrightarrow & \mathrm{Map}_d & \xrightarrow[ev]{} & S^2
\end{array}$$

in which the horizontal sequences are evaluation fibre sequences. The result follows from the induced diagram of homotopy groups, by using Lemmas 4.1, 4.2, 4.3 and the Five Lemma. This completes the proof of (1).

(2) Suppose that $d \geq 3$ and $k \geq 3$ are integers. Consider the commutative diagram of principal bundles

$$\begin{array}{ccccc}
\mathrm{Hol}_1^* & \longrightarrow & \mathrm{Hol}_d^* & \xrightarrow{p_d} & \mathrm{Hol}_1^* \backslash \mathrm{Hol}_d^* \\
\downarrow & & {\scriptstyle j_d}\downarrow & & {\scriptstyle \tilde{j}_d}\downarrow{\scriptstyle \approx} \\
\mathrm{Hol}_1 & \longrightarrow & \mathrm{Hol}_d & \xrightarrow{q_d} & \mathrm{Hol}_1 \backslash \mathrm{Hol}_d
\end{array}$$

where j_d is a natural inclusion map. In the induced homotopy exact sequences, since $\mathrm{Hol}_1^* \simeq \mathrm{S}^1$, $(\mathrm{p}_d)_*: \pi_k(\mathrm{Hol}_d^*) \to \pi_k(\mathrm{Hol}_1^* \backslash \mathrm{Hol}_d^*)$ is an isomorphism for $k \geq 3$. Hence $(\mathrm{j}_d)_* \circ (\mathrm{p}_d)_*^{-1} \circ (\tilde{\mathrm{j}}_d)_*^{-1}: \pi_k(\mathrm{Hol}_1 \backslash \mathrm{Hol}_d) \to \pi_k(\mathrm{Hol}_d)$ gives a splitting of $(\mathrm{q}_d)_*$. So we have

$$\pi_k(\mathrm{Hol}_d) = \pi_k(\mathrm{Hol}_d^*) \oplus \pi_k(\mathrm{Hol}_1).$$

Because $\mathrm{Hol}_1 \simeq \mathbf{R}P^3$, $\pi_k(\mathrm{Hol}_1) = \pi_k(\mathrm{S}^3)$ and this completes the proof of (2).

(3) It follows from Theorem 0 that $\pi_k(\mathrm{Hol}_d^*) = \pi_{k+2}(\mathrm{S}^2)$ for $k < d$ and the assertion easily follows from (2).

Remark. The above method allows one to deduce the result of Epshtein ([6]) that $\pi_1(\mathrm{Hol}_d)$ is $\mathbf{Z}/2d$ from the result of Jones (see [8]) that $\pi_1(\mathrm{Hol}_d^*) = \mathbf{Z}$.

5. The C_2-operad structure on $\coprod_{d\geq 0}\mathrm{Hol}_d^*$

Consider $\coprod_{d\geq 0}\mathrm{Hol}_d^*$, the disjoint union of the based rational functions of degree d. It is known that this is a C_2-operad space ([2]). Let $\mu_d\colon F(\mathbf{C}, d)\times(\mathrm{S}^1)^d \to \mathrm{Hol}_d^*$ be the structure map, where we identify Hol_1^* up to homotopy with S^1. Let $i_d\colon \mathrm{Hol}_d^* \to \Omega_d^2\mathrm{S}^2$ and $i\colon \mathrm{Hol}^* \to \Omega^2\mathrm{S}^2$ be the inclusion maps. It is known that i is a C_2-map up to homotopy ([2]). It follows from the May-Milgram model of $\Omega^2\Sigma^2 X$ that we can identify $\Omega^2\mathrm{S}^3$ with $J(\mathrm{S}^1)$, where $J(\mathrm{X})$ denotes the space

$$J(\mathrm{X}) = \coprod_{d\geq 1}(\mathrm{F}(\mathbf{C}, d)\times_{\Sigma_d}\mathrm{X}^d)/\sim$$

and $\sim$ is a well known equivalence relation.

The following observation of Professor F. R. Cohen shows that the C_2-structure on $\Omega_0^2\mathrm{S}^2$ is incompatible with the one on $\coprod_{d\geq 0}\mathrm{Hol}_d^*$. (This contradicts the statement of [3] that diagram (3.4) of that paper is homotopy commutative.)

Let $J_d(\mathrm{X})$ be the d-th term of the May-Milgram filtration on $J(\mathrm{X})$.

