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Certain unstable modular algebras over the mod p Steenrod algebra

By

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1. Introduction

Let p be an odd prime. We assume that all spaces are completed at p by means of the Bousfield-Kan [4]. In this paper, a cohomology is taken with Z/p coefficients unless otherwise specified, and $H^*(-)$ means $H^*(-;Z/p)$. Let $\mathscr{A}_p$ be the mod p Steenrod algebra and $\mathscr{K}$ denote the category of unstable $\mathscr{A}_p$-algebras. The objects of $\mathscr{K}$ are called $\mathscr{K}$-algebras. For a space X, $H^*(X)$ is a $\mathscr{K}$-algebra. It is known, however, that a $\mathscr{K}$-algebra need not be of the form $H^*(X)$.

A $\mathscr{K}$-algebra A is said to be *realizable* if A is represented as the cohomology of some space, that is, there exists a space X with $A \cong H^*(X)$ as $\mathscr{K}$-algebras. The realizability of an algebra is one of the major problems in the unstable homotopy theory. There are, indeed, many results, such as the Steenrod problem [6], the Cooke conjecture [1], and others.

In this paper we investigate the realizability of the following algebras for $n \geq 1$:

$$A_n = Z/p[x_{2n}] \otimes \Lambda(y_{2n+1}, z_{2n+2p-1})$$

with Steenrod operation actions $\beta(x_{2n}) = y_{2n+1}$ and $\mathscr{P}^1(y_{2n+1}) = z_{2n+2p-1}$. Our first result gives a necessary condition for A_n to be a $\mathscr{K}$-algebra:

Theorem A. *If A_n is a $\mathscr{K}$-algebra, then $n = p^i$ for some $i \geq 0$.*

By Theorem A, we concentrate on the algebras of the following form:

$$B_i = A_{p^i} = Z/p[x_{2p^i}] \otimes \Lambda(y_{2p^i+1}, z_{2p^i+2p-1})$$

with $\beta(x_{2p^i}) = y_{2p^i+1}$ and $\mathscr{P}^1(y_{2p^i+1}) = z_{2p^i+2p-1}$.

Actually, the $\mathscr{K}$-structure of B_i is uniquely determined for $i > 0$ (see §2). On the other hand, B_0 has two $\mathscr{K}$-structures and the realizability of B_0 has completely determined by [2] (see Theorem 3.1). We show the $\mathscr{K}$-algebra B_1 is realizable as the cohomology of some H-spaces (see Proposition 3.2).

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The $\mathscr{K}$-algebra B_2 is realizable as follows: Let $X(p)$ be the H-space constructed by Harper [7] so that $H^*(X(p)) \cong \Lambda(u_3, u_{2p+1}) \otimes Z/p[u_{2p+2}]/(u_{2p+2}^p)$ with $\mathscr{P}^1(u_3) = u_{2p+1}$ and $\beta(u_{2p+1}) = u_{2p+2}$. Then the three-connective cover of $X(p)$ realizes B_2, namely we have

$$H^*(X(p)\langle 3\rangle) \cong B_2.$$

Thus the realizability of A_n is completely determined by the following:

Theorem B. *If B_i is realizable as the cohomology of a space, then $i=0$, 1 or 2.*

We shall prove Theorem B using the work of Lannes about his T-functor [8], which has been remarkable in the recent study of unstable homotopy theory.

This paper is organized as follows: In §2 and §3, we prove Theorem A and show the realizability of B_1, respectively. §4 is devoted to the proof of Theorem B.

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2. Proof of Theorem A

In this section we prove Theorem A, that is, if the algebra A_n with the given Steenrod operation actions is a $\mathscr{K}$-algebra, then $n = p^i$ for some $i \geq 0$.

First we show that the ideal $I = (y_{2n+1}, z_{2n+2p-1})$ generated by y_{2n+1} and $z_{2n+2p-1}$ is closed under the action of $\mathscr{A}_p$. If $\alpha \in I$, then $\beta(\alpha)$, $\mathscr{P}^{p^i}(\alpha) \in I$ for $i \geq 0$ since $\beta(y_{2n+1}) = \beta(z_{2n+2p-1}) = 0$ and $(\mathscr{P}^{p^i}(\alpha))^p = \mathscr{P}^{p^{i+1}}(\alpha^p) = 0$. Hence $Z/p[x_{2n}] \cong A_n/I$ has a $\mathscr{K}$-structure, and this implies that $n = p^i r$ for some $i \geq 0$ and $r \mid (p-1)$. Thus, to complete the proof, we have only to show that $r = 1$.

