J. Math. Kyoto Univ. (JMKYAZ)
37-2 (1997) 251-259

The quasi KO_*-types of weighted projective spaces

By

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0. Introduction

Let KO and KU be the real and the complex K-spectrum respectively. For any CW-spectrum X its KU-homology KU_*X is regarded as a ($Z/2$-graded) abelian group with involution because the complex K-spectrum KU possesses the conjugation $\psi_C^{-1}: KU \to KU$. Given CW-spectra X and Y we say that X is quasi KO_*-equivalent to Y if there exists an equivalence $f: KO\wedge X \to KO \wedge Y$ of KO-module spectra (see [5]). If X is quasi KO_*-equivalent to Y, then KO_*X is isomorphic to KO_*Y as a KO_*-module, and in addition KU_*X is isomorphic to KU_*Y as an abelian group with involution ψ_C^{-1}.

Let $\tilde{P}^n = \tilde{P}^n(q_0, \cdots, q_n)$ be the (complex) weighted projective space of type $(q_0, \cdots, q_n)$, and $CP^n = \tilde{P}^n(1, \cdots, 1)$ the usual complex projective space. The KU-cohomology of $\tilde{P}^n$ has been computed in [1]. Our purpose here is to determine the quasi KO_*-type of $\tilde{P}^n$. In the special case $(q_0, \cdots, q_n) = (1, \cdots, 1)$, it is known that CP^n is quasi KO_*-equivalent to $\bigvee_m C_\eta$ or $\bigvee_m C_\eta \vee \Sigma^{2n}$ according as $n=2m$ or $2m+1$ where Σ^k is the k-dimensional sphere spectrum and C_η denotes the cofiber of the stable Hopf map $\eta: \Sigma^1 \to \Sigma^0$ (cf. [3, Theorem 2] and [5, Corollary 2.5]).

In §1 we recall some results about the KU-cohomology of a weighted projective space $\tilde{P}^n = \tilde{P}^n(q_0, \cdots, q_n)$ from [1]. In §2 we investigate the behaviour of the conjugation ψ_C^{-1} on $KU^*\tilde{P}^n$ in order to determine the quasi KO_*-type of $\tilde{P}^n$. In §3 we describe generators of the KO-cohomology group $KO^*\tilde{P}^n$.

1. A weighted projective space $\tilde{P}^n$

Let n be a positive integer and $(q_0, \cdots, q_n)$ a tuple of positive integers. Consider the following operation of the multiplicative group $C^* = C\setminus\{0\}$ on

Communicated by Prof. A. Kono, July 12, 1996

the space $(C^{n+1})^*=C^{n+1}\backslash\{0\}$:

$$\lambda \cdot (x_0, \cdots, x_n)=(\lambda^{q_0}x_0, \cdots, \lambda^{q_n}x_n). \tag{1}$$

The associated topological quotient space is called the (complex) weighted projective space of type $(q_0, \cdots, q_n)$ and it is denoted by

$$\tilde{P}^n=\tilde{P}^n(q_0, \cdots, q_n).$$

Of course, $\tilde{P}^n(1, \cdots, 1)$ is the usual complex projective space CP^n.

We call a tuple $(q_0, \cdots, q_n)$ "well-ordered" if q_i divides q_{i-1} for all $i(1\leq i\leq n)$. Given a tuple $(q_0, \cdots, q_n)$ the integer $l_k=l_k(q_0, \cdots, q_n)$ for $0\leq k\leq n$ is defined as the least common multiple (lcm) of all integers

$$\Big(\prod_{0\leq\alpha\leq k} q_{i_\alpha}\Big)/\gcd\{q_{i_\alpha} \mid 0\leq\alpha\leq k\}, \quad \text{with} \quad 0\leq i_0<i_1<\cdots<i_k\leq n$$

where gcd stands for the greatest common divisor. Using the integer $l_k=l_k(q_0, \cdots, q_n)$ we define integers

$$\bar{q}_i=l_{i+1}/l_i \quad (0\leq i\leq n-1), \quad \bar{q}_n=\gcd\{q_0, \cdots, q_n\}. \tag{2}$$

Then the tuple $(\bar{q}_0, \cdots, \bar{q}_n)$ is well-ordered and $l_k(\bar{q}_0, \cdots, \bar{q}_n)=l_k(q_0, \cdots, q_n)$ for all $k(0\leq k\leq n)$ (cf.[1, 4.10]). Suppose that $\nu_p(q_0)\geq\nu_p(q_1)\geq\cdots\geq\nu_p(q_n)$ for a prime p where ν_p is the p-valuation. Then the p-valuation of l_k is $\nu_p(l_k)=\nu_p(q_0)+\nu_p(q_1)+\cdots+\nu_p(q_{k-1})$. Therefore $l_k=q_0q_1\cdots q_{k-1}$, and hence $\bar{q}_k=q_k$, if $(q_0, \cdots, q_n)$ is well-ordered.