Proposition 5.1. *There is no homotopy equivalence*

$$\theta\colon \mathrm{Map}_2^*(S^2, S^2) = \Omega_2^2 S^2 \to \Omega^2 S^3$$

such that the following diagram is homotopy commutative:

$$\begin{array}{ccccc}
F(\mathbf{C}, 2)\times_{\Sigma_2}(S^1)^2 & \xrightarrow{\mu_2} & \mathrm{Hol}_2^* & \xrightarrow{i_2} & \Omega_2^2 S^2 \\
\big\downarrow{\scriptstyle j_2} & & & & \big\downarrow{\scriptstyle \theta} \\
J_2(S^1) & \longrightarrow & J(S^1) & \xrightarrow{\simeq} & \Omega^2 S^3
\end{array}$$

Here j_2 denotes the inclusion map.

Proof. Suppose that the above diagram is homotopy commutative. Since $\mathrm{F}(\mathbf{C}, 2)\simeq S^1$, there is a non-zero element $e_1\otimes e_1^2\in \mathrm{H}_3(\mathrm{F}(\mathbf{C}, 2)\times_{\Sigma_2}(\mathrm{S}^1)^2, \mathbf{Z}/2) = \mathbf{Z}/2$ which represents a generator. Using the above diagram, if we put $\alpha = (\mu_2)_*(e_1\otimes e_1^2)$, we have

$$0 \neq \mathrm{Q}_1(e_1) = (\theta\circ i_2)_*(\alpha)\in \mathrm{H}_3(\Omega^2\mathrm{S}^3, \mathbf{Z}/2).$$

Since $\mathrm{Q}_1(e_1)\in \mathrm{H}_3(\Omega^2\mathrm{S}^3, \mathbf{Z}/2)$ is primitive, and θ is an equivalence, the element $(i_2)_*\alpha = \theta_*^{-1}(\mathrm{Q}_1(e_1))\in \mathrm{H}_3(\mathrm{Hol}_2^*, \mathbf{Z}/2) = \mathbf{Z}/2$ is also primitive (the image of a primitive element is primitive). By [8], $(i_2)_*$ is injective, so α is primitive. Thus the generator of $\mathrm{H}^3(\mathrm{Hol}_2^*, \mathbf{Z}/2) = \mathbf{Z}/2$ is indecomposable. On the other hand, because the universal cover of Hol_2^* is homotopy equivalent to S^2 and $\pi_1(\mathrm{Hol}_2^*) = \mathbf{Z}$, there is a fibration

$$\mathrm{S}^2 \to \mathrm{Hol}_2^* \to B\mathbf{Z} \simeq S^1.$$

Consider the mod 2 Serre spectral sequence of this fibration. This collapses at the E_2 level, and the generator of $H^3(\mathrm{Hol}_2^*, \mathbf{Z}/2) = \mathbf{Z}/2$ is decomposable. This is a contradiction.

Remark. The above result implies that the C_2-structure of Hol_d^* and that of $\Omega^2 S^3$ are not compatible (up to homotopy) at least when $d = 2$. This was also pointed out in [4].

Acknowlegements. The first, second and fourth authors are indebted to the Mathematics Department of Tokyo Institute of Technology for its hospitality. The application to C_2-operad structures is due to Professor F. R. Cohen. A gap in an earlier version of the proof of Theorem 3 was filled thanks to Professor A. Kono, who suggested the present version. The authors are grateful to Professors Cohen and Kono for their kind assistance.

Department of Mathematics,
University of Rochester,
Rochester, New York 14627,
USA

Department of Mathematics,
Toyama International University

Department of Mathematics,
Tokyo Institute of Technology

Department of Mathematics,
The University of Electro-Communications

References

[1] M. F. Atiyah and N. J. Hitchin The Geometry and Dynamics of Magnetic Monopoles, Princetton Univ. Press, 1988.
[2] C. P. Boyer and B. M. Mann, Monopoles, non-linear σ models and two-fold loop spaces, Commun. Math. Phys., **115** (1988), 571–594.
[3] F. R. Cohen, R. L. Cohen, B. M. Mann and R. J. Milgram, The topology of rational functions and divisors of surfaces, Acta Math., **166** (1991), 163–221.
[4] R. L. Cohen and D. H. Shimamoto, Rational functions, labelled configurations and Hilbert schemes, J. Lond. Math. Soc., **43** (1991), 509–528.
[5] S. K. Donaldson, Nahm's equations and the classification of monopoles, Commun. Math. Phys., **96** (1984), 387–407.
[6] S. I. Epshtein, Fundamental groups of spacs of coprime polynomials, Functional Analysis and its Applications, **7** (1973), 82–83.
[7] P. J. Hilton and J. H. C. Whitehead, Note on Whitehead product, Annals of Math., **58** (1953), 429–442.
[8] G. B. Segal, The topology of spaces of rational functions, Acta Math., **143** (1979), 39–72.
[9] H. Toda, Composition Methods in Homotopy Groups of Spheres, Vol. 49, Princeton Univ. Press, 1962.
[10] G. W. Whitehead, On products in homotopy groups, Annals of Math., **47** (1946), 460–475.