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Entropy numbers in L^p-spaces for averages of rotations

By

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1. Introduction

Let (E, d) be a metric space with finite diameter D. Let us denote for any $0<\varepsilon\leq D$, by $N(E, d, \varepsilon)$ the minimal covering number (possibly infinite) of E by d-open balls of radius ε. These numbers, called entropy numbers of (E, d), analysing the global scattering of the space (E, d), are classical tools of analysis. In a recent work ([2], Theorem 1.3), mainly devoted to the study of the regularity of gaussian processes indexed by product sets, Talagrand proved an estimation of the entropy numbers related to averages of hilbertian contractions. More precisely, let $(H, \|\cdot\|)$ be a Hilbert space and $U: H\to H$ a contraction of H. Put for any $x\in H$

$$\forall n\geq 1 \quad A_n^U(x)=\frac{1}{n}\sum_{j=0}^{n-1}U^j(x) \quad A^U(x)=\{A_n^U(x),\ \ n\geq 1\}. \tag{A1}$$

Then, there exists a universal constant $K>0$ such that

$$\forall x\in H \text{ with } \|x\|=1, \quad \forall 0<\varepsilon\leq 1, \quad N(A^U(x),\ \|\cdot\|,\ \varepsilon)\leq\frac{K}{\varepsilon^2} \tag{A2}$$

That result allowed him to solve a question raised by the author in [**4**], but for L^2-spaces and ergodic averages only. A complete answer based on a different method, the Stein's randomization technic, is provided in ([**3**], Theorem 3.2).

That estimate is also optimal. Let $\mathbf{T}=[-\pi, \pi]$ be the circle. Put,

$$\forall n\geq 1, \quad \forall \theta\in\mathbf{T}, \quad V_n(\theta)=\frac{1}{n}\sum_{j=0}^{n-1}e^{ij\theta}. \tag{A3}$$

By the spectral lemma,

$$\|A_n^U(f)-A_m^U(f)\|^2\leq\int_{\mathbf{T}}|V_n(\theta)-V_m(\theta)|^2\mu_f(d\theta), \tag{A4}$$

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A closer look into Talagrand's proof reveals that (A2) is deduced from the following stronger property: to any nonnegative bounded measure ν on $\mathbf{T}$ and $0<\varepsilon\leq\nu(\mathbf{T})$ can be a finite set of numbers F associated, such that

$$\mathrm{Card}\,(F)\leq K\Big(\frac{\nu(\mathbf{T})}{\varepsilon}\Big)^2, \tag{A5}$$

$$\forall n\geq 1,\quad \exists m\in F\quad \Big|\int_{\mathbf{T}}|V_n(\theta)-V_m(\theta)|^2\nu(d\theta)\leq K\varepsilon^2, \tag{A6}$$

where K is a universal constant. By Hahn-Jordan decomposition theorem, that result extend to arbitrary bounded measures on $\mathbf{T}$ by replacing $\nu(\mathbf{T})$ by $\|\nu\|$ everywhere. It is also clear from (A4) that (A5), (A6) implies (A2).

It is quite natural to inquire how Talagrand's result can be extended for mean averages of L^p-contractions. Let T denotes at first an L^p-contraction and use the notation (A1). We may ask whether

Problem 1: there exists a universal constant $K>0$ such that

$$\forall x\in L^p \text{ with } \|x\|_p\leq 1,\quad \forall\, 0<\varepsilon\leq 1,\quad N(A^T(x),\|\cdot\|^p,\varepsilon)\leq\frac{K}{\varepsilon^p}.$$

We may also weaken *Problem* 1 by only asking an (ε^{-p}) behavior for the covering numbers:

Problem 2: is it true that

$$\forall x\in L^p \text{ with } \|x\|_p=1,\quad \forall\, 0<\varepsilon\leq 1,\quad N(A^T(x),\|\cdot\|_p,\varepsilon)\leq\frac{K(x)}{\varepsilon^p},$$

where $K(x)$ depends on x only?

