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Inductive limit of general linear groups

By

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0. Introduction

Let $G = \varinjlim G_n$ be the inductive limit of an inductive system of topological groups $G_n \to G_{n+1}$ $(n \in \boldsymbol{N})$.

Recently, Tatsuuma ([4] or [5]) remarked:

(1) The inductive limit topology of G does not always give a topological group structure on G.

(2) If G_n is locally compact for any n, then the inductive limit topology defines a topological group structure on G.

According to his second remark, for example, $GL(\boldsymbol{C}) = \varinjlim GL_n(\boldsymbol{C})$ is a topological group by the inductive limit topology. However, even in this particular case, it is not quite straightforward to show the above fact.

On the other hand, for a time, the author uses the topological group structure of $GL(\boldsymbol{C})$ defined by the following way (3). This topology has a merit in application that one can explicitly write down a fundamental system of neighbourhoods of the unity.

(3) The full matrix algebra $M_n(\boldsymbol{C})$ is a Banach algebra, so the inductive limit locally convex space structure is considered on $M(\boldsymbol{C}) = \varinjlim M_n(\boldsymbol{C})$ [1]. Embed $M(\boldsymbol{C})$ in $\mathrm{End}_{\boldsymbol{C}}(\boldsymbol{C}^{\oplus\infty})$, where $\boldsymbol{C}^{\oplus\infty}$ is the countable direct sum of $\boldsymbol{C}$. Translate the topology on $M(\boldsymbol{C})$ to that on $1 + M(\boldsymbol{C})$, then reduce it to $GL(\boldsymbol{C}) \subset 1 + M(\boldsymbol{C})$. Thus we get a topological group structure on $GL(\boldsymbol{C})$.

In this paper we shall describe the following results.

(i) In §1, we shall show that the topology of $GL(\boldsymbol{C})$ defined in (3), in fact, coincides with the inductive limit topology as topological spaces.

(ii) In §2, we discuss a similar problem for $GL_n(\Lambda)$, where $\Lambda = C(X, \boldsymbol{C})$ is the Banach algebra of complex valued continuous functions on a compact space X. In this case, since Λ is not locally compact, the inductive limit topology as topological spaces does not give a group topology. But the topology obtained by (3) (but using Λ instead of $\boldsymbol{C}$) gives a group topology on $GL(\Lambda)$, and it coincides with the BS-topology in Tatsuuma's sense.

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(iii) In §3, apart from general linear groups, we shall show partial converse to Tatsuuma's remark (2) in a general setting. Namely, we prove that under some additional conditions, for an inductive sequence of *non-locally compact* groups, the inductive limit topology as topological spaces is not a group topology. The proof can be applied to all examples previously known.

1. Inductive limit of $GL_n(C)$

Consider the inductive limit of general linear groups;

$$G_n = GL_n(C), \qquad G = GL(C) = \varinjlim_n GL_n(C),$$

and that of full matrix algebras;

$$M_n = M_n(C), \qquad M = M(C) = \varinjlim_n M_n(C).$$

Here for $n < m$ we identify $x \in G_n$ with $\begin{pmatrix} x & 0 \\ 0 & 1_{m-n} \end{pmatrix} \in G_m$, and $x \in M_n$ with $\begin{pmatrix} x & 0 \\ 0 & 0_{m-n} \end{pmatrix} \in M_m$.

For the inductive limit topology as topological spaces, the system of neighbourhoods of 1 is given by;

$$\mathfrak{U}_1 = \{U \mid 1 \in U \subset G, \ {}^\forall n \in N \ U \in G_n \text{ is open in } G_n\}.$$

We consider the inductive limit locally convex vector topology on M, and translate it to $1+M$, then reduce the topology on $GL \subset 1+M$.

The system of neighbourhoods of 1 in the obtained topology is given by;

$$\mathfrak{U}_2 = \{U(\{\varepsilon_n\}_{n=1}^{\infty}) \mid \varepsilon_1 \geq \varepsilon_2 \geq \cdots > 0\},$$

$$\text{where } U(\{\varepsilon_n\}_{n=1}^{\infty}) = \left\{ 1 + \sum_n x_n \in G \,\middle|\, x_n \in M_n \ \sum_n \frac{\|x_n\|_n}{\varepsilon_n} < 1 \right\}.$$

($\|x_n\|_n$ is the operator norm of x_n in $M_n(C)$.)

The purpose of this section is to prove that these two topologies coincide.

First we prove that G becomes a topological group in the topology $\mathfrak{U}_2$. It is enough to check the following (1) through (5).

(1) $1 \in U(\{\varepsilon_n\})$. This is obvious.

(2) $U(\{\mathrm{Min}(\varepsilon_n, \varepsilon'_n)\}) \subset U(\{\varepsilon_n\}) \cap U(\{\varepsilon'_n\})$. This is also obvious.

(3) We show ${}^\forall\{\varepsilon_n\}_{n=1}^{\infty}$, ${}^\exists\{\varepsilon'_n\}_{n=1}^{\infty}$, $U(\{\varepsilon'_n\})^{-1} \subset U(\{\varepsilon_n\})$.