Denote by γ the canonical line bundle over CP^n and set $a=[\gamma]-1\in KU^0CP^n$. Then it is well known that the (reduced) KU-cohomology group $KU^*CP^n_+\cong Z[a]/(a^{n+1})$ where CP^n_+ is the disjoint union of CP^n and a point. Consider the map $\varphi=\varphi(q_0, \cdots, q_n): CP^n\to\tilde{P}^n$ defined by $\varphi[x_0, \cdots, x_n]=[x_0^{q_0}, \cdots, x_n^{q_n}]$. According to [1, Theorem 3.4] the map φ induces a monomorphism $\varphi^*: KU^*\tilde{P}^n\to KU^*CP^n$ and there exists a Z-basis $\{T_1, \cdots, T_n\}$ of $KU^*\tilde{P}^n$ such that

$$\varphi^*(T_i)=Q_{\bar{q}_0}(a)Q_{\bar{q}_1}(a)\cdots Q_{\bar{q}_{i-1}}(a) \tag{3}$$

where $Q_k(a)=(1+a)^k-1$. Thus $KU^0\tilde{P}^n$ is a free abelian group of rank n and $KU^1\tilde{P}^n=0$.

The group Z/q of the q th roots of the unity in C^* acts on the space $(C^{n+1})^*$ as in (1). The quotient space denoted by $\tilde{L}^n=\tilde{L}^n(q\,; q_0, \cdots, q_n)$ is called the weighted lens space of type $(q\,; q_0, \cdots, q_n)$. For the natural surjection $\theta:\tilde{L}^n\to\tilde{P}^n$ and the canonical inclusion $i_{n+1}:\tilde{P}^n\to\tilde{P}^{n+1}$ the following sequence is a cofibering (cf. [4, Assertion. 1]) :

$$\tilde{L}^n(q\,; q_0, \cdots, q_n)\xrightarrow{\theta}\tilde{P}^n(q_0, \cdots, q_n)\xrightarrow{i_{n+1}}\tilde{P}^{n+1}(q_0, \cdots, q_n, q). \tag{4}$$

In particular, if q divides all $q_i(0\leq i\leq n)$ then Z/q acts trivially on the space $(C^{n+1})^*$ and hence $\tilde{L}^n=(C^{n+1})^*$ is (homotopy) equivalent to Σ^{2n+1}.

2. The quasi KO_*-type of $\tilde{P}^n$

In order to determine the quasi KO_*-type of $\tilde{P}^n$ we shall use the following theorem (cf. [2, Theorem 3.2] or [5, Theorem 2.4]).

Theorem 2.1. *Let X be a CW-spectrum such that KU^0X is a free abelian group and $KU^1X=0$. Then X is quasi KO_*-equivalent to a certain wedge sum of copies of C_η and $\Sigma^{2s}(0\leq s\leq 3)$ where C_η denotes the cofiber of the stable Hopf map $\eta: \Sigma^1\to\Sigma^0$.*

Remark. The conjugation map ψ_C^{-1} acts on KU^0X as follows:

$$\psi_C^{-1}=\begin{cases} 1 & \text{when } X=\Sigma^0 \text{ or } \Sigma^4, \\ -1 & \text{when } X=\Sigma^2 \text{ or } \Sigma^6, \\ \begin{pmatrix} -1 & -1 \\ 0 & 1\end{pmatrix} \text{ (denoted by } \rho) & \text{when } X=C_\eta. \end{cases} \tag{5}$$

The following lemma asserts that for our purpose any tuples $(q_0, \cdots, q_n)$ may be restricted to well-ordered ones.

Lemma 2.2. *$\tilde{P}^n(q_0, \cdots, q_n)$ is quasi KO_*-equivalent to $\tilde{P}^n(\bar{q}_0, \cdots, \bar{q}_n)$.*

Proof. The map associated with any permutation

$$f: \tilde{P}^n(q_0, \cdots, q_n)\to\tilde{P}^n(q_{i_0}, \cdots, q_{i_n})$$

is clearly a homeomorphism. Therefore we may assume that $\nu_2(q_0)\geq\nu_2(q_1)\geq\cdots\geq\nu_2(q_n)$ where ν_2 is the 2-valuation. It is easily seen that $\nu_2(\bar{q}_i)=\nu_2(q_i)$ for all $i(0\leq i\leq n)$. We put $\xi_i=l\bar{q}_i/q_i$ and $\bar{\xi}_i=lq_i/\bar{q}_i(0\leq i\leq n)$ with $l=\mathrm{lcm}\{q_0, \cdots, q_n\}/2^{\nu_2(q_0)}$ and consider the following two maps between $\tilde{P}^n=\tilde{P}^n(q_0, \cdots, q_n)$ and $\bar{P}^n=\tilde{P}^n(\bar{q}_0, \cdots, \bar{q}_n)$:

$$g: \tilde{P}^n\to\bar{P}^n,\ (x_0, \cdots, x_n)\mapsto(x_0^{\xi_0}, \cdots, x_n^{\xi_n}),$$
$$h: \bar{P}^n\to\tilde{P}^n,\ (x_0, \cdots, x_n)\mapsto(x_0^{\bar{\xi}_0}, \cdots, x_n^{\bar{\xi}_n}).$$

According to Theorem 2.1 $\tilde{P}^n$ and $\bar{P}^n$ are quasi KO_*-equivalent to certain wedge sums Y_n and $\bar{Y}_n$ of C_η and $\Sigma^{2s}(0\leq s\leq 3)$ respectively. Since gh, hg: $(x_i)\mapsto(x_i^{l^2})$ and l is odd, $g: \tilde{P}^n\to\bar{P}^n$ is a 2-equivalence. Therefore Y_n must be coincide with $\bar{Y}_n$.

Given CW-spectra X and Y we say that X has the same $\mathscr{C}$-type as Y if KU_*X is isomorphic to KU_*Y as an abelian group with involution ψ_C^{-1} (cf. [2, 4.1]).