It is instructive to observe that *Problem* 2 can be answered affirmatively when the spectral measure of T at x is suggiciently smooth, assuming for the rest of the paper that T is a rotation on T. Let $\{\Phi_j,\ j\in\mathbf{Z}\}$ be the family characters of $\mathbf{T}$ and corresponding eigenvalues $\{\alpha_j,\ j\in\mathbf{Z}\}$ of T. Let

$$x=\sum_{j\in\mathbf{Z}}c_j\Phi_j$$

be an element of $L^p(\mu)$ and denotes

$$Q(x)=\Big(\sum_{i\in\mathbf{Z}}\sum_{j\in\Delta_i}c_j\Phi_j|^2\Big)^{\frac{1}{2}} \tag{2}$$

its *square function*. The so-called dyadics intervals Δ_i, $(i\in\mathbf{Z})$ are defined as follows

$$\Delta_i=\begin{cases}\{2^{i-1},2^{i-1}+1,\ldots,2^i-1\} & \textit{if}\quad i>0,\\ 0 & \textit{if}\quad i=0,\\ -\Delta_{|i|} & \textit{if}\quad i<0,\end{cases}$$

Then, by (A3)

$$y=A_N^T(x)-A_M^T(x)=\sum_{j\in\mathbf{Z}}c_j\Phi_j(V_N(\alpha_j)-V_M(\alpha_j)).$$

According to the Littlewood-Paley theory (see [**1**], Chapter 1, p.4)

$$A_p\|y\|_p\leq\|Q(y)\|_p\leq B_p\|y\|_p, \tag{3}$$

where A_p, B_p are universal constants. By Cauchy-Schwarz inequality

$$\Big|\sum_{j\in\Delta_i}c_j\Phi_j(V_N(\alpha_j)-V_M(\alpha_j))\Big|^2\leq\Big(\sum_{j\in\Delta_i}|c_j|^2\Big)\Big(\sum_{j\in\Delta_i}|V_N(\alpha_j)-V_M(\alpha_j)|^2\Big).$$

Hence,

$$Q(y)^2\leq\sum_{i\in\mathbf{Z}}\Big(\sum_{j\in\Delta_i}|c_j|^2\Big)\Big(\sum_{j\in\Delta_i}|V_N(\alpha_j)-V_M(\alpha_j)|^2\Big).$$

Assume now that x satisfies

$$m=\sum_{j\in\mathbf{Z}}|j||c_j|^2<\infty, \tag{4}$$

and let ν denotes the bounded measure on $\mathbf{T}$ defined by

$$\nu=\sum_{i\in\mathbf{Z}}\Big(\sum_{j\in\Delta_i}|c_j|^2\Big)\sum_{j\Delta_i}\delta_{\alpha_j}.$$

We have

$$Q(y)^2\leq\int_{\mathbf{T}}|V_N(\alpha)-V_M(\alpha)|^2\nu(d\alpha)$$

Thus,

$$\|A_N^T(x)-A_M^T(x)\|_p\leq A_p^{-1}\Big(\int_0^{2\pi}|V_N(\alpha)-V_M(\alpha)|^2\nu(d\alpha)\Big)^{\frac{1}{2}},$$

which implies with (A5), (A6) that

$$N(A^T(x),\|\cdot\|_p,\varepsilon)\leq\frac{Km^2}{\varepsilon^2}, \tag{5}$$

for all $0<\varepsilon<m$, where K is an absolute constant.