For given $\varepsilon_1 \geq \varepsilon_2 \geq \cdots > 0$, choose $\{\varepsilon'_n\}$ such that

$$0 < \varepsilon'_1 < 1 \qquad \frac{\varepsilon'_1}{1-\varepsilon'_1} < \frac{\varepsilon_1}{2}$$

$$0 < \varepsilon'_2 < \mathrm{Min}(\varepsilon'_1, 1-\varepsilon'_1) \qquad \frac{\varepsilon'_1 + \varepsilon'_2}{1-(\varepsilon'_1+\varepsilon'_2)} - \frac{\varepsilon'_1}{1-\varepsilon'_1} < \frac{\varepsilon_2}{2^2}$$

$$0<\varepsilon_3'<\mathrm{Min}(\varepsilon_2', 1-\varepsilon_1'-\varepsilon_2') \qquad \frac{\varepsilon_1'+\varepsilon_2'+\varepsilon_3'}{1-(\varepsilon_1'+\varepsilon_2'+\varepsilon_3')}-\frac{\varepsilon_1'+\varepsilon_2'}{1-(\varepsilon_1'+\varepsilon_2')}<\frac{\varepsilon_3}{2^2}$$
$$\vdots$$

For $x=1+\sum_{n:\text{finite sum}} x_n$, suppose that $\sum_n \frac{\|x_n\|_n}{\varepsilon_n'}<1$. Then

$$x^{-1}=1+\sum_{k=1}^{\infty}(-1)^k\left(\sum_n x_n\right)^k=1+\sum_{k=1}^{\infty}\sum_{i_1\cdots i_k}(-x_{i_1})\cdots(-x_{i_k}).$$

We put $\mathrm{Max}(i_1,\cdots,i_k)=n$, then we have $\|x_{i_1}\cdots x_{i_k}\|_n\le\|x_{i_1}\|_n\cdots\|x_{i_k}\|_n=\|x_{i_1}\|_{i_1}\cdots\|x_{i_k}\|_{i_k}\le\varepsilon_{i_1}'\cdots\varepsilon_{i_k}'$, so the norm of the sum of all such monomials is evaluated as $\le(\varepsilon_1'+\cdots\varepsilon_n')^k-(\varepsilon_1'+\cdots+\varepsilon_{n-1}')^k$.

Varying k, we denote with x_n' for the sum of all monomials satisfying $\mathrm{Max}(i_1,\cdots,i_k)=n$ (infinite sum), then we get

$$\|x_n'\|_n\le\frac{\varepsilon_1'+\cdots+\varepsilon_n'}{1-(\varepsilon_1'+\cdots+\varepsilon_n')}-\frac{\varepsilon_1'+\cdots+\varepsilon_{n-1}'}{1-(\varepsilon_1'+\cdots+\varepsilon_{n-1}')}<\frac{\varepsilon_n}{2^n}.$$

Thus we have

$$x^{-1}=1+\sum_n x_n' \qquad \sum_n\frac{\|x_n'\|_n}{\varepsilon_n}<\sum_{n-1}^{\infty}\frac{1}{2^n}=1.$$

This means that $U(\{\varepsilon_n'\})^{-1}\subset U(\{\varepsilon_n\})$.

(4) We shall show that ${}^\forall\{\varepsilon_n\}_{n=1}^{\infty}$, ${}^\exists\{\varepsilon_n'\}_{n=1}^{\infty}$, $U(\{\varepsilon_n'\})^2\subset U(\{\varepsilon_n\})$.

For a given $\varepsilon_1\ge\varepsilon_2\ge\cdots>0$, choose $\{\varepsilon_n'\}$ such that

$$\varepsilon_1'>0 \qquad 2\varepsilon_1'+\varepsilon_1'^2<\varepsilon_1/2$$
$$0<\varepsilon_2'<\varepsilon_1' \qquad 2\varepsilon_2'+2\varepsilon_1'\varepsilon_2'+\varepsilon_2'^2<\varepsilon_2/2^2$$
$$0<\varepsilon_3'<\varepsilon_2' \qquad 2\varepsilon_3'+2\varepsilon_1'\varepsilon_3'+2\varepsilon_2'\varepsilon_3'+\varepsilon_3'^2<\varepsilon_3/2^3$$
$$\vdots$$

For $x=1+\sum_{n:\text{finite sum}} x_n$ and $y=1+\sum_{n:\text{finite sum}} y_n$, we assume $\sum_n\frac{\|x_n\|_n}{\varepsilon_n'}<1$ and $\sum_n\frac{\|y_n\|_n}{\varepsilon_n'}<1$. We write $x_n'=x_n+y_n+\sum_{k=1}^{n-1}(x_ky_n+x_ny_k)+x_ny_n$, then $xy=1+\sum_{n:\text{finite sum}} x_n'$ and

$$\|x_n'\|_n\le 2\varepsilon_n'+2\sum_{k=1}^{n-1}\varepsilon_k'\varepsilon_n'+\varepsilon_n'^2<\varepsilon_n/2^n$$

This means that $U(\{\varepsilon_n'\})^2\subset U(\{\varepsilon_n\})$.

(5) We show ${}^\forall g\in G$, ${}^\forall\{\varepsilon_n\}_{n=1}^{\infty}$, ${}^\exists\{\varepsilon_n'\}_{n=1}^{\infty}$, $gU(\{\varepsilon_n'\})g^{-1}\subset U(\{\varepsilon_n\})$.