Proposition 2.3. *Let $(q_0, \cdots, q_n)$ be a well-ordered tuple and put $c_k=(q_0+\cdots+q_{k-1})/q_k$ for $1\leq k\leq n-1$. Then the weighted projective space $\tilde{P}^n=\tilde{P}^n(q_0, \cdots, q_n)$ has the same $\mathscr{C}$-type as the following cell complex*

$$Y_n=\Sigma^2\underset{\theta_1}{\cup}e^4\underset{\theta_2}{\cup}e^6\underset{\theta_3}{\cup}\cdots\underset{\theta_{n-1}}{\cup}e^{2n}$$

where $\theta_k=\eta$ if c_k is odd, and $\theta_k=0$ if c_k is even.

Remark. Note that $\theta_k=\eta$ if $\theta_{k-1}=0$ and q_{k-1}/q_k is odd, and $\theta_k=0$ otherwise. Let $S=\{1\leq s\leq n \mid c_{s-1}$ and c_s are even$\}$ and $T=\{1\leq t\leq n-1 \mid c_t$ is odd$\}$ where we understand that c_0 and c_n are even. Then Y_n is just the wedge sum $\vee_{t\in T}\Sigma^{2t}C_\eta\vee\vee_{s\in S}\Sigma^{2s}$.

We shall prove Proposition 2.3 by induction on n below, but first show how to obtain the quasi KO_*-type of $\tilde{P}^n$ applying Proposition 2.3.

Theorem 2.4. *Let $(q_0, \cdots, q_n)$ be a well-ordered tuple of positive integers. Then the weighted projective space $\tilde{P}^n=\tilde{P}^n(q_0, \cdots, q_n)$ is quasi KO_*-equivalent to the wedge sum $Y_n=\vee_{t\in T}\Sigma^{2t}C_\eta\vee\vee_{s\in S}\Sigma^{2s}$ given in the above remark.*

Proof. In order to prove our theorem by induction on n we consider the following diagram

$$\begin{array}{ccccc} \Sigma^{2n+1}KO & \xrightarrow{1\wedge\theta} & KO\wedge\tilde{P}^n & \xrightarrow{1\wedge i_{n+1}} & KO\wedge\tilde{P}^{n+1} \\ \uparrow\iota & & \uparrow f & & \\ \Sigma^{2n+1} & \xrightarrow{\theta_n} & Y_n & \longrightarrow & Y_{n+1} \end{array}$$

Here $f: Y_n\to KO\wedge\tilde{P}^n$ is a quasi KO_*-equivalence and $\iota: \Sigma^0\to KO$ is the unit of KO. For each component of Y_n we have

$$KO_{2n+1}\Sigma^{2s}\cong\begin{cases} Z/2(\text{generated by } \eta) & \text{if } n\equiv s \bmod 4 \\ 0 & \text{if otherwise,} \end{cases}$$
$$KO_{2n+1}C_\eta=0.$$

Since $\tilde{P}^{n+1}$ and Y_{n+1} have the same $\mathscr{C}$-type, we observe that the map $\iota\wedge\theta$: $\Sigma^{2n+1}\to KO\wedge\tilde{P}^n$ is trivial if and only if $\theta_n=0$. Therefore the square in the above diagram becomes commutative after changing the quasi KO_*-equivalence $f: Y_n\to KO\wedge\tilde{P}^n$ suitably if necessary. So there exists a quasi KO_*-equivalence $g: Y_{n+1}\to KO\wedge\tilde{P}^{n+1}$ as desired.

Let $(q_0, \cdots, q_{n-1}, q_n, q_{n+1})$ be a tuple such that $(q_0, \cdots, q_n)$ is well-ordered and q_{n+1} divides q_{n-1}. Since $l_k(q_0, \cdots, q_n)=l_k(q_0, \cdots, q_n, q_{n+1})$ for $0\leq k\leq n$ it follows that $i_{n+1}^*T_i=T_i(1\leq i\leq n)$ and $i_{n+1}^*T_{n+1}=0$ for the canonical inclusion

$i_{n+1}: \tilde{P}^n(q_0, \cdots, q_n) \to \tilde{P}^{n+1}(q_0, \cdots, q_n, q_{n+1})$ where T_i's are the generators of $KU^0\tilde{P}^{n+1}$ and $KU^0\tilde{P}^n$ given in (3). In order to prove Proposition 2.3 by induction on n we may assume that the conjugation ψ_C^{-1} on $KU^0\tilde{P}^n \cong KU^0 Y_n$ is expressed as a certain direct sum J_n of ± 1 and $\pm\rho$ after the basis $\{T_1, \cdots, T_n\}$ is replaced by $\{T_1', \cdots, T_n'\}$ where $T_i' = T_i +$ a linear combination of $\{T_{i+1}, \cdots, T_n\}$. Then the conjugation ψ_C^{-1} on $KU^0\tilde{P}^{n+1}$ behaves as $\psi_C^{-1} T_i' = J_n T_i' + \gamma_i T_{n+1}$ for some integer $\gamma_i (1 \leq i \leq n)$ and $\psi_C^{-1} T_{n+1} = (-1)^{n+1} T_{n+1}$.