We remark that the generator $x_{2p^i r}$ can be taken to satisfy

$$\mathscr{P}^{p^i}(x_{2p^i r}) = r x_{2p^i r}^{s+1} \tag{2.1}$$

for $s = (p-1)/r$. In fact, using the variation of a result of Adams-Wilkerson as in [3, Th. 4.2] (see also [1, Th. 2.1]), $Z/p[x_{2p^i r}]$ is isomorphic to $Z/p[t_{2p^i}]^{Z/r}$ with $\mathscr{P}^{p^i}(t_{2p^i}) = t_{2p^i}^p$ as $\mathscr{K}$-algebras, where Z/r acts as ring automorphisms and as the usual multiplication on t_{2p^i}.

Now we divide the proof into two cases for $i > 0$ and $i = 0$. First assume that $i > 0$. Then, there is an Adem relation

$$\mathscr{P}^{p^i}\beta = \mathscr{P}^1 \beta \mathscr{P}^{p^i - 1} + \beta \mathscr{P}^{p^i}. \tag{2.2}$$

Using (2.1) and applying the operations of (2.2) on $x_{2p^i r}$, we have

$$\mathscr{P}^{p^i}(y_{2p^i r+1}) = (r-1) x_{2p^i r}^s y_{2p^i r+1}. \tag{2.3}$$

For the dimensional reason, we can put $\mathscr{P}^{p^i}(z_{2p^i r+2p-1}) = a x_{2p^i r}^s z_{2p^i r+2p-1}$ for some $a \in Z/p$. Then applying (2.2) to $z_{2p^i r+2p-1}$, we have $a = 0$. Thus

$$(2.4) \qquad \mathscr{P}^{p^i}(z_{2p^ir+2p-1})=0.$$

When $i>1$, there is an Adem relation $\mathscr{P}^p\mathscr{P}^{p^i-p+1}+\mathscr{P}^1\mathscr{P}^{p^i}=\mathscr{P}^{p^i}\mathscr{P}^1$, and we apply these on y_{2p^ir+1}. Then, using also (2.3) and (2.4), we have $\mathscr{P}^1((r-1)x^s_{2p^ir}y_{2p^ir+1}) =\mathscr{P}^{p^i}(z_{2p^ir+2p-1})=0$. Since $\mathscr{P}^1(x^s_{2p^ir}y_{2p^ir+1})=x^s_{2p^ir}z_{2p^ir+2p-1}\neq 0$, we can conclude that $r=1$. When $i=1$, applying the operations in the Adem relation $\mathscr{P}^p\mathscr{P}^{p+1} =\mathscr{P}^{2p+1}+\mathscr{P}^{2p}\mathscr{P}^1$ on y_{2pr+1}, we obtain $\mathscr{P}^1\mathscr{P}^{2p}(y_{2pr+1})=-(r-1)x^{2s}_{2pr}z_{2pr+2p-1}$. On the other hand, using the Adem relation $\mathscr{P}^p\mathscr{P}^p=2\mathscr{P}^{2p}+\mathscr{P}^{2p-1}\mathscr{P}^1$, we get $\mathscr{P}^1\mathscr{P}^{2p}(y_{2pr+1})=((r-1)(r-2)/2)x^{2s}_{2pr}z_{2pr+2p-1}$. Thus we also have the result $r=1$ in this case, which completes the proof for $i>0$.

Next consider the case $i=0$. Applying the Adem relation

$$(2.5) \qquad 2\mathscr{P}^1\beta\mathscr{P}^1=\mathscr{P}^1\mathscr{P}^1\beta+\beta\mathscr{P}^1\mathscr{P}^1$$

on x_{2r}, we have

$$(2.6) \qquad \mathscr{P}^1(z_{2r+2p-1})=2(r-1)x^s_{2r}z_{2r+2p-1}-r(r-1)x^{2s}_{2r}y_{2r+1}.$$

We apply (2.5) on y_{2r+1}, and see that $\beta\mathscr{P}^1(z_{2r+2p-1})=0$. By (2.6), we also have $\beta\mathscr{P}^1(z_{2r+2p-1})=2(r-1)sx^{s-1}_{2r}y_{2r+1}z_{2r+2p-1}$. From these equations, we can conclude that $r=1$ since $s\neq 0$. Hence we have completed the proof of Theorem A.