For a given $g\in G$, we have $g\in GL_k(\boldsymbol{C})$ for some k.

For any $U(\{\varepsilon_n\})\in\mathfrak{U}_2$, take $\{\varepsilon'_n\}$ as $\varepsilon'_n=\dfrac{\varepsilon_{\mathrm{Max}(n,k)}}{\mathrm{Max}(1,\|g\|_k)\mathrm{Max}(1,\|g^{-1}\|_k)}$.

Suppose that $x=1+\sum_n x_n\in U(\{\varepsilon'_n\})$.

If we put $x'_1=\cdots=x'_{k-1}=0$, $x'_k=g(x_1+\cdots+x_k)g^{-1}$, and $x'_n=gx_ng^{-1}$ $(n\geq k+1)$, then we have $gxg^{-1}=1+\sum_n x'_n$, and

$$\|x'_n\|_n\leq\|x_n\|_n\mathrm{Max}(1,\|g\|_k)\mathrm{Max}(1,\|g^{-1}\|_k)\qquad\text{for } n\geq k+1$$

$$\|x'_k\|_k\leq\|x_1+\cdots+x_k\|_k\|g\|_k\|g^{-1}\|_k\leq\sum_{i=1}^k\|x_i\|_i\|g\|_k\|g^{-1}\|_k.$$

This implies that $\dfrac{\|x'_n\|_n}{\varepsilon_n}\leq\dfrac{\|x'_n\|_n}{\varepsilon'_n}$ for $n\geq k+1$ and $\dfrac{\|x'_k\|_k}{\varepsilon_k}\leq\sum_{i=1}^k\dfrac{\|x_i\|_i}{\varepsilon'_i}$. This means that $gU(\{\varepsilon'_n\})g^{-1}\subset U(\{\varepsilon_n\})$.

From Tatsuuma's paper, G becomes a topological group in the inductive limit topology $\mathfrak{U}_1$. Using this result, we will prove that $\mathfrak{U}_1$ coincides with $\mathfrak{U}_2$.

First we prove that

$$\forall n,\qquad U(\{\varepsilon_n\})\cap G_n=\left\{1+\sum_{k=1}^n x_n\in G\,\middle|\,x_k\in M_k\ \sum_{k=1}^n\frac{\|x_k\|_k}{\varepsilon_k}<1\right\}.$$

$\supset$ is obvious. Conversely, assume $1+\sum_{k=1}^N x_k\in U(\{\varepsilon_n\})\cap G_n$. It is enough to consider only when $N>n$. We put $x'_k=x_k$ $(k<n)$, $x'_n=\sum_{k=n}^N x_k$. Since ε_k decreases monotonically, $1>\sum_k\dfrac{\|x_k\|_k}{\varepsilon_k}>\sum_k\dfrac{\|x_k\|_k}{\varepsilon_{\mathrm{Min}(k,n)}}\geq\sum_{k=1}^n\dfrac{\|x'_k\|_k}{\varepsilon_k}$, because $\|x'_n\|_n=\|x'_n\|_N\leq\sum_{k=n}^N\|x_k\|_k$. So $\subset$ has been proved.

$U(\{\varepsilon_n\})\cap G_n$ is an open neighbourhood of 1 in G_n. So $U(\{\varepsilon_n\})$ is a neighbourhood of 1 in the inductive limit topology of G. That is, $\mathfrak{U}_1$ is stronger than $\mathfrak{U}_2$.

Conversely, take an arbitrary $U\in\mathfrak{U}_1$. Since G is a topological group in the topology $\mathfrak{U}_1$, we have

$$\begin{array}{ll}\exists U_1\in\mathfrak{U}_1 & U_1^2\subset U\\ \exists U_2\in\mathfrak{U}_1 & U_2^2\subset U_1\\ \exists U_3\in\mathfrak{U}_1 & U_3^2\subset U_2\end{array}$$

Then we have $U\supset U_1^2\supset U_2^2U_1\supset U_3^2U_2U_1\supset\cdots$. Since $U_n\cap G_n$ is open in G_n, we get $\exists\varepsilon_n>0$, $U_n\supset\{1+x\mid x\in M_n,\ \|x\|_n<\varepsilon_n\}$.

We can choose $\{\varepsilon_n\}_{n=1}^\infty$ so that $\varepsilon_n<\dfrac{1}{2^n}$, $\varepsilon_1>\varepsilon_2>\cdots>0$. Choose $\{\varepsilon'_n\}$ as follows;

$$\begin{aligned}
&\varepsilon_1 > \varepsilon_1' > 0 \\
&\mathrm{Min}(\varepsilon_1', \varepsilon_2) > \varepsilon_2' > 0 \quad (1-\varepsilon_1')^{-1}\varepsilon_2' < \varepsilon_2 \\
&\mathrm{Min}(\varepsilon_2', \varepsilon_3) > \varepsilon_3' > 0 \quad \{1-(\varepsilon_1'+\varepsilon_2')\}^{-1}\varepsilon_3' < \varepsilon_3 \\
&\mathrm{Min}(\varepsilon_3', \varepsilon_4) > \varepsilon_4' > 0 \quad \{1-(\varepsilon_1'+\varepsilon_2'+\varepsilon_3')\}^{-1}\varepsilon_4' < \varepsilon_4 \\
&\qquad\qquad\qquad\qquad\vdots
\end{aligned}$$