Lemma 2.5. *Let $(q_0, \cdots, q_{n-1}, q_n, q_{n+1})$ be a tuple such that $(q_0, \cdots, q_n)$ is well-ordered and q_{n+1} divides q_{n-1}. When the conjugation ψ_C^{-1} behaves as $\psi_C^{-1} T_i' = J_n T_i' + \gamma_i T_{n+1}$ on $KU^0\tilde{P}^{n+1}$, $q\gamma_i$ is divisible by q_{n-1} for any i $(1 \leq i \leq n)$ and in particular $q\gamma_n = (-1)^{n+1}(q_0 + \cdots + q_{n-1})$ where $q = \operatorname{lcm}\{q_n, q_{n+1}\}$.*

Proof. Consider $\bar{P}^{n+1} = \tilde{P}^{n+1}(q_0, \cdots, q_{n-1}, q_{n-1}, q_{n-1})$ as well as $\tilde{P}^{n+1} = \tilde{P}^{n+1}(q_0, \cdots, q_{n-1}, q_n, q_{n+1})$. Recall that $\{T_1', \cdots, T_n', T_{n+1}\}$ and $\{\bar{T}_1', \cdots, \bar{T}_n', \bar{T}_{n+1}\}$ form base of $KU^0\tilde{P}^{n+1}$ and $KU^0\bar{P}^{n+1}$ respectively where $\varphi^* T_{n+1} = q_0 \cdots q_{n-1} q a^{n+1}$ and $\varphi^* \bar{T}_{n+1} = q_0 \cdots q_{n-1} q_{n-1} a^{n+1}$. Since the conjugation ψ_C^{-1} on $KU^0\bar{P}^{n+1}$ behaves as $\psi_C^{-1} \bar{T}_i' = J_n \bar{T}_i' + \zeta_i \bar{T}_{n+1}$ for some integer ζ_i it follows immediately that $q\gamma_i = q_{n-1}\zeta_i$ for any $i (1 \leq i \leq n)$. In the special case $i = n (T_n' = T_n)$ we have

$$\varphi^* T_n = l_n a^n + L a^{n+1}, \quad \varphi^* T_{n+1} = l_{n+1} a^{n+1}$$

where $l_n = q_0 \cdots q_{n-1}$, $l_{n+1} = q_0 \cdots q_{n-1} q_n$ and $L = l_n (q_0 + \cdots + q_{n-1} - n)/2$. Note that $\psi_C^{-1} a = (1+a)^{-1} - 1$. This implies that

$$\psi_C^{-1} a^n = (-1)^n a^n + (-1)^{n+1} n a^{n+1}, \quad \psi_C^{-1} a^{n+1} = (-1)^{n+1} a^{n+1}.$$

Since $\psi_C^{-1} \varphi^* = \varphi^* \psi_C^{-1}$, we see that $\gamma_n = (-1)^{n+1}(q_0 + \cdots + q_{n-1})/q$.

Proof of Proposition 2.3. By induction on n we shall show that the conjugation ψ_C^{-1} on $KU^0\tilde{P}^{n+1}$ is normalized as a desired direct sum J_{n+1} of ± 1 and $\pm\rho$ after the basis $\{T_1', \cdots, T_n', T_{n+1}\}$ is replaced by $\{T_1'', \cdots, T_n'', T_{n+1}\}$ where $T_i'' = T_i' + \delta_i T_{n+1}$ for some integer $\delta_i (1 \leq i \leq n)$. Set $\alpha_i = \nu_2(q_i)$ for simplicity.

i) The "$\theta_{n-1} = 0$" case: In this case $\tilde{P}^n$ has the same $\mathscr{C}$-type as $Y_n = Y_{n-1} \vee \Sigma^{2n}$. Therefore the conjugation ψ_C^{-1} on $KU^0\tilde{P}^n$ is $J_n = J_{n-1} \oplus (-1)^n$. Then the conjugation ψ_C^{-1} on $KU^0\tilde{P}^{n+1}$ behaves as follows:

$$\psi_C^{-1} T_i' = J_{n-1} T_i' + \gamma_i T_{n+1} \quad \text{if} \quad 1 \leq i \leq n-1,$$
$$\psi_C^{-1} T_n' = (-1)^n T_n' + \gamma_n T_{n+1} \quad \text{and} \quad \psi_C^{-1} T_{n+1} = (-1)^{n+1} T_{n+1}.$$

We first assume that $\alpha_{n-2} > \alpha_{n-1}$. If $\alpha_{n-1} = \alpha_n$, then $(q_0 + \cdots + q_{n-1})/q_n$ is odd, and hence, by Lemma 2.5, so is γ_n. Hence the conjugation ψ_C^{-1} is congruent to $J_{n-1} \oplus (-1)^n \rho$ and $Y_{n+1} = Y_{n-1} \vee \Sigma^{2n} C_\eta$. If $\alpha_{n-1} > \alpha_n$, then all of γ_i are even from Lemma 2.5. Therefore the conjugation ψ_C^{-1} is congruent to $J_n \oplus (-1)^{n+1}$ and

$Y_{n+1}=Y_n\vee\Sigma^{2n+2}$. We next assume that $\alpha_{n-2}=\alpha_{n-1}$, and hence $\theta_{n-2}=\eta$. Then there exists an odd integer $m\geq 3$ such that $\alpha_{n-m-1}>\alpha_{n-m}=\cdots=\alpha_{n-2}=\alpha_{n-1}$. From Lemma 2.5 it follows that γ_n is even if and only if $\alpha_{n-1}>\alpha_n$. Now our result is shown similarly to the first case "$\alpha_{n-2}>\alpha_{n-1}$".