If x is exactly a dyadic polynomial, says

$$\sum_{j\Delta_{i_0}}c_j\Phi_j,$$

then by (2), $Q(x)=|x|$ so that (3) is empty and the problem of estimating L^p-norms of x (*a fortiori* $A_N^T(x)-A_M^T(x)$) remains entire. Things are changing if instead of searching to measure the L^p-size of $A^T(x)$ by means of the L^p-norm of x, one searches a control in terms of the conjugate norm of the Fourier coedfficients of x. This point of view is justified by the theorem of Hausdorff-Young. We can indeed prove the following

Theorem 1. *Let $2 \leq p < \infty$ and q with $\frac{1}{p}+\frac{1}{q}=1$; there exists a universal constant K_p such that for any rotation T on* $\mathbf{T}$

$$\forall x=\sum_{i\in\mathbf{Z}} c_i\Phi_i \text{ with } \|(c_i)\|_q=1, \quad \forall 0<\varepsilon\leq 1, \quad N(A^T(x), \|\cdot\|_p, \varepsilon) \leq \frac{K_p}{\varepsilon^p}. \tag{6}$$

2. Proof

The proof will require to adapt to the L^p-setting and to modify the tools of Talagrand's proof at many places.
Let $\{\alpha_j, j\in\mathbf{Z}\}$ be the corresponding eigenvalues of T.

By invoking a plain argument of density, it is enough to prove (6) for all x of the type $x=\sum_{0\geq i<N} c_i\Phi_i$, $N\geq 1$. Recall that

$$y=A_N^T(x)-A_M^T(x)=\sum_{0\leq j<N} c_j\Phi_j(V_N(\alpha_j)-V_M(\alpha_j)).$$

Since y is a finite linear combination of the Φ_i's, by Hausdorff-Young theorem and by the very proof of Riesz's Theorem (see [**5**], Chapter IX, par. 9.1, 9.2, 9.3)

$$\|y\|_p \leq \|((V_N(\alpha_j)-V_M(\alpha_j))c_j)_{0\leq i<N}\|_q.$$

In other words

$$\|A_N^T(x)-A_M^T(x)\|_p \leq \left(\int_{\mathbf{T}} |V_N(\alpha)-V_M(\alpha)|^q \mu_x(d\alpha)\right)^{\frac{1}{q}}, \tag{6}$$

where we put $\mu_x=\sum_{0\leq i<N}|c_i|^q\delta_{\alpha_i}$.

Talagrand's proof involves a regularization of the measure μ_x, which we write simply μ in what follows. Put

$$\forall l\geq 1, \quad J_l=\{\theta\in\mathbf{T} \mid 2^{-l}\pi<|\theta|\leq 2^{-l+1}\pi\}, \quad a_l=\mu(J_l). \tag{7}$$

The sequence $\{a_n, n\geq 1\}$ is indeed regularized as follows; set

$$\forall l\geq 1, \quad b_l=\sum_{k=1}^{\infty} 2^{-|k-l|}a_k. \tag{8}$$

In the next lemma, we collect a few properties of that regularization

Lemma 2.

$$\sum_{l\geq 1} a_l \leq 1, \tag{$\mathscr{P}1$}$$

$$\sum_{l\geq 1} b_l \leq 3, \tag{$\mathscr{P}2$}$$

$$\forall\, l\geq 1,\ 0\leq a_l\leq b_l\leq 1, \tag{$\mathscr{P}3$}$$

$$\forall\, l\geq 1,\ \ \frac{1}{2}\leq\frac{b_{l+1}}{b_l}\leq 2, \tag{$\mathscr{P}4$}$$

$$(b_l 2^{ql},\ l\geq 1)\ \textit{is strictly increasing and unbounded.} \tag{$\mathscr{P}5$}$$

Proof. The three first properties are obvious. It is enough to observe that

$$\sum_{l=1}^{\infty} b_l=\sum_{l=1}^{\infty}\sum_{k=1}^{\infty} a_k 2^{-|k-l|}=\sum_{k=1}^{\infty} a_k \sum_{l=1}^{\infty} 2^{-|k-l|}\leq 3\sum_{k=1}^{\infty} a_k.$$