3. Realization of B_0 and B_1

By Theorem A, the realizability of A_n is concentrated on the following cases:

$$B_i=A_{p^i}=\mathbf{Z}/p[x_{2p^i}]\otimes\Lambda(y_{2p^i+1},z_{2p^i+2p-1}) \quad \text{for } i\geq 0$$

with $\beta(x_{2p^i})=y_{2p^i+1}$ and $\mathscr{P}^1(y_{2p^i+1})=z_{2p^i+2p-1}$.

First we consider the realizability of B_0. By (2.6) we have $\mathscr{P}^1(z_{2p+1})=0$, and for the dimensional reason and unstability, we see that the $\mathscr{A}_p$-actions on B_0 are completely determined except for $\mathscr{P}^p(z_{2p+1})$. Let $B(p)$ be the H-space introduced by Mimura-Toda [9] so that $H^*(B(p))\cong\Lambda(u_3,u_{2p+1})$ with $\mathscr{P}^1(u_3)=u_{2p+1}$, and $B(p)\langle 3;p\rangle$ denote the homotopy fiber of the map of degree p

$$[p]:B(p)\to K(\mathbf{Z},3).$$

Then the following results of Aguadé-Broto-Santos [2] completely determine the realizability of B_0, by which it turns out that there are just two $\mathscr{K}$-structures on B_0:

Theorem 3.1 ([2]). (1) *On the* $\mathscr{K}$-algebra B_0, $\mathscr{P}^p(z_{2p+1})=0$ *or* $x_2^{p(p-1)}z_{2p+1}$.

(2) *If* $\mathscr{P}^p(z_{2p+1})=x_2^{p(p-1)}z_{2p+1}$, *then the* $\mathscr{K}$*-algebra* B_0 *cannot be realizable as a cohomology of some space.*

(3) *If* $\mathscr{P}^p(z_{2p+1})=0$, *then the* $\mathscr{K}$*-algebra* B_0 *is realizable as the cohomology of* $B(p)\langle 3;p\rangle$, *namely*

$$H^*(B(p)\langle 3;p\rangle)\cong B_0.$$

(4) *If there is a space* X *so that* $H^*(X) \cong B_0$ *as* $\mathscr{K}$*-algebras, then* $X \simeq B(p)\langle 3;p\rangle$ *up to p-completion.*

For $i>0$, if we impose the unstability condition on B_i, the $\mathscr{A}_p$-actions on B_i are completely determined except for $\mathscr{P}^{p^i}(y_{2p^i+1})$ and $\mathscr{P}^{p^i}(z_{2p^i+2p-1})$ by dimensional reason. But it follows $\mathscr{P}^{p^i}(y_{2p^i+1}) = \mathscr{P}^{p^i}(z_{2p^i+2p-1}) = 0$ from (2.3) and (2.4). Thus, B_i for $i>0$ has a unique $\mathscr{K}$-structure.

For the realizability of B_1, we have the following:

Proposition 3.2. *The* $\mathscr{K}$*-algebra* B_1 *is realizable as the cohomology of an H-space.*

Proof. There is an H-space $Y(p)$ satisfying $H^*(Y(p)) \cong \Lambda(u_3, u_{4p-1})$. In fact, $Y(3) = G_2$, the exceptional Lie group, if $p=3$. For $p \geq 5$, as a special case of [5], we have an H-space $Y(p)$ which contains the cell complex

$$S^3 \cup_\alpha e^{4p-1},$$

where $\alpha \in \pi_{4p-2}(S^3) \cong \boldsymbol{Z}/p$ is the generator. Computing the Serre spectral sequence, we see that the three-connective cover $Y(p)\langle 3\rangle$ of $Y(p)$ realizes B_1, namely we have

$$H^*(Y(p)\langle 3\rangle) \cong B_1,$$

which completes the proof.

4. Proof of Theorem B

We use the Lannes theory concerning the T-functor in the proof of Theorem B. Thus, we recall the theory first. The functor $T: \mathscr{K} \to \mathscr{K}$ is the left adjoint of the functor $H^*(B\boldsymbol{Z}/p) \otimes -$, that is, there is an adjoint isomorphism $\mathrm{Hom}_{\mathscr{K}}(T(A), B) \cong \mathrm{Hom}_{\mathscr{K}}(A, H^*(B\boldsymbol{Z}/p) \otimes B)$ for $\mathscr{K}$-algebras A and B.