ii) The "$\theta_{n-1}=\eta$" case: In this case $\tilde{P}^n$ has the same $\mathscr{C}$-type as $Y_n=Y_{n-2}\vee\Sigma^{2n-2}C_\eta$ and $\alpha_{n-2}=\alpha_{n-1}$. We first assume that $\alpha_{n-1}>\alpha_n$. Then $Y_{n+1}=Y_n\vee\Sigma^{2n+2}$ as is shown in the "$\theta_{n-1}=0$" case. We next assume that $\alpha_{n-1}=\alpha_n$. Then there exists an even integer $m\geq 2$ such that $\alpha_{n-m-1}>\alpha_{n-m}=\cdots=\alpha_{n-1}=\alpha_n$. From induction hypothesis the conjugation ψ_C^{-1} on $KU^0\tilde{P}^n$ is $J_n=J_{n-m}\oplus(-1)^{n+1}(\rho\oplus\cdots\oplus\rho)$. Then the conjugation ψ_C^{-1} on $KU^0\tilde{P}^{n+1}$ behaves as follows:

$$\psi_C^{-1}T_i'=(-1)^{n+1}T_i'+\gamma_iT_{n+1},\qquad \psi_C^{-1}T_j'=(-1)^nT_j'+\gamma_jT_{n+1},$$
$$\psi_C^{-1}T_h'=\mp(T_h'+T_{h+1}')+\gamma_hT_{n+1},\quad \psi_C^{-1}T_{h+1}'=\pm T_{h+1}'+\gamma_{h+1}T_{n+1}.$$

for some $i, j\geq n-m$ and $h\leq n-1$. For $\psi_C^{-1}=\begin{pmatrix}(-1)^{n+1} & \gamma_i\\ 0 & (-1)^{n+1}\end{pmatrix}$ on the $(i, n+1)$-th component we have $\gamma_i=0$ because $(\psi_C^{-1})^2=1$. On the $(h, h+1, n+1)$-th component $\psi_C^{-1}=\begin{pmatrix}\pm\rho & \gamma\\ 0 & (-1)^{n+1}\end{pmatrix}$ is congruent to $(\pm\rho)\oplus(-1)^{n+1}$ where $\gamma=\begin{pmatrix}\gamma_h\\ \gamma_{h+1}\end{pmatrix}$. We here consider $\tilde{P}^{n+2}(q_0, \cdots, q_n, q_{n+1}, q_{n+2})$ with $q_{n+2}=q_n$. Then the conjugation ψ_C^{-1} on the $(j, n+1, n+2)$-th component is expressed as

$$\psi_C^{-1}=\begin{pmatrix}(-1)^n & \gamma_j & \zeta_j\\ 0 & (-1)^{n+1} & \zeta_{n+1}\\ 0 & 0 & (-1)^n\end{pmatrix}$$

for some integer ζ_j and $\zeta_{n+1}=(-1)^n(q_0+\cdots+q_n)/q_n$ by Lemma 2.5. Note that $\gamma_j\zeta_{n+1}=(-1)^n2\zeta_j$ because $(\psi_C^{-1})^2=1$. This equality implies that γ_j must be even since ζ_{n+1} is odd. Therefore $\psi_C^{-1}=\begin{pmatrix}(-1)^n & \gamma_j\\ 0 & (-1)^{n+1}\end{pmatrix}$ on the $(j, n+1)$-component is congruent to $(-1)^n\oplus(-1)^{n+1}$. Consequently we see that the conjugation ψ_C^{-1} is congruent to $J_n\oplus(-1)^{n+1}$ and $Y_{n+1}=Y_n\vee\Sigma^{2n+2}$.

3. The group $KO^*\tilde{P}^n$

We have the Bott cofiber sequence

$$\Sigma^1KO\xrightarrow{\eta\wedge 1}KO\xrightarrow{\varepsilon_U}KU\xrightarrow{\varepsilon_O\beta}\Sigma^2KO$$

where $\beta: KU\to\Sigma^2KU$ denotes the inverse of the Bott periodicity and ε_U: KO $\to$KU is the complexification and ε_O: KU$\to$KO is the realification. As is well known, the equalities $\varepsilon_O\varepsilon_U=2$ and $\varepsilon_U\varepsilon_O=1+\psi_C^{-1}$ hold. Let $\eta\in\pi_1KO\cong Z/2$, $\eta^2\in\pi_2KO\cong Z/2$ and $\xi\in\pi_4KO\cong Z$ be the generators such that $\xi^2=4B_R\in\pi_8KO\cong\pi_0KO\cong Z$ where B_R denotes the Bott periodicity element. Hereafter we shall drop B_R writing $\xi^2=4$ instead of $\xi^2=4B_R$. Let $(q_0, \cdots, q_n)$ be a well-ordered tuple of positive integers and $\tilde{P}^n=\tilde{P}^n(q_0, \cdots, q_n)$. Recall that $KU^0\tilde{P}^n$ is a free abelian group with basis $\{T_1, \cdots, T_n\}$ and $KU^1\tilde{P}^n=0$. By a routine computation we can obtain

Lemma 3.1. *Operating* $\varepsilon_U\varepsilon_O=1+\psi_C^{-1}$ *on* $KU^{ev}\tilde{P}^n$ *we have the following equalities*: $(1+\psi_C^{-1})\beta^nT_n=2\beta_nT_n$, $(1+\psi_C^{-1})\beta^nT_{n-1}=c_{n-1}\beta^nT_n$, $(1+\psi_C^{-1})\beta^{n-1}T_n=0$ *and* $(1+\psi_C^{-1})\beta^{n-1}T_{n-1}=2\beta^{n-1}T_{n-1}-c_{n-1}\beta^{n-1}T_n$ *where* $c_k=(q_0+\cdots+q_{k-1})/q_k$.