We prove property ($\mathscr{P}4$). One the one hand

$$b_{l+1}=\sum_{k=1}^{\infty} a_k 2^{-|k-l-1|}=a_1 2^{-l}+\cdots+a_l 2^{-1}+a_{l+1}+a_{l+2}2^{-1}+a_{l+3}2^{-2}+\cdots,$$

and on the other one,

$$b_l=\sum_{k=1}^{\infty} a_k 2^{-|k-l|}=a_1 2^{-l+1}+\cdots+a_l+a_{l+1}2^{-1}+a_{l+2}2^{-2}+a_{l+3}2^{-3}+\cdots.$$

Since $\frac{b_l}{2}=a_1 2^{-l}+\cdots+a_l 2^{-1}+a_{l+1}2^{-2}+a_{l+2}2^{-3}+a_{l+3}2^{-4}+\cdots$, we have

$$\frac{b_l}{2}\leq b_{l+1}.$$

Besides, $2b_l=a_1 2^{-l+2}+\cdots+2a_l+a_{l+1}+a_{l+2}2^{-1}+a_{l+3}2^{-2}+\cdots$, hence also,

$$2b_l\geq b_{l+1}.$$

And ($\mathscr{P}5$) follows from $b_{l+1}2^{q(l+1)}\geq b_l 2^{q(l+1)-1}>b_l 2^{ql}$.

Fix now $0<\varepsilon\leq 1$. For any $k\leq 1$, let then $m(k)$ denotes the least integer greater than 1 and verifying

$$b_{m(k)}2^{qm(k)}\geq 2^{qk}\varepsilon^p. \tag{9}$$

The next lemma shows that both sequences $\{m(k),\ k\geq 1\}$, $\{b_{m(k)},\ k\geq 1\}$ are very regular

Lemma 3.

$$\forall\, k\geq 1,\quad m(k)\leq m(k+1), \tag{$\mathscr{P}6$}$$

Put

$k^*=$ the greatest integer such that $m(k^*)=1$.

Then k^* is finite and

$$k^* \leq \frac{1}{\log 2} \log \frac{2}{\varepsilon^{\frac{p}{q}}}. \qquad (\mathscr{P}7)$$

$$\forall k \geq k^*, \quad m(k)+1 \leq m(k+2), \qquad (\mathscr{P}8)$$

$$\forall k \geq 1, \quad m(k+1) \leq m(k)+j_q, \qquad (\mathscr{P}9)$$

where j_q denotes the least integer j such that $j \leq (j-1)q$.

$$\forall k \geq 1, \quad 2^{-j_q} \leq \frac{b_{m(k+1)}}{b_{m(k)}} \leq 2^{j_q} \qquad (\mathscr{P}10)$$

Proof. *a*) By definition of $m(k+1)$, $b_{m(k+1)}2^{qm(k+1)} \geq 2^{q(k+1)}\varepsilon^p > 2^{qk}\varepsilon^p$. It follows that $m(k+1) \geq m(k)$.

b) By definition of $m(k+2)$ and by ($\mathscr{P}5$), $b_{m(k+2)} \leq 2b_{m(k+2)-1}$. Hence,

$$b_{m(k+2)-1}2^{q(m(k+2)-1)} \geq 2^{q-1}2^{qk}\varepsilon^p,$$

which shows $m(k+2)-1 \geq m(k)$.

c) By definition of $m(k)$ this time, $b_{m(k)}2^{qm(k)} \geq 2^{qk}\varepsilon^p$. For any $j \geq 1$, $b_{m(k)} \leq 2^j b_{m(k)+j}$, we thus have

$$b_{m(k)+j}2^{q(m(k)+j)+j-(j-1)q} \geq 2^{q(k+1)}\varepsilon^p,$$

which shows by taking $j=j_q$ that $m(k)+j_q \geq m(k+1)$. Finally the last inequality follows from the three previous.