Suppose that $x = 1 + \sum_{n=1}^{N} x_n \in U(\{\varepsilon_n'\})$, then we get $x\left(1 + \sum_{n=1}^{N-1} x_n\right)^{-1} \in U_N$ as follows.

$$x\left(1 + \sum_{n=1}^{N-1} x_n\right)^{-1} = 1 + x_N\left(1 + \sum_{n=1}^{N-1} x_n\right)^{-1},$$

$$\left\| x_N\left(1 + \sum_{n=1}^{N-1} x_n\right)^{-1} \right\|_N \leq \varepsilon_N'\left(1 - \sum_{n=1}^{N-1} \varepsilon_n'\right)^{-1} < \varepsilon_N.$$

In a similar way, we get

$$\left(1 + \sum_{n=1}^{N-1} x_n\right)\left(1 + \sum_{n=1}^{N-2} x_n\right)^{-1} \in U_{N-1}.$$

In this way we finally get

$$x \in U_N \cdot U_{N-1} \cdots U_1 \subset U.$$

This implies that $U(\{\varepsilon_n'\}) \subset U$, so $\mathfrak{U}_2$ is stronger than $\mathfrak{U}_1$. Therefore two topologies coincide.

2. The case of $GL(\Lambda)$, $\Lambda = C(X, \boldsymbol{C})$

In this section, let X be a compact topological space, $\Lambda = C(X, \boldsymbol{C})$ be the set of all complex valued continuous functions on X, then Λ becomes a Banach algebra over $\boldsymbol{C}$ with the uniforfm norm $\|f\| = \mathrm{Max}_{x \in X} |f(x)|$. Λ^n is also a Banach space with the norm $\|a\|_n = \mathrm{Max}_i \|a_i\|$ for $a = (a_i) \in \Lambda^n$. The full matrix algebra $M_n(\Lambda)$ also becomes a Banach algebra over $\boldsymbol{C}$ with the operator norm on $M_n(\Lambda)$. As was in the case of $M(\boldsymbol{C})$, we consider the inductive limit locally convex vector topology on $M(\Lambda) = \varinjlim M_n(\Lambda)$, then $M(\Lambda)$ becomes a topological ring. Translating this topology to $1 + M(\Lambda)$ and reducing it on $GL(\Lambda)$, we get a topological group $GL(\Lambda)$.

We shall define an isomorphism $M_n(\Lambda) \to C(X, M_n(\boldsymbol{C}))$ by identifying $(a_{ij}(x))$ with $x \mapsto (a_{ij}(x))$. This is an isometric isomorphism as Banach algebras as shown below. The norm of $(a_{ij}(x))$ in $M_n(\Lambda)$ is given by, for $t=(t_j)\in\Lambda^n$;

$$\sup_{\|t\|_n\leq 1}\left\|\sum_{j=1}^{n} a_{ij}(x)t_j(x)\right\|_n = \sup_{\|t\|_n\leq 1}\ \operatorname*{Max}_{i}\ \operatorname*{Max}_{x\in X}\left|\sum_{j=1}^{n} a_{ij}(x)t_j(x)\right|$$
$$=\sup\left\{\left|\sum_{j=1}^{n} a_{ij}(x)t_j(x)\right|\ \middle|\ \text{condition } (*)\right\},$$

where the condition$(*)$ is given by $1\leq i\leq n$, $x\in X$, and $\operatorname*{Max}_{j}\ \operatorname*{Max}_{x\in X}|t_j(x)|\leq 1$.

On the other hand, the norm of $(a_{ij}(x))$ in $C(X,M_n(\boldsymbol{C}))$ is given by, for $\tau=(\tau_j)\in\boldsymbol{C}^n$;

$$\operatorname*{Max}_{x\in X}\|a_{ij}(x)\|_{M_n(\boldsymbol{C})}=\operatorname*{Max}_{x\in X}\ \operatorname*{Max}_{\|\tau\|_n\leq 1}\left\|\sum_{j=1}^{n} a_{ij}(x)\tau_j\right\|_n=\operatorname*{Max}_{x\in X}\ \operatorname*{Max}_{\|\tau\|_n\leq 1}\ \operatorname*{Max}_{i}\left|\sum_{j=1}^{n} a_{ij}(x)\tau_j\right|$$
$$=\operatorname{Max}\left\{\left|\sum_{j=1}^{n} a_{ij}(x)\tau_j\right|\ \middle|\ \text{condition}(**)\right\},$$

where the condition$(**)$ is given by $1\leq i\leq n$, $x\in X$, and $\operatorname*{Max}_{j}|\tau_j|\leq 1$.

Especially putting $t_j(x)=\text{const.}=\tau_j$, we see that the norm in $M_n(\Lambda)$ is larger than or equal to the norm in $C(X,M_n(\boldsymbol{C}))$. Conversely we choose $i_0, x_0, \{t_j^0(x)\}_{j=1}^n$ such that $\left|\sum_{j=1}^{n} a_{i_0,j}(x_0)t_j^0(x_0)\right|$ is arbitrarily close to the norm in $M_n(\Lambda)$. Putting $t_j^0(x_0)=\tau_j$, we easily see that the norm in $C(X,M_n(\boldsymbol{C}))$ is larger than or equal to the norm in $M_n(\Lambda)$. Thus both norms coincide.