For a $\mathscr{K}$-map $f: A \to H^*(B\boldsymbol{Z}/p)$, its adjoint restricts to a $\mathscr{K}$-map $T(A)^0 \to \boldsymbol{Z}/p$, where $T(A)^0$ is the subalgebra of $T(A)$ of elements of degree 0. The connected component $T_f(A)$ of $T(A)$ corresponding to f is defined by $T_f(A) = T(A) \otimes_{T(A)^0} \boldsymbol{Z}/p$, and there is a natural $\mathscr{K}$-map $\varepsilon_f: A \to T_f(A)$.

The evaluation map $e: B\boldsymbol{Z}/p \times \mathrm{Map}(B\boldsymbol{Z}/p, X) \to X$ induces a $\mathscr{K}$-map e^*, and taking the adjoint of this yields a $\mathscr{K}$-map $\lambda: T(H^*(X)) \to H^*(\mathrm{Map}(\boldsymbol{Z}/p, X))$. For a map $\phi: B\boldsymbol{Z}/p \to X$, there is a $\mathscr{K}$-map $\lambda_{\phi^*}: T_{\phi^*}(H^*(X)) \to H^*(\mathrm{Map}(B\boldsymbol{Z}/p, X)_\phi)$ considering componentwise. Then, by definition, the composite $\lambda_{\phi^*}\varepsilon_{\phi^*}$ is induced by the evaluation $e_\phi: \mathrm{Map}(B\boldsymbol{Z}/p, X)_\phi \to X$ at the base point. The following theorem is due to Lannes:

Theorem 4.1 ([8]). *For a map* $\phi: B\boldsymbol{Z}/p \to X$, *if* $T_{\phi^*}(H^*(X))^1 = 0$, *then* $\lambda_{\phi^*}: T_{\phi^*}(H^*(X)) \to H^*(\mathrm{Map}(B\boldsymbol{Z}/p, X)_\phi)$ *is an isomorphism.*

Moreover, for each $\mathscr{K}$-algebra A, T_f can be considered as a functor from $\mathscr{K}(A)$

to $\mathscr{K}(T_f(A))$, where $\mathscr{K}(A)$ denotes the subcategory of $\mathscr{K}$ each of whose objects has an A-module structure compatible with its $\mathscr{K}$-structure.

We also regard $T_f(M)$ as an object of $\mathscr{K}(A)$ through the natural $\mathscr{K}$-map $\varepsilon_f: A \to T_f(A)$ for any object M of $\mathscr{K}(A)$, and $\varepsilon_f: M \to T_f(M)$ becomes a morphism of $\mathscr{K}(A)$-algebras. It is well known that T_f is exact, and commutes with suspensions and tensor products.

To prove Theorem B, we need the T-functor for B_i. As is known, $H^*(BZ/p) \cong \Lambda(w_1) \otimes Z/p[w_2]$ with $\beta(w_1) = w_2$. Now we define a $\mathscr{K}$-map $f: B_i \to H^*(BZ/p)$ as $f(x_{2p^i}) = w_2^{p^i}$ and $f(y_{2p^i+1}) = f(z_{2p^i+2p-1}) = 0$.

Proposition 4.2. *$\varepsilon_f: B_i \to T_f(B_i)$ is an isomorphism.*

Proof. Let $C_i = Z/p[x_{2p^i}] \otimes \Lambda(y_{2p^i+1})$, and $k: B_i \to C_i$ be the quotient map. Then it is obvious that $k^*: \mathrm{Hom}_{\mathscr{K}}(C_i, H^*(BZ/p)) \to \mathrm{Hom}_{\mathscr{K}}(B_i, H^*(BZ/p))$ is an isomorphism. Thus, by the results of Aguadé-Broto-Notbohm [1], $T_f(C_i) \cong T_g(C_i)$ for a non trivial map $g: C_i \to H^*(BZ/p)$, and $\varepsilon_g: C_i \to T_g(C_i)$ is an isomorphism. Since T_f is exact, we have the following commutative diagram whose horizontal arrows are exact sequences of $\mathscr{K}(B_i)$-algebras:

$$\begin{array}{ccccccccc} 0 & \to & z_{2p^i+2p-1}C_i & \to & B_i & \to & C_i & \to & 0 \\ & & \downarrow{\scriptstyle \varepsilon_f} & & \downarrow{\scriptstyle \varepsilon_f} & & \downarrow{\scriptstyle \varepsilon_f}\,\cong & & \\ 0 & \to & T_f(z_{2p^i+2p-1}C_i) & \to & T_f(B_i) & \to & T_f(C_i) & \to & 0. \end{array} \tag{4.1}$$