Define $f: \mathbf{N} \setminus \{0, 1\} \to \mathbf{R}^+$ as follows. For any $n \geq 2$, let $k \geq 1$ be defined by $2^k \leq n < 2^{k+1}$. Put

$$f(n) = \sum_{l<k} b_{m(l)} + (2^{-k}n-1)\, b_{m(k)}. \qquad (10)$$

Then f is strictly increasing and increases with constant jumps equal to $2^{-k}b_{m(k)}$ in $[2^k+1, 2^{k+1}]$. Further f is bounded and $f(n) \leq k^*b_1+6$, if $n \geq 2$. Indeed

$$f(n) \leq \sum_{k=1}^{\infty} b_{m(k)} \leq k^*b_1 + \sum_{\substack{k \geq k^* \\ k \text{ even}}} b_{m(k)} + \sum_{\substack{k \geq k^* \\ k \text{ odd}}} b_{m(k)} \leq k^*b_1+6.$$

We build the set $F \subset \mathbf{N}$ as follows: we put $[1, 2^{k^*+1}]$ in F. Then we put $n \geq 2^{k^*+1}$ in F whenever for some integer $r \geq 1$,

$$f(n-1) \leq r\varepsilon^p \quad r\varepsilon^p \leq f(n). \qquad (10')$$

The theorem will be proved if we show

$$\mathrm{Card}(F) \leq \frac{K_p}{\varepsilon^p}, \tag{11}$$

$$\forall n \geq 2, \quad \exists n' \in F : \| A_n^T(f) - A_{n'}^T(f) \|_p \leq K_p \varepsilon. \tag{12}$$

where K_p is a constant depending on p only (which may change at each occurence). First we show (11). If $k \leq k^*$, then estimation ($\mathscr{P}7$) is enough to conclude. If $k > k^*$, observe that if l is minimal for the relation

$$2^l (2^{-k} b_{m(k)}) \geq \varepsilon^p,$$

then we breing a point n of $[2^k, 2^k+1]$ in F. Estimate l: by ($\mathscr{P}5$) $l \leq qm(k) - (q-1)k$. By ($\mathscr{P}5$) again,

$$b_{m(k)-1} 2^{q(m(k)-1)} \leq 2^{qk} \varepsilon^p,$$

thus

$$l \geq qm(k) - (q-1)k - q - 1$$

The number of points of $[2^k, 2^{k+1}]$ belonging to F is

$$\frac{\mathrm{Card}([2^k, 2^{k+1}])}{2^l} \leq \frac{2^k}{2^{qm(k)-(q-1)k-q-1}} \leq 8 \frac{b_{m(k)}}{\varepsilon^p}.$$

The total number of $n \geq 2^k$ with $k \geq k^*$ belonging to F is thus less than

$$8 \sum_{k^* < k \leq k^+} \frac{b_{m(k)}}{\varepsilon^p} \leq \frac{8}{\varepsilon^p} \sum_{k^* < k} b_{m(k)} \leq \frac{16}{\varepsilon^p} \sum_{l \geq 1} b_l \leq \frac{48}{\varepsilon^p},$$

where k^+ is the greatest integer such that $8 \frac{b_{m(k)}}{\varepsilon^p} \geq 1$. Hence (11) is proved. We turn to (12), which proof relies on estimates concerning the kernels $V_n(\theta)$.

Lemma 4. *For any $\theta \in \mathbf{T}$ and $n, m \geq 1$*

$$|V_n(\theta)| \leq \frac{K}{n|\theta|} \wedge 1, \tag{13}$$

$$|V_n(\theta) - V_m(\theta)| \leq K|\theta||n-m|. \tag{14}$$

Proof. The first assertion follows from the inequality

$$\forall \theta \in \mathbf{T}, \quad |e^{i\theta} - 1| \geq K|\theta|.$$

We show (14). Put

$$\phi(x) = \frac{1}{x}(e^{ix\theta} - 1), \quad x > 0.$$

Then for any $\theta \in \mathbf{T}$ and $n, m \geq 1$

$$|V_n(\theta)-V_m(\theta)|=\frac{|\phi(n)-\phi(m)|}{|e^{i\theta}-1|}$$
$$\leq|n-m|\frac{\sup_{n\wedge m<x<n\vee m}|\phi'(x)|}{|e^{i\theta}-1|}$$
$$\leq\frac{|n-n|}{K|\theta|}\sup_{n\wedge m<x<n\vee m}|\phi'(x)|.$$