Since $GL_n(\Lambda)$ and $C(X,GL_n(\boldsymbol{C}))$ are the multiplicative groups of all invertible elements of $M_n(\Lambda)$ and $C(X,M_n(\boldsymbol{C}))$ respectively, they are mutually isomorphic as topological groups. Taking the inductive limit, we can embed $GL(\Lambda)$ into $C(X,GL(\boldsymbol{C}))$.

Theorem 1. *For the topology induced from the inductive limit locally convex vector topology of $M(\Lambda)$, $GL(\Lambda)$ is isomorphic to $C(X,GL(\boldsymbol{C}))$ as topological groups.*

Proof. First we shall show that the embedding above is surjective, so that it is actually an isomorphism as abstract groups. For $f\in C(X,GL(\boldsymbol{C}))$, $f(X)$ is a compact subset of $GL(\boldsymbol{C})$. If we show that $f(X)\subset GL_n(\boldsymbol{C})$ for some n, since f is continuous in the norm $\|\cdot\|_n$ of $GL_n(\boldsymbol{C})$, we have $f\in GL_n(\Lambda)\subset GL(\Lambda)$.

Thus it is sufficent to prove that a compact subset C of $GL(\boldsymbol{C})$ is contained in $GL_n(\boldsymbol{C})$ for some n. Assume that $^\forall n\in\boldsymbol{N}$, $^\exists c_n\in C$, $c_n\notin GL_n(\boldsymbol{C})$. Choose ε_n such that $\operatorname*{Max}_{\operatorname{Max}(i,j)>n}|c_{nij}-\delta_{i,j}|>\varepsilon_n \operatorname{Max}_{m\leq n}\|c_m\|>0$, ε_n can be taken to be monotonically decreasing. For this $\{\varepsilon_n\}$, we consider the neighbourhood of 1,

$$U=U(\{\varepsilon_n\}_{n=1}^{\infty})=\left\{1+\sum_n x_n\ \middle|\ x_n\in M_n(\boldsymbol{C}),\ \sum_n\frac{\|x_n\|_n}{\varepsilon_n}<1\right\}.$$

We shall show that for every $m, c_n \notin c_m U$ for sufficiently large n, so that C is not compact. Assume that $c_m \in GL_k(C)$, $n > k$, and $c_m^{-1} c_n = 1 + \sum_{j=1}^{l} x_j$, $x_j \in M_j(C)$, $l > n$.Since $c_n \in 1 + M_n(C) + c_m(x_{n+1} + \cdots + x_l)$, we get $\varepsilon_n \leq \|x_{n+1} + \cdots + x_l\|_l \leq \|x_{n+1}\|_{n+1} + \cdots + \|x_l\|_l$. $\therefore \sum_{j=n+1}^{l} \frac{\|x_j\|_j}{\varepsilon_j} \geq \frac{1}{\varepsilon_n} \sum_{j=n+1}^{l} \|x_j\|_j \geq 1$. Thus $c_m^{-1} c_n \notin U$.

Next we shall show that the topologies of $GL(\Lambda)$ and $C(X, GL(C))$ coincide. The system of neighbourhoods of 1 in $GL(\Lambda)$ is obtained by

$$\mathfrak{U}_1 = \{U_1(\{\varepsilon_n\}_{n=1}^{\infty}) \mid \varepsilon_1 \geq \varepsilon_2 \geq \cdots > 0\}.$$

$$\text{where } U_1(\{\varepsilon_n\}_{n=1}^{\infty}) = \left\{1 + \sum_n f_n \,\middle|\, f_n \in M_n(\Lambda),\ \sum_n \frac{\|f_n\|_n}{\varepsilon_n} < 1\right\}.$$

The sysytem of neighbourhoods of 1 in $C(X, M_n(C))$ is obtained by

$$\mathfrak{U}_2 = \{U_2(\{\varepsilon_n\}_{n=1}^{\infty}) \mid \varepsilon_1 \geq \varepsilon_2 \geq \cdots > 0\}.$$

$$\text{where } U_2(\{\varepsilon_n\}_{n=1}^{\infty}) = \{f \mid {}^{\forall}x \in X,\ f(x) \in U(\{\varepsilon_n\}_{n=1}^{\infty})\},$$

$$U(\{\varepsilon_n\}_{n=1}^{\infty}) = \left\{y = 1 + \sum_n y_n \,\middle|\, y_n \in M_n(C)\ \sum_n \frac{\|y_n\|_n}{\varepsilon_n} < 1\right\}.$$

We shall show that $U_1(\{\varepsilon_n\}) \subset U_2(\{\varepsilon_n\})$ below.

Suppose $f \in U_1(\{\varepsilon_n\})$, then for any $x \in X$, putting $y_n = f_n(x)$, we see $f(x) \in U(\{\varepsilon_n\})$ because $\|f_n(x)\|_n \leq \|f_n\|_n$, so that $f \in U_2(\{\varepsilon_n\})$.

Conversely we shall show that $U_2\left(\left\{\frac{\varepsilon_n}{2^n}\right\}\right) \subset U_1(\{\varepsilon_n\})$.