Since $z_{2p^i+2p-1}C_i \cong \Sigma^{2p^i+2p-1}C_i$ as $\mathscr{K}(B_i)$-algebras and T_f commutes with suspensions, we have $T_f(z_{2p^i+2p-1}C_i) \cong z_{2p^i+2p-1}C_i$. Hence we can conclude that $\varepsilon_f: B_i \to T_f(B_i)$ is an isomorphism by the diagram (4.1), which completes the proof.

Proof of Theorem B. We assume that B_i is realizable, that is, $B_i \cong H^*(X)$ for some space X. A result of Lannes [8] implies that there is a map $\phi: BZ/p \to X$ such that $\phi^* = f$, and then the evaluation map $e_\phi: \mathrm{Map}(BZ/p, X)_\phi \to X$ is a homotopy equivalence by Theorem 4.1 and Proposition 4.2. Let $\iota: BZ/p \to \mathrm{Map}(BZ/p, X)_\phi$ be the adjoint of $\phi\omega$, where ω is the multiplication map for the H-structure of BZ/p. We have the following commutative diagram of fibrations:

$$\begin{array}{ccccccc} BZ/p & = & BZ/p & \to & EBZ/p & \to & B^2Z/p \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \iota} & & \downarrow & & \| \\ X & \underset{\simeq}{\overset{e_\phi}{\leftarrow}} & M & \to & M_{hBZ/p} & \overset{j}{\to} & B^2Z/p, \end{array} \tag{4.2}$$

where $M = \mathrm{Map}(BZ/p, X)_\phi$ and $M_{hBZ/p} = EBZ/p \times_{BZ/p} M$ is the Borel construction. We consider the Serre spectral sequence of the bottom fibration whose E_2-term is given as $E_2^{*,*} = H^*(B^2Z/p) \otimes B_i$.

As is known, $H^*(B^2Z/p) \cong Z/p[\eta_2, \beta\mathscr{P}^{\Delta_j}\beta\eta_2 \mid j \geq 0] \otimes \Lambda(\beta\eta_2, \mathscr{P}^{\Delta_j}\beta\eta_2 \mid j \geq 0)$, where

$\mathscr{P}^{\Delta_j}=\mathscr{P}^{p^j}\cdots\mathscr{P}^1$ and η_2 denotes the fundamental class. We fix the basis Γ of the vector space $H^*(B^2\boldsymbol{Z}/p)$ by taking all monomials of η_2, $\beta\mathscr{P}^{\Delta_j}\beta\eta_2$, $\beta\eta_2$ and $\mathscr{P}^{\Delta_j}\beta\eta_2$ for $j\geq 0$. For the $\mathscr{A}_p$-actions on indecomposables, by definition and unstability, we have $\mathscr{P}^{p^{j+1}}(\mathscr{P}^{\Delta_j}\beta\eta_2)=\mathscr{P}^{\Delta_{j+1}}\beta\eta_2$ and $\mathscr{P}^1(\mathscr{P}^{\Delta_j}\beta\eta_2)=0$. Furthermore, we need the following:

Lemma 4.3 ([1]).

(1) $$\mathscr{P}^1(\beta\mathscr{P}^{\Delta_j}\beta\eta_2)=\begin{cases}0 & \textit{if } j=0,\\ (\beta\mathscr{P}^{\Delta_{j-1}}\beta\eta_2)^p & \textit{if } j>0.\end{cases}$$

(2) $$\mathscr{P}^{p^{j+1}}(\beta\mathscr{P}^{\Delta_j}\beta\eta_2)=\beta\mathscr{P}^{\Delta_{j+1}}\beta\eta_2 \quad \textit{for } j\geq 0.$$

(3) $$\mathscr{P}^{p^k}(\mathscr{P}^{\Delta_j}\beta\eta_2)=\mathscr{P}^{p^k}(\beta\mathscr{P}^{\Delta_j}\beta\eta_2)=0 \quad \textit{for } k\neq 0, j+1.$$

From the diagram (4.2), we have $\tau(x_{2p^i})=\mathscr{P}^{\Delta_{i-1}}\beta\eta_2+\delta_{2p^i+1}$ since $\phi^*(x_{2p^i})=w_2^{p^i}$ and $\tau(w_2^{p^i})=\mathscr{P}^{\Delta_{i-1}}\beta\eta_2$, where τ denotes the transgression and δ_{2p^i+1} is some decomposable element in $H^*(B^2\boldsymbol{Z}/p)$. From now on, we assume that $i\geq 3$, and deduce a contradiction from this assumption.