But $\phi'(x)=\frac{-(1-ix\theta)e^{ix\theta}-1}{x^2}$, for any $x>0$ and $\theta\in\mathbf{T}$. For any $z\in\mathbf{C}$ with $|z|\leq 1$,

$$|(1-z)e^z-1|\leq K|z|^2,$$

where K is some numerical constant. Combining now these estimates, we get for all $n, m\geq 1$ and $|\theta|\leq\frac{1}{n\vee m}$,

$$|V_n(\theta)-V_m(\theta)|\leq K|\theta||n-m|,$$

which proves (14) if $|\theta|\leq\frac{1}{n\leq m}$. Observe now for any $n>m$ and $\theta\in\mathbf{T}$,

$$|V_n(\theta)-V_m(\theta)|=\left|\left(\frac{1}{n}-\frac{1}{m}\right)\sum_{j=0}^{m-1}e^{ij\theta}+\frac{1}{n}\sum_{j=m}^{n-1}e^{ij\theta}\right|$$
$$\leq\left|\frac{1}{n}-\frac{1}{m}\right|m+\frac{1}{n}|n-m|$$
$$=\frac{2|n-m|}{n},$$

and thus for any $n, m\geq 1$ and $\theta\in\mathbf{T}$

$$|V_n(\theta)-V_m(\theta)|\leq K\left(|\theta|\wedge\frac{1}{n\vee m}\right)|n-m|.$$

Put for any $k\geq 1$,

$$I_k=\bigcup_{l\leq m(k)}J_l,\quad \widehat{I}_k=\bigcup_{l>m(k)}J_l. \tag{15}$$

In the sequel of the proof the two following estimates are used

E1. *Let $k\geq k^*$ and $n\geq 2^k$; then*

$$\int_{I_k}|V_n(\theta)|^q\mu_x(d\theta)\leq K_p\varepsilon^p.$$

Proof. By means of lemma 4

$$\int_{I_k}|V_n(\theta)|^q\mu x(d\theta)\leq\frac{K_p}{n^q}\sum_{l\leq m(k)}\int_{J_l}\frac{1}{|\theta|^q}\mu(d\theta),$$
$$\leq\frac{K_p}{n^q}\sum_{l\leq m(k)}2^{ql}a_l$$

$$\leq \frac{K_p}{n^q} \sum_{l \leq m(k)} 2^{ql} b^l$$

$$\leq \frac{K_p}{n^q} b_{m(k)} 2^{qm(k)}$$

$$\leq K_p \varepsilon^p.$$

E2. *For any $k \geq 1$ and $n, m \geq 1$*

$$\int_{\hat{I}_k} |V_n(\theta) - V_m(\theta)|^q \mu(d\theta) \leq K_p |n-m|^q b_{m(k)} 2^{-qm(k)}.$$

Proof. By means of lemma 4

$$\int_{\hat{I}_k} |V_n(\theta) - V_m(\theta)|^q \mu(d\theta) \leq K_p |n-m|^q \sum_{l > m(k)} \int_{J_l} |\theta|^q \mu(d\theta)$$

$$\leq K_p |n-m|^q \leq \sum_{l > m(k)} 2^{-ql} b_l,$$

$$\leq K_p |n-m|^q b_{m(k)} 2^{-qm(k)}.$$

We can now pass to the proof of (12). Consider $n \geq 2^{k^*+1}$, and let $k \geq k^*$ such that $2^k \leq n \leq 2^{k+1}$. Let $r \geq 1$ denotes the greatest integer such that

$$f(n) \geq r\varepsilon^p.$$

Let $m \geq 1$ denotes the smallest integer satisfying

$$f(m) \geq r\varepsilon^p.$$

Then m is well defined and belongs to F by definition. We will see that

$$\int_{\mathbf{T}} |V_n(\theta) - V_m(\theta)|^q \mu(d\theta) \leq K_p \varepsilon^q. \tag{16}$$

This point will achieve the proof.