Assume that $f \in U_2\left(\left\{\frac{\varepsilon_n}{2^n}\right\}\right)$ and $f(X) \subset GL_N(C)$.

For any $x \in X$, we can write as $f(x) = 1 + \sum_{n=1}^{N} y_n$, $y_n \in M_n(C)$, $\|y_n\|_n < \frac{\varepsilon_n}{2^n}$. For a fixed $x_0 \in X$, we choose such $\{y_n\}$. For $n < N$ we define $y_n(x) = y_n$, $y_N(x) = f(x) - 1 - \sum_{n=1}^{N-1} y_n$. Since $y_N(x_0) = y_N$, for an open neighbourhood U_{x_0} of x_0, we see $\|y_N(x)\|_N < \frac{\varepsilon_N}{2^N}$ for $x \in U_{x_0}$. Since X is compact, there exists a finite set A such that $\{U_{x_\alpha}\}_{\alpha \in A}$ is an open covering of X. With respect to this open covering, consider a decomposition of unity $\{f_\alpha\}_{\alpha \in A}$. ($f_\alpha(x)$ is a continuous function $X \to [0, 1]$; $f_\alpha(x) = 0$ for $x \notin U_{x_\alpha}$ and $\sum_\alpha f_\alpha = 1$.) For $n \leq N$, we put $f_n(x) = \sum_\alpha y_{\alpha,n}(x) f_\alpha(x)$. For $n < N$, we have $\|f_n(x)\| \leq \sum_\alpha \|y_{\alpha,n}\| f_\alpha(x) < \frac{\varepsilon_n}{2^n}$. Since $\|y_{\alpha,N}(x)\| < \frac{\varepsilon_n}{2^N}$ if $f_\alpha(x) \neq 0$, this inequality also holds for $n = N$. So $f = 1 + \sum_{n=1}^{N} f_n$ with $\Sigma_{n=1}^{N} \frac{\|f_n\|_n}{\varepsilon_n} \leq \Sigma_{n=1}^{N} \frac{1}{2^n} < 1$, which means

$f \in U_1(\{\varepsilon_n\})$.

Thus the two topologies coincide.

Remark. The topology above is defined for any $GL(\Lambda)$, where Λ is an arbitrary Banach algebra. This topology also coincides with the BS-topology in Tatsuuma's sense. Similarly as in §1, we see that this topology is the strongest topology in group topologies. Since the BS-topology is also so, both topologies coincide. But we have to check the Tatsuuma's condition (PTA)

$$^\forall n,\ ^\forall U,\ ^\exists V \subset U,\ V = V^{-1},\ ^\forall m > n,\ ^\forall W,\ ^\exists W',\ W'V \subset VW.$$

(U, V are neighbourhoods of 1 in $GL_n(\Lambda)$, W, W' are neighbourhoods of 1 in $GL(\Lambda)$.)

Suppose that $V \subset \{1+x \mid \|x\|_n < 1\}$ and $V = V^{-1}$ holds. If $W \supset \{1+y \mid \|y\|_m < \varepsilon\}$, it is enough to choose $W' = \{1+y \mid \|y\|_m < \frac{\varepsilon}{4}\}$ as follows. Let $w = 1+y \in W'$, $\|y\|_m < \frac{\varepsilon}{4}$. For $v \in V$, we only have to show $v^{-1}wv \in W$ because $wv = v(v^{-1}wv)$. But it is O.K. because $v^{-1}wv = 1 + v^{-1}yv$ and $\|v^{-1}yv\|_m \leq \|v^{-1}\|_m \|y\|_m \|v\|_m \leq 4\|y\|_m < \varepsilon$ ($\because \|v\|_m \leq 1 + \|x\|_m = 1 + \|x\|_n < 2$).

Next we ask if this topology coincides with the inductive limit topology as topological spaces. This is true if and only if $GL(\Lambda)$ becomes a topological group with the inductive limit topology. According to Tatsuuma's paper, if all G_n are locally compact, then $G = \varinjlim G_n$ becomes a topological group with respect to the inductive limit topology as topological spaces. However, it can also be showed that locally compactness is close to a necessary condition.

Up to the present, we know two kinds of countr examples.

Tatsuuma's counter example: the additive group of $\boldsymbol{Q}^n$(or $\boldsymbol{Q} \times \boldsymbol{R}^n$).

Hirai-Shimomura's counter example: For a σ compact differentiable manifold M, the group $Diff_0(M)$ of all diffeomorphisms which are the identity map outside some compact set.

Here we can add one more couter example, namely the general linear group $GL(\Lambda)$ for a non-locally compact Banach algebra Λ. The proof of "not being a topological group" is carried out only using non-locally compactness, as shown in the next section.

3. The converse of the Tatsuuma's theorem

We assume from now on that for $n < m$, $G_n \hookrightarrow G_m$ is a continuous injective homomorphism of topological groups. As Tatsuuma proved, if all G_n are locally compact, G becomes a topological group with respect to the inductive limit topology as topological spaces. This theorem can be generalized as follows.