We set

$$\theta_{2p^i+2p^2}=(\beta\mathscr{P}^{\Delta_{i-3}}\beta\eta_2)^{p^2}+\mathscr{P}^{\Delta_1}\beta(\delta_{2p^i+1})$$

in $H^{2p^i+2p^2}(\beta^2\boldsymbol{Z}/p)$. Since $j^*(\theta_{2p^i+2p^2})=\mathscr{P}^{\Delta_1}\beta(j^*(\mathscr{P}^{\Delta_{i-1}}\beta\eta_2+\delta_{2p^i+1}))=0$, there exists an element of total degree $2p^i+2p^2-1$ which kills $\theta_{2p^i+2p^2}$ in the spectral sequence. On the other hand, we shall show that $\theta_{2p^i+2p^2}$ cannot be killed in the spectral sequence, which causes a contradiction.

First, we remark the following:

Lemma 4.4. *When we represent $\theta_{2p^i+2p^2}$ as a linear combination with basis Γ, it must contain the term $(\beta\mathscr{P}^{\Delta_{i-3}}\beta\eta_2)^{p^2}$.*

Proof. If $i\neq 4$, then we have the conclusion since we can see that $\mathscr{P}^{\Delta_1}\beta(\delta_{2p^i+1})$ does not contain the term $(\beta\mathscr{P}^{\Delta_{i-3}}\beta\eta_2)^{p^2}$ by the $\mathscr{K}$-structure of $H^*(B^2\boldsymbol{Z}/p)$. Thus we assume that $i=4$. We set

$$\alpha_{2p^4+1}=(\beta\mathscr{P}^{\Delta_2}\beta\eta_2)(\beta\mathscr{P}^{\Delta_1}\beta\eta_2)^{p^2-p-2}(\mathscr{P}^{\Delta_1}\beta\eta_2)(\beta\mathscr{P}^1\beta\eta_2),$$

$$\beta_{2p^4+1}=(\beta\mathscr{P}^{\Delta_2}\beta\eta_2)(\beta\mathscr{P}^{\Delta_1}\beta\eta_2)^{p^2-p-1}(\mathscr{P}^1\beta\eta_2),$$

and

$$\gamma_{2p^4+1}=(\mathscr{P}^{\Delta_2}\beta\eta_2)(\beta\mathscr{P}^{\Delta_1}\beta\eta_2)^{p^2-p-1}(\beta\mathscr{P}^1\beta\eta_2).$$

Then, for the dimensional reason, we can put $\delta_{2p^4+1}=a\alpha_{2p^4+1}+b\beta_{2p^4+1}+c\gamma_{2p^4+1}+\bar{\delta}_{2p^4+1}$ for some $a,b,c\in\boldsymbol{Z}/p$, where $\bar{\delta}_{2p^4+1}$ is an element which does not contain the term α_{2p^4+1}, β_{2p^4+1} or γ_{2p^4+1}. We note that $\mathscr{P}^{\Delta_1}\beta(\alpha_{2p^4+1})$, $\mathscr{P}^{\Delta_1}\beta(\beta_{2p^4+1})$ and $\mathscr{P}^{\Delta_1}\beta(\gamma_{2p^4+1})$ contain the term $(\beta\mathscr{P}^{\Delta_1}\beta\eta_2)^{p^2}$ while $\mathscr{P}^{\Delta_1}\beta(\bar{\delta}_{2p^4+1})$ does not contain this term.

Using $\mathscr{P}^1(x_{2p^4})=\mathscr{P}^p(x_{2p^4})=0$ and the $\mathscr{K}$-structure of $H^*(B^2\boldsymbol{Z}/p)$, we can show that $a=b=c=0$ by a routine calculations. Then $\mathscr{P}^{\Delta_1}\beta(\delta_{2p^4+1})=\mathscr{P}^{\Delta_1}\beta(\bar{\delta}_{2p^4+1})$ does not contain the term $(\beta\mathscr{P}^{\Delta_1}\beta\eta_2)^{p^2}$, and we have the required conclusion.