Let $k' \geq 1$ be such that $2^{k'} \leq m < 2^{k'+1}$. Clearly, $k' \leq k$. We will distinguish three cases: $(k=k')$, $(k=k'+1)$ and $(k>k'+1)$ as in the original proof.

First case: $(k=k')$

Then, we have

$$\varepsilon^p \geq f(n) - f(m) = 2^{-k}(n-m) b_{m(k)}.$$

By means of (E_1) and the relation $|x+y|^q \leq c_q (|x|^q + |y|^q)$ (c_q is a constant depending on q only)

$$\int_{\mathbf{T}} |V_n(\theta) - V_m(\theta))|^q \mu(d\theta) \leq 2c_p K_p \varepsilon^p + \int_{\hat{I}_k} |V_n(\theta) - V_m(\theta)|^q \mu(d\theta).$$

By means of (E_2)

$$\int_{\hat{I}_k} |V_n(\theta) - V_m(\theta)|^q \mu(d\theta)$$

$$\le K_p (n-m)^q 2^{-qm(k)} b_{m(k)}$$

$$\le K_p \frac{\varepsilon^{qp} b^2{}_{m(k)}}{2^{qm(k)-qk} b^q_{m(k)}}$$

$$\le K_p \varepsilon_{pq-p} b^{2-q}_{m(k)} \le K_p \varepsilon^{pq} = K_p \varepsilon^q$$

and achieves the proof in that case.

Second case: $(k'+1<k)$

Then,

$$\varepsilon^p \ge f(n) - f(m) = (2^{-k}n-1) b_{m(k)} + \sum_{k'<l<k} b_{m(l)} + (2-2^{-k}m) b_{m(k')}.$$

Hence,

$$\varepsilon^p \ge \sum_{k'<l<k} b_{m(l)},$$

and by $(\mathscr{P}10)$

$$5\varepsilon^p \ge \sum_{k'\le l<k} b_{m(l)},$$

and by $(\mathscr{P}9)$ and $(\mathscr{P}10)$

$$\sum_{k'\le l<k} b_l \le 100\varepsilon^p. \tag{17}$$

Recall that $I_{k'} = \cup_{l\le m(k')} J_l$, $I_k = \cup_{l\le m(k)} J_l$, $\hat{I}_k = \cup_{l>m(k)} J_l$. Put $I = \cup_{m(k')<l\le m(k)} J_l$. Then,

$$\int_{\mathbf{T}} |V_n(\theta) - V_m(\theta)|^q \mu(d\theta) = \left(\int_{\hat{I}_{k'}} + \int_{\hat{I}_k} + \int_I\right) |V_n(\theta) - V_m(\theta)|^2 \mu(d\theta).$$

Since $m(k') \le m(k)$, we have $I_{k'} \subset I_k$; hence by means of estimate $(E1)$,

$$\int_{I_{k'}} |V_n(\theta) - V_m(\theta)|^q \mu(d\theta) \le c_q \left(\int_{I_{k'}} |V_m(\theta)|^q \mu(d\theta) + \int_{I_{k'}} |V_n\theta)|^q \mu(d\theta)\right) \le K_p \varepsilon^p. \tag{18}$$

Also,

$$\int_I |V_n(\theta) - V_m(\theta)|^q \mu(d\theta) \le K_p \sum_{k'\le l<k} b_l \le K_p \varepsilon^p. \tag{19}$$

From estimate $(E2)$

$$\int_{\hat{I}_k} |V_n(\theta) - V_m(\theta)|^q \mu(d\theta) \le K_p (n-m)^q 2^{-qm(k)} b_{m(k)}$$

$$\leq K_p 2^{q(k-m(k))} b_{m(k)}$$

$$\leq \frac{K_p b_{m(k)}^2}{\varepsilon^p}$$

$$\leq K_p \varepsilon^{2p-p} \leq K_p \varepsilon^p \leq \varepsilon^q. \tag{20}$$

Putting together estimates (18), (19), (20) achieves the proof in that case too.