Theorem 2. *Under the following condition* (LC), *G becomes a topological group with the inductive limit topology as topological spaces.*

(LC) $\qquad \forall n, \exists U, \exists m>n,\ \bar{U}^{(m)}$ *is compact.*

where U is a neighbourhood of 1 *in G_n and $\bar{U}^{(m)}$ is the closure of U in G_m.*

Remark. If (LC) holds for some $m>n$, then it also holds for (n',m') where $n'\leq n$, $m\leq m'$, as shown below. $U\cap G_{n'}$ is a neighbourhood of 1 in $G_{n'}$, and $\overline{U\cap G_{n'}}^{(m)}\subset \bar{U}^{(m)}$. Since $\bar{U}^{(m)}$ is compact in G_m, it is compact also in $G_{m'}$ so it is closed in $G_{m'}$ and contains $\overline{U\cap G_{n'}}^{(m')}$.

Proof of the theorem. By taking a subsequence of $\{G_n\}$, we can assume that some neighbourhood U of 1 in G_n is relatively compact in G_{n+1}. We divide the proof of the theorem into two steps. We shall show that the Tatsuuma's condition (PTA) holds for $\{G_n\}$ (and the BS-topology can be defined), and that the inductive limit topology as topological spaces coincides with the BS-topology.

STEP 1. Let U be a neighbourhood of 1 in G_n, such that the closure $\bar{U}$ of U in G_{n+1} is compact.

For an open neighbourhood W of 1 in G_{n+1}, UW is open and contains the compact set $\bar{U}$, so that $W'\bar{U}\subset UW$ for some neighbourhood W' of 1.

STEP 2. It is enough to show that any open neighbourhood O of 1 in the inductive limit topology is also a neighbourhood of 1 in the BS-topology. Namely, it is enough to find a sequence $\{W_n\}$ of symmetric neighbourhoods of 1 in G_n such that

$$(*)\qquad \forall n,\ W_nW_{n-1}\cdots W_2W_1^2W_2\cdots W_{n-1}W_n\subset O_n(=O\cap G_n).$$

First we choose W_1 as follows. Let V_2 be a neighbourhood of 1 in G_2 such that $V_2^3\subset O_2$, then $\overline{V_2^2}^{(2)}\subset O_2$. Choose W_1 such that $V_2\cap G_1\supset W_1(=W_1^{-1})$ and $\bar{W}_1^{(2)}$ is compact, then we see that $(\bar{W}_1^{(2)})^2=\overline{W_1^2}^{(2)}\subset \overline{V_2^2}^{(2)}\subset O_2$.

By induction with respect to n, we shall prove that

$$(**)\qquad \bar{W}_n^{(n+1)}\cdots \bar{W}_2^{(3)}(\bar{W}_1^{(2)})^2\bar{W}_2^{(3)}\cdots \bar{W}_n^{(n+1)}\subset O_{n+1}$$

Then $(*)$ holds since $O_{n+1}\cap G_n=O_n$. We write the left hand side of $(**)$ by K_{n+1}. Assume that K_{n+1} is a compact subset of G_{n+1}, then we can choose a neighbourhood V_{n+2} of 1 in G_{n+2} so that $V_{n+2}K_{n+1}V_{n+2}\subset O_{n+2}$. If $V_{n+2}'^2\subset V_{n+2}$ holds, then $\overline{V_{n+2}'}^{(n+2)}\subset V_{n+2}$. Choose W_{n+1} such that $V_{n+2}'\cap G_{n+1}\supset W_{n+1}=W_{n+1}^{-1}$ and $\bar{W}_{n+1}^{(n+2)}$ is compact. Since $\bar{W}_{n+1}^{(n+2)}\subset \overline{V_{n+2}'}^{(n+2)}\subset V_{n+2}$, we see that $\bar{W}_{n+1}^{(n+2)}K_{n+1}\bar{W}_{n+1}^{(n+2)}\subset O_{n+2}$. Putting the left hand side as K_{n+2}, this completes the induction.

Rermark. If G_n is a closed subgroup of G_{n+1} (G_n is closed in G_{n+1} and homeomorphically embedded), the assumption of the theorem is equivalent to the locally compactness of each G_n. (For a neighbourhood U of 1 in G_n, $\bar{U}^{(n+1)}$ is contained in G_n and coincides with $\bar{U}^{(n)}$.)

Theorem 3. *Suppose that the injection $G_n \subsetneq G_{n+1}$ is a homeomorphism (i.e. G_n is a subgroup of G_{n+1} as topological groups). If G_1 is open in G_n for all n, then the inductive limit topology is a group topology. (The condition can be weakened as "$^{\exists}n, {}^{\forall}m>n$, G_n is open in G_m".)*

Proof. Since G_1 is an open subgroup of G_n, the system of neighbourhoods $\mathfrak{U}$ of 1 in G_1 is also so in G_n. Therefore $\mathfrak{U}$ is also the system of neighbourhoods of 1 in the inductive limit topology. Of the five conditions for $\mathfrak{U}$ to define a grouup topology, (1) $^{\forall}U\in\mathfrak{U}, 1\in U$, (2) $^{\forall}U, V\in\mathfrak{U}$, $^{\exists}W\in\mathfrak{U}$, $W\subset U\cap V$, (3) $^{\forall}U\in\mathfrak{U}$, $^{\exists}V\in\mathfrak{U}$, $V^{-1}\subset U$, (4) $^{\forall}U\in\mathfrak{U}$, $^{\exists}V\in\mathfrak{U}$, $V^2\subset U$ are obviously satisfied. For the last condition (5) $^{\forall}g\in G$, $^{\forall}U\in\mathfrak{U}$, $^{\exists}V\in\mathfrak{U}$, $gVg^{-1}\subset U$, since $g\in G_n$ for some n and $\mathfrak{U}$ is the system of neighbourhoods of 1 in G_n, it is obviously satisfied.