For the dimensional reason, the element which hits $\theta_{2p^i+2p^2}$ must have one of the following forms:

$$\lambda_{2p^2-1}\otimes x_{2p^i}, \quad \kappa_{2p^2-2}\otimes y_{2p^i+1}, \quad \nu_{2p^2-2p}\otimes z_{2p^i+2p-1}.$$

If $i\geq 4$, then any element of the above form cannot hit $\theta_{2p^i+2p^2}$ by Lemma 4.4 and the dimensional reason.

For $i=3$, the only possible case $\theta_{2p^3+2p^2}$ can be hit is that $\kappa_{2p^2-2}=(\beta\mathscr{P}^1\beta\eta_2)^{p-1}+\bar{\kappa}_{(2p+2)(p-1)}$ and $\tau(y_{2p^3+1})$ contain the term $(\beta\mathscr{P}^1\beta\eta_2)^{p^2-p+1}$, where $\bar{\kappa}_{(2p+2)(p-1)}\in H^*(B^2\boldsymbol{Z}/p)$ is some element which does not contain the term $(\beta\mathscr{P}^1\beta\eta_2)^{p-1}$. But we have the following:

Lemma 4.5. *When we represent $\tau(y_{2p^3+1})$ as a linear combination with basis Γ, it does not contain the term $(\beta\mathscr{P}^1\beta\eta_2)^{p^2-p+1}$.*

Proof. Since $\tau(y_{2p^3+1})=\beta\mathscr{P}^{\Delta_2}\beta\eta_2+\beta(\delta_{2p^3+1})$, it is sufficient to show that δ_{2p^3+1} does not contain the term $(\beta\mathscr{P}^1\beta\eta_2)^{p^2-p}(\mathscr{P}^1\beta\eta_2)$. For the dimensional reason, we can put $\delta_{2p^3+1}=d(\beta\mathscr{P}^1\beta\eta_2)^{p^2-p}(\mathscr{P}^1\beta\eta_2)+\bar{\delta}_{2p^3+1}$ for some $d\in\boldsymbol{Z}/p$. Then we have $\mathscr{P}^p(\tau(x_{2p^3}))=d(\mathscr{P}^{\Delta_1}\beta\eta_2)(\beta\mathscr{P}^1\beta\eta_2)^{p^2-p}+\mathscr{P}^p(\bar{\delta}_{2p^3+1})$, where $\mathscr{P}^p(\bar{\delta}_{2p^3+1})$ does not contain the term $(\mathscr{P}^{\Delta_1}\beta\eta_2)(\beta\mathscr{P}^1\beta\eta_2)^{p^2-p}$. This implies that $d=0$ since $\mathscr{P}^p(x_{2p^3})=0$, and we have the required conclusion.

Then, this causes a contradiction, and we have completed the proof of Theorem B.

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References

[1] J. Aguadé, C. Broto and D. Notbohm, Homotopy classification of spaces with interesting cohomology and a conjecture of Cooke, Part I, Topology **33** (1994), 455–492.

[2] J. Aguadé, C. Broto and M. Santos, Fake three connected coverings of Lie groups, Duke Math. J. **80** (1995), 91–103.

[3] J. Aguadé and L. Smith, Modular Cohomology Algebras, Amer. J. Math. **107** (1985), 507–530.

[4] A. Bousfield and D. Kan, Homotopy Limits, Completion and Localizations, Springer Lecture Notes in Math. **304** (1972).

[5] G. E. Cooke, J. R. Harper and A. Zabrodsky, Torsion Free Mod *p* *H*-spaces of Low Rank,

Topology **18** (1979), 349–359.
[6] W. G. Dwyer, H. Miller and C. W. Wilkerson, The homotopical uniqueness of classifying spaces, Topology **31** (1992), 29–45.
[7] J. R. Harper, *H*-spaces with Torsion, Memoirs Amer. Math. Soc. **223** (1979).
[8] J. Lannes, Sur les espaces fonctionnels dont la source est le classifiant d'un *p*-groupe abélien élémentaire, Publ. Math. I. H. E. S. **75** (1992), 135–244.
[9] M. Mimura and H. Toda, Cohomology operations and the homotopy of compact Lie groups-I, Topology **9** (1970), 317–336.