Consider the following cofiber sequence

$$\Sigma^{2n-1}\xrightarrow{\theta}\tilde{P}^{n-1}\xrightarrow{i_n}\tilde{P}^n\xrightarrow{j_n}\Sigma^{2n}.$$

The map j_n induces elements of the group $KO^{ev}\tilde{P}^n$ as follows:

$$R_n=j_n^*1\in KO^{2n}\tilde{P}^n,\ R_n'=j_n^*\eta^2\in KO^{2n-2}\tilde{P}^n \quad\text{and}\quad R_n''=j_n^*\xi\in KO^{2n-4}\tilde{P}^n.$$

There hold the following relations: $2R_n=\varepsilon_O\beta^nT_n$, $\varepsilon_UR_n=\beta^nT_n$, $R_n'=\varepsilon_O\beta^{n-1}T_n$, $\varepsilon_UR_n'=0$, $R_n''=\varepsilon_O\beta^{n-2}T_n$ and $\varepsilon_UR_n''=2\beta^{n-2}T_n$. In particular, R_n' and R_n'' are contained in the image $\varepsilon_O(KU^{ev}\tilde{P}^n)\subset KO^{ev}\tilde{P}^n$.

Assume that c_{n-1} is odd. Then we have the following cofiber sequence

$$\Sigma^{2n-3}C_\eta\longrightarrow\tilde{P}^{n-2}\xrightarrow{i_ni_{n-1}}\tilde{P}^n\xrightarrow{k_n}\Sigma^{2n-2}C_\eta$$

inducing a split exact sequence

$$0\longrightarrow KO^*\Sigma^{2n-2}C_\eta\longrightarrow KO^*\tilde{P}^n\longrightarrow KO^*\tilde{P}^{n-2}\longrightarrow 0.$$

Set $T_{n-1}'=T_{n-1}+(1-c_{n-1})/2T_n$. Using the realification ε_O: $KU\to KO$ we consider the following elements in $KO^{ev}\tilde{P}^n$:

$$S_n=\varepsilon_O\beta^nT_{n-1}'\in KO^{2n}\tilde{P}^n,\qquad S_n'=\varepsilon_O\beta^{n-1}T_{n-1}'\in KO^{2n-2}\tilde{P}^n,$$
$$S_n''=\varepsilon_O\beta^{n-2}T_{n-1}'\in KO^{2n-4}\tilde{P}^n,\ S_n'''=\varepsilon_O\beta^{n-3}T_{n-1}'\in KO^{2n-6}\tilde{P}^n.$$

Since $\varepsilon_US_n=\beta^nT_n=\varepsilon_UR_n$ and $2\varepsilon_US_n''=2\beta^{n-2}T_n=\varepsilon_UR_n''$ it follows that $S_n-R_n=\eta^*x$ for some $x\in KO^{2n+1}\tilde{P}^n\cong KO^{2n+1}\tilde{P}^{n-2}$ and $2S_n''-R_n''=\eta^*y$ for some $y\in KO^{2n-3}\tilde{P}^n\cong KO^{2n-3}\tilde{P}^{n-2}$. Therefore we may employ the elements S_n, S_n', S_n'' and S_n''' instead of a basis of the image $k_n^*(KO^{ev}\Sigma^{2n-2}C_\eta)\subset KO^{ev}\tilde{P}^n$.

Lemma 3.2. *Let* $(q_0, \cdots, q_n, \cdots, q_{n+m})$ *be a well-ordered tuple such that* $c_n=(q_0+\cdots+q_{n-1})/q_n$ *is even. For any* $m\geq 0$ *there exists an element* $T_{n,n+m}=T_n+a_1T_{n+1}+\cdots+a_mT_{n+m}\in KU^0\tilde{P}^{n+m}$ *satisfying* $\varepsilon_O\beta^{n+1}T_{n,n+m}=0$, *where* $a_1=-c_n/2$ *and* a_{2j} *is taken to be* 0 *or* 1. *In particular,* $a_{4k}=a_{4k+2}=0$ *if* c_{n+4k} *is even.*

Proof. By induction on m we shall construct a desired element $T_{n,n+m}$. Obviously $\varepsilon_O\beta^{n+1}T_n=0$ and $(1+\psi_C^{-1})\beta^{n+1}T_{n,n+1}=0$, which implies that $\varepsilon_O\beta^{n+1}T_{n,n+1}=0$. Under induction hypothesis we here assume that there exists an element $T_{n,n+m-1}\in KU^0\tilde{P}^{n+m-1}$ satisfying $\varepsilon_O\beta^{n+1}T_{n,n+m-1}=0$.

i) The $m=4k+2$ case: Take $a_{4k+2}=0$ if $\varepsilon_O\beta^{n+1}T_{n,n+4k+1}=0\in KO^{2n+2}\tilde{P}^{n+4k+2}$ and $a_{4k+2}=1$ if otherwise. Setting $T_{n,n+4k+2}=T_{n,n+4k+1}+a_{4k+2}T_{n+4k+2}\in KU^0\tilde{P}^{n+4k+2}$, it follows immediately that $\varepsilon_O\beta^{n+1}T_{n,n+4k+2}=0$.