Now we consider the converse of the above two theorems. From now on, we assume the additional condition: "Every G_n has a countable system of neighbourhoods of 1", then the converse holds as the next theorem claims

Theorem 4. *We assume that the injection $G_n \subsetneq G_{n+1}$ is a homeomorphism and every G_n has a countable system of neighbourhoods of 1. We assume that neither condition of the above two theorems holds. Namely we assume that*

(1) $^{\exists}n_0$, $^{\forall}U$ *(a neighbourhood of 1 in G_{n_0})*, $^{\forall}m>n_0$, *$\bar{U}^{(m)}$ is not compact in G_m.*

(2) $^{\forall}n$, $^{\exists}m>n$, *G_n is not open in G_m.*

Then the inductive limit topology is not a group topology.

Remark. If we assume that G_n is a closed subgroup of G_{n+1}, then (1) is equivalent to that some G_n is not locally compact.

All the counter examples previously known are in the situation of this theorem, so its proof is valid for all such examples. We also see that *the locally-compactness* is very close to a necessary and sufficient condition.

Proof. The inductive limit topology does not change if we take a subsequence of $\{G_n\}$. Thus we can assume that $n_0=1$ in the condition (1), and that "G_n is not open in G_{n+1}" in the condition (2).

Suppose that the inductive limit topology is a group topology. For any neighbourhood U of 1, there exists a neighbourhood V of 1 such that $V^2\subset U$. Then putting $V\cap G_n=V_n$, we get the next result.

$$\text{“}{}^{\forall}n,\ {}^{\exists}V_n \text{ (a neighbourhood of 1 in } G_n),\ V_1V_n\subset U\cap G_n\text{”}$$

Thus the theorem will be proved if we find a sequence U_n of open neighbourhoods of 1 in G_n which satisfies

$$\text{“}1 \in U_1,\ U_{n+1} \cap G_n = U_n,\ {}^{\forall}V_1 \text{ (a neighbourhood of 1 in } G_1),$$
$$^{\exists}n > 1,\ {}^{\forall}V_n \text{ (a neighbourhood of 1 in } G_n),\ V_1 V_n \not\subset U_n\text{”}.$$

We write a countable system of neighbourhoods of 1 in G_1 as $\{V_{1,j}\}_{j=1}^{\infty}$. It is enough to find U_n inductively so that the following condition $(*)$ holds:

$(*)$ $\qquad {}^{\forall}n > 1,\ {}^{\forall}V_n$ (a neighbourhood of 1 in G_n), $V_{1,n}V_n \not\subset U_n$

($(*)$ is sufficient, since ${}^{\forall}V_1$ (a neighbourhood of 1 in G_1), ${}^{\exists}n$, $V_{1,n} \subset V_1$).

We choose an open neighbourhood U_1 of 1 in G_1 arbitrarily. (For example $U_1 = G_1$). We assume that U_k is defined for $k < n$.

From the assumption of the theorem, G_{n-1} is not open in G_n, and since G_n has a countable system of neighbourhoods of 1, ${}^{\exists}\{x_j\}_{j=1}^{\infty}$, $x_j \in G_n \backslash G_{n-1}$, $\lim_{j \to \infty} x_j = 1$.

Now since $\overline{V_{1,n}}^{(n)}$ is not compact in G_n, again by G_n having a countable system of neighbourhoods of 1, ${}^{\exists}\{y_j\}_{j=1}^{\infty}$, $y_j \in V_{1,n}$, $\{y_j\}$ does not have an accumulating point in G_n. We put $z_j = y_j x_j$, then $\{z_j\}$ does not have an accumulating point in G_n. ($\because z = \lim_{k \to \infty} y_{j_k} x_{j_k}$ implies $z = \lim_{k \to \infty} y_{j_k}$, since $\lim_{k \to \infty} x_{j_k} = 1$. Therefore $Z = \{z_j | 1 \leq j < \infty\}$ is closed in G_n.

Since $z_j \notin G_{n-1}$ ($\because x_j \notin G_{n-1}$ and $y_j \in G_1 \subset G_{n-1}$), $Z \cap G_{n-1} = \phi$. Therefore $G_n \backslash Z \supset G_{n-1} \supset U_{n-1}$. Since the injection $G_{n-1} \subsetneq G_n$ is homeomorphic, ${}^{\exists}U'_n$ (an open subset of G_n) $U'_n \cap G_{n-1} = U_{n-1}$. Then $U_n = U'_n \cap (G_n \backslash Z)$ is an open subset of G_n and $U_n \cap G_{n-1} = U_{n-1}$ holds. Since ${}^{\forall}j, z_j = y_j x_j \notin U_n$ and $x_j \to 1$ in $G_n, y_j \in V_{1,n}$, we obtain the desired result "${}^{\forall}V_n$ (a neighbourhood of 1 in G_n), $V_{1,n}V_n \not\subset U_n$".

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