Third case: $(k=k'+1)$

At first

$$\left(\int_{\mathbf{T}} |V_n(\theta) - V_m(\theta)|^q \mu(d\theta)\right)^{\frac{1}{q}}$$

$$\leq \left(\int_{\mathbf{T}} |V_n(\theta) - V_{2^k}(\theta)|^q \mu(d\theta)\right)^{\frac{1}{q}} + \left(\int_{\mathbf{T}} |V_{2^k}(\theta) - V_m(\theta)|^q \mu(\mathrm{d}\theta)\right)^{\frac{1}{q}}.$$

As in the first case, estimations $(E1)$, $(E2)$ show

$$\int_{\mathbf{T}} |V_n(\theta) - V_{2^k}(\theta)|^q \mu(d\theta) \leq K_p(\varepsilon^p + |n-2^k|^q 2^{-qm(k)} b_{m(k)}),$$

$$\int_{\mathbf{T}} |V_m(\theta) - V_{2^k}(\theta)|^q \mu(d\theta) \leq K_p(\varepsilon^p + |m-2^k|^q 2^{-qm(k)} b_{m(k)}),$$

Since f is increasing and $m < 2^k \leq \mathrm{n}$

$$\varepsilon^p \geq f(n) - f(m) \geq f(n) - f(2^k) = (2^{-k} n - 1) b_{m(k)}$$

Thus $\varepsilon^p \geq (2^{-k} n - 1) b_{m(k)}$ *and*,

$$(n-2^k)^q 2^{-qm(k)} b_{m(k)} \leq 2^{q(k-m(k))} b_{m(k)} |2^{-k} n - 1|^p$$

$$\leq K_p \varepsilon^{pq} 2^{q(k-m(k))} b_{m(k)}^{1-q}$$

$$\leq K_p \frac{\varepsilon^p q b_{m(k)}^{2-q}}{2^{-q(k-m(k))} b_{m(k)}}.$$

$$\leq K_p \varepsilon^{pq-p} b_{m(k)}^{2-q} \leq \varepsilon^q$$

Hence

$$\int_{\mathbf{T}} |V_n(\theta) - V_{2^k}(\theta)|^q \mu(d\theta) \leq K_p \varepsilon^q.$$

Similarly

$$\varepsilon^p \geq f(n) - f(m) \geq f(2^k) - f(m).$$

But

$$f(m) = \sum_{l<k-1} b_{m(l)} + (2^{-k+1} m - 1) b_{m(k-1)}$$

$$f(2^k)=\sum_{l\le k-1} b_{m(l)}.$$

$$\begin{aligned}\varepsilon^p\ge f(2^k)-f(m)&=b_{m(k-1)}-(2^{-k+1}m-1)\,b_{m(k-1)}\\ &=b_{m(k-1)}2^{-k}m\\ &\ge\frac{1}{2}b_{m(k)}2^{-k}m.\end{aligned}$$

Thus

$$\begin{aligned}(m-2^k)^q2^{-qm(k)}b_{m(k)}&\le 2^{q(k-m(k))}b_{m(k)}|2^{-k}m-1|^p\\ &\le K_p\varepsilon^{pq}2^{q(k-m(k))}b_{m(k)}^{1-q}\\ &\le K_p\frac{\varepsilon^{pq}b_{m(k)}^{2-q}}{2^{-q(k-m(k))}b_{m(k)}}.\\ &\le K_p\varepsilon^{pq-p}b_{m(k)}^{2-q}\le\varepsilon^q\end{aligned}$$

And finally,

$$\int_{\mathbf{T}}|V_m(\theta)-V_{2^k}(\theta)|^q\mu(d\theta)\le K_p\varepsilon^q,$$

which achieves the proof in the last remainding case.

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