ii) The $m=4k+3$ case: Note that $(1+\psi_C^{-1})\beta^{n+1}T_{n,n+4k+2}=b_{n+4k+2}\beta^{n+1}T_{n+4k+3}\in KU^{2n+2}\tilde{P}^{n+4k+3}$ for some integer b_{n+4k+2}. Since there exists an integer b such that $\varepsilon_O\beta^{n+1}T_{n,n+4k+2}=b\varepsilon_O\beta^{n+1}T_{n+4k+3}\in KO^{2n+2}\tilde{P}^{n+4k+3}$, we see that $b_{n+4k+2}=2b$ is even. Setting $T_{n,n+4k+3}=T_{n,n+4k+2}-b_{n+4k+2}/2T_{n+4k+3}\in KU^0\tilde{P}^{n+4k+3}$, it is obvious that $\varepsilon_O\beta^{n+1}T_{n,n+4k+3}=0$.

iii) The $m=4k$ case: Setting $T'_{n,n+4k}=T_{n,n+4k-1}+aT_{n+4k}\in KU^0\tilde{P}^{n+4k}$ for any integer a, we see that $\varepsilon_O\beta^{n+1}T'_{n,n+4k}=0$. The integer a_{4k} will be determined in iv).

iv) The $m=4k+1$ case: Note that $(1+\psi_C^{-1})\beta^{n+1}T_{n,n+4k-1}=b_{n+4k}\beta^{n+1}T_{n+4k+1}\in KU^{2n+2}\tilde{P}^{n+4k+1}$ for some integer b_{n+4k}. Consider $\bar{P}^{n+4k+2}=\tilde{P}^{n+4k+2}(q_0, \cdots, q_{n+4k-1}, q_{n+4k}, q_{n+4k}, q_{n+4k})$. Then $(1+\psi_C^{-1})\beta^{n+1}T_{n,n+4k-1}=b_{n+4k}\beta^{n+1}T_{n+4k+1}-d\beta^{n+1}T_{n+4k+2}$, $(1+\psi_C^{-1})\beta^{n+1}T_{n+4k+1}=2\beta^{n+1}T_{n+4k+1}-(c_{n+4k}+1)\beta^{n+1}T_{n+4k+2}$ in $KU^{2n+2}\bar{P}^{n+4k+2}$. Since $(\psi_C^{-1})^2=1$ it is immediate that $b_{n+4k}(c_{n+4k}+1)=2d$. This implies that c_{n+4k} is odd if b_{n+4k} is odd. Take $a_{4k}=0$, $a_{4k+1}=-b_{n+4k}/2$ when b_{n+4k} is even, and $a_{4k}=1$, $a_{4k+1}=-(b_{n+4k}+c_{n+4k})/2$ when b_{n+4k} is odd. Setting $T_{n,n+4k+1}=T_{n,n+4k-1}+a_{4k}T_{n+4k}+a_{4k+1}T_{n+4k+1}\in KU^0\tilde{P}^{n+4k+1}$, it follows immediately that $(1+\psi_C^{-1})\beta^{n+1}T_{n,n+4k+1}=0$, and hence $\varepsilon_O\beta^{n+1}T_{n,n+4k+1}=0$.

Assume that c_{n+4k} is even. In this case b_{n+4k} is even, so a_{4k} is taken to be 0 as in iv). If c_{n+4k+1} is odd, then $\tilde{P}^{n+4k+2}$ is quasi KO_*-equivalent to $\tilde{P}^{n+4k}\vee\Sigma^{2n+8k+2}C_\eta$. Hence we can see that $\varepsilon_O\beta^{n+1}T_{n,n+4k+1}=0\in KO^{2n+2}\tilde{P}^{n+4k+2}$. So a_{4k+2} is taken to be 0 as in i). When c_{n+4k+1} is even, we consider $\bar{P}^{n+4k+2}=\tilde{P}^{n+4k+2}(q_0, \cdots, q_{n+4k-1}, q_{n+4k}, q_{n+4k}, q_{n+4k})$. Then the canonical map $\pi: \tilde{P}^{n+4k+2}\to\bar{P}^{n+4k+2}$ induces a homomorphism $\pi^*: KO^{2n+2}\bar{P}^{n+4k+2}\to KO^{2n+2}\tilde{P}^{n+4k+2}$ carrying $\varepsilon_O\beta^{n+1}T_{n,n+4k+1}$ to $\varepsilon_O\beta^{n+1}T_{n,n+4k+1}$. Since $\varepsilon_O\beta^{n+1}T_{n,n+4k+1}=0$ in $KO^{2n+2}\bar{P}^{n+4k+2}$, a_{4k+2} is taken to be 0 even if c_{n+4k+1} is even.

Corollary 3.3. *Let $(q_0, \cdots, q_n, \cdots, q_{n+m})$ be a well-ordered tuple such that $c_n=(q_0+\cdots+q_{n-1})/q_n$ is even. For any $m\geq 1$ there exists an element $R_{n,n+m}\in KO^{2n}\tilde{P}^{n+m}$ such that $\varepsilon_UR_{n,n+m}=\beta^nT_{n,n+m}$, $i^*_{n+m}R_{n,n+m}=R_{n,n+m-1}$ or $R_{n,n+m-1}+R'_{n+m-1}$ if $m\equiv 2$ mod 4 and c_{n+m-1} is odd, and $i^*_{n+m}R_{n,n+m}=R_{n,n+m-1}$ if otherwise. Here $R_{n,n}=R_n=j^*_n1\in KO^{2n}\tilde{P}^n$ and $R'_{n+m-1}=j^*_{n+m-1}\eta^2\in KO^{2n+2m-4}\tilde{P}^{n+m-1}\cong KO^{2n}\tilde{P}^{n+m-1}$.*

Proof. The induced homomorphism $i^*_{n+m}: KO^{2n+1}\tilde{P}^{n+m}\to KO^{2n+1}\tilde{P}^{n+m-1}$ is an epimorphism unless $m\equiv 2$ mod 4 and c_{n+m-1} is odd. In this case we can easily find an element $R_{n,n+m}\in KO^{2n}\tilde{P}^{n+m}$ satisfying $\varepsilon_UR_{n,n+m}=\beta^nT_{n,n+m}$ and $i^*_{n+m}R_{n,n+m}=R_{n,n+m-1}$. Assume that $m\equiv 2$ mod 4 and c_{n+m-1} is odd. Since $(i_{n+m}i_{n+m-1})^*: KO^{2n+1}\tilde{P}^{n+m}\to KO^{2n+1}\tilde{P}^{n+m-2}$ is an isomorphism, we can find an element $R_{n,n+m}\in KO^{2n}\tilde{P}^{n+m}$ such that $\varepsilon_UR_{n,n+m}=\beta^nT_{n,n+m}$ and $i^*_{n+m-1}i^*_{n+m}R_{n,n+m}=i^*_{n+m-1}R_{n,n+m-1}$. The last equality implies that $i^*_{n+m}R_{n,n+m}=R_{n,n+m-1}+xR'_{n+m-1}$ for some $x \in Z/2$.

Remark. In Corollary 3.3 we can uniquely choose an element $R_{n, n+m}$ unless $m \equiv 1 \bmod 4$ and c_{n+m-1} is even. On the other hand, we can choose just two elements $R_{n, n+m}$ and $R_{n, n+m} + R'_{n+m}$ if $m \equiv 1 \bmod 4$ and c_{n+m-1} is even.

If c_n is even, we set

$$\begin{aligned} R'_{n, n+m} &= \varepsilon_O \beta^{n-1} T_{n, n+m} \in KO^{2n-2} \tilde{P}^{n+m}, \\ R''_{n, n+m} &= \varepsilon_O \beta^{n-2} T_n \quad \in KO^{2n-4} \tilde{P}^{n+m}. \end{aligned} \tag{6}$$

Note that $\eta^2 R_{n, n+m} = \varepsilon_O \beta^{-1} \varepsilon_U R_{n, n+m} = R'_{n, n+m}$ where $R_{n, n+m} \in KO^{2n} \tilde{P}^{n+m}$ is obtained in Corollary 3.3. If c_{n-1} is odd, we set

$$\begin{aligned} S_{n, n+m} &= \varepsilon_O \beta^n T'_{n-1} \quad \in KO^{2n} \tilde{P}^{n+m}, \\ S'_{n, n+m} &= \varepsilon_O \beta^{n-1} T_{n-1} \in KO^{2n-2} \tilde{P}^{n+m}, \\ S''_{n, n+m} &= \varepsilon_O \beta^{n-2} T'_{n-1} \in KO^{2n-4} \tilde{P}^{n+m}, \\ S'''_{n, n+m} &= \varepsilon_O \beta^{n-3} T_{n-1} \in KO^{2n-6} \tilde{P}^{n+m}. \end{aligned} \tag{7}$$

By virtue of Theorem 2.4 we can now give generators of $KO^* \tilde{P}^n$ as follows (cf. [3]):

Theorem 3.4. *Let $(q_0, \cdots, q_n)$ be a well-ordered tuple of positive integers. For the weighted projective space $\tilde{P}^n = \tilde{P}^n(q_0, \cdots, q_n)$ the group $KO^* \tilde{P}^n \cong \bigoplus_{0 \leq i \leq 7} KO^i \tilde{P}^n$ is generated by the following elements:*

$$R_{s, n},\ \eta R_{s, n},\ R'_{s, n},\ R''_{s, n},\ S_{t+1, n},\ S'_{t+1, n},\ S''_{t+1, n},\ S'''_{t+1, n}$$

where s and t run over $S = \{1 \leq s \leq n\,;\ c_{s-1}$ and c_s are even$\}$ and $T = \{1 \leq t \leq n-1\,;\ c_t$ is odd$\}$ respectively.

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References

[1] A. Al Amrani, Complex K-theory of weighted projective spaces, J. Pure and Applied Algebra, **93** (1994) 113–127.
[2] A.K. Bousfield, A Classification of K-local spectra, J. Pure and Applied Algebra, **66** (1990) 121–163.
[3] M. Fujii, Ko-groups of projective spaces, Osaka J. Math., **4** (1967) 141–149.
[4] T. Kawasaki, Cohomology of twisted projective spaces and lens complexes, Math. Ann., **206** (1973) 243–248.
[5] Z. Yosimura, Quasi K-homology equivalences, I, Osaka J. Math., **27** (1990) 465–498.