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Module derivations and the adjoint action of a finite loop space

By

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0. Introduction

Let G be a finite loop space, in other words, a loop space with the homotopy type of a finite CW complex. Following Milnor's description of universal bundles over spaces [13], we regard the finite loop space as a topological group up to homotopy. Let LG be the loop group, which is the space of free loops on G, and ΩG the subgroup of LG consisting of based loops. We can identify the loop group LG with the product $\Omega G \times G$ as a space. The map $\Phi : \Omega G \times G \to LG$ defined by $\Phi(l, g)(t) = l(t) \cdot g$ is a homeomorphism which guarantees the identification. We define the adjoint action $Ad : G \times \Omega G \to \Omega G$ of G on ΩG by $Ad(g, l) = glg^{-1}$. If we give the space $\Omega G \times G$ the group structure so that $(l, g) \cdot (l', g') = (lAd(g, l'), gg')$, then the homeomorphism Φ is regarded as an isomorphism $\Omega G \times G \cong LG$. Therefore one may expect that it is useful for studies on both geometry and topology for loop groups to consider the adjoint action of G on ΩG. In fact, by calculating the adjoint action $Ad^* : H^*(\Omega G; \mathbf{Z}/p) \to H^*(G; \mathbf{Z}/p) \otimes H^*(\Omega G; \mathbf{Z}/p)$, we can determine the Hopf algebra structure of $H^*(LG; \mathbf{Z}/p)$. Moreover this result enables us to calculate the E_2-term of the Rothenberg-Steenrod spectral sequence converging to the p cohomology of the classifying space BLG of the loop group LG. Consequently, our knowledge on the structure of $H^*(BLG; \mathbf{Z}/p)$ will contribute to studies on characteristic classes of loop group bundles.

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1. Aims and results

Let G be a compact simply connected Lie group. In [7], Kono and Kozima give a good characterization of the triviality of the adjoint action $Ad^* : H^*(\Omega G; \mathbf{Z}/p) \to H^*(G; \mathbf{Z}/p) \otimes H^*(\Omega; \mathbf{Z}/p)$. Before stating the main theorem

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in [7], we explain briefly a fibration whose total space is the classifying space of a loop group. Let G be a finite loop space. Since ΩG is a closed normal subgroup of the loop group LG, we have a principal G-bundle $G \to B\Omega G \to BLG$. Hence BLG has the homotopy type of $B\Omega G \times_G EG$. Since the classifying space $B\Omega G$ is homotopy equivalent to G by a G-adjoint action preserving map, it follows that BLG has the homotopy type of $G \times_G EG$. Here the G-adjoint action $ad : G \times G \to G$ is defined by $(g, h) \mapsto ghg^{-1}$. Thus we have a fibration $G \xrightarrow{j} BLG \to BG$.

Theorem 1.1 ([7]). *Let G be a compact, simply connected Lie group and p a prime. Then the following three conditions are equivalent.*

(i) *$H^*(G; \mathbf{Z})$ has no p-torsion.*

(ii) *$Ad^* = pr_2^* : H^*(\Omega G; \mathbf{Z}/p) \to H^*(G; \mathbf{Z}/p) \otimes H^*(\Omega G; \mathbf{Z}/p)$, where pr_2 is the projection to the second factor.*

(iii) *There is an isomorphism of $H^*(BG; \mathbf{Z}/p)$-algebras $\phi : H^*(BLG; \mathbf{Z}/p) \cong H^*(BG; \mathbf{Z}/p) \otimes H^*(G; \mathbf{Z}/p)$ which satisfies $inc_2^* \circ \phi = j^*$, where $inc_2 : G \to BG \otimes G$ is the inclusion onto the second factor.*

The proof of Theorem 1.1 is based on studies on the classification of simple Lie groups and their cohomologies.

In [3], Iwase has generalized Theorem 1.1 to the case where G is a finite loop space and p is odd.

Theorem 1.2 ([3], Theorem 2.2, Theorem 2.3). *Let G be a simply connected finite loop space. For all odd prime p, the condition* (i), (ii) *and* (iii) *in Theorem* 1.1 *are also equivalent to any of the following four conditions:*

(iv) *The induced homomorphism $j^* : H^*(BLG; \mathbf{Z}/p) \to H^*(G; \mathbf{Z}/p)$ is surjective;*

(v) *The Hopf algebra $H^*(G; \mathbf{Z}/p)$ is primitively generated;*

(vi) *The Hopf algebra $H^*(G; \mathbf{Z}/p)$ is cocomutative;*

(vii) *There is an isomorphism of $H^*(BG; \mathbf{Z}/p)$-modules $\phi : H^*(BLG; \mathbf{Z}/p) \cong H^*(BG; \mathbf{Z}/p) \otimes H^*(G; \mathbf{Z}/p)$ which satisfies $inc_2^* \circ \phi = j^*$.*

Recently, Iwase and Kono have considered a generalization of Theorem 1.1 for the case where G is a simply connected finite loop space and $p = 2$.

Theorem 1.3 ([4]). *Let G be a simply connected finite loop space. At the prime* 2, *the conditions* (i), (ii) *and* (iii) *in Theorem* 1.1 *are equivalent.*

Let m be the product of G and $T : G \times G \to G \times G$ the switching mapping. In the proofs of Theorems 1.2 and 1.3, an explicit homotopy $H : I \times G \times G \to BLG$ which connects the map $j \circ m$ and $j \circ m \circ T$ plays an important role. In particular, the homotopy is needed to show the existence of 1-implication in $H^*(BLG; \mathbf{Z}/2)$, which is a key to prove Theorem 1.3. Therefore we can describe that the generalization of Theorem 1.1 to the case in which G is a finite loop space is completely made with a homotopy theoretic approach. On the other hand, we can also prove algebraically Theorem 1.2 by making use of the Eilenberg-Moore spectral sequence of the cobar type converging to $H^*(EG \times_G G; \mathbf{Z}/p) = H^*(BLG; \mathbf{Z}/p)$. (See Remark 1.4 below.)

One of our aims in this manuscript is to give another proof of Theorem 1.3 without using the explicit homotopy $H : I \times G \times G \to BLG$. One may describe that the approach is of homological algebra. Our great tools in the proof are the Rothenberg-Steenrod spectral sequence $\{{}_{RS}E_r^{*,*}(BLG), d_r\}$ converging to $H^*(BLG; \mathbf{Z}/p)$ whose E_2-term is isomorphic as an algebra to $Cotor_{H^*(LG;\mathbf{Z}/p)}(\mathbf{Z}/p, \mathbf{Z}/p)$ and the module derivation $\mathscr{D}_{BG} : H^*(BG; \mathbf{Z}/2) \to H^*(LBG; \mathbf{Z}/2)$ defined by $\int_{S^1} \circ\, ev^*$, where $ev : S^1 \times LBG \to BG$ is the evaluation map and $\int_{S^1}$ is the integration along S^1. In general, a module derivation is defined as a linear map with the degree -1 from an algebra A over a field to an A-module which satisfies the Leibniz rule on A. For the explicit definition see Section 3.

Theorem 1.2 asserts that the conditions (i)–(vii) form one equivalence class when the prime p is odd. The second aim is to consider how many equivalence classes are formed form the conditions (i)–(vii) in the case $p = 2$. To this end, in Remark 1.4 below, we clarify whether some results which are applied to prove Theorem 1.2 hold for $p = 2$. At the same time, we prove Theorem 1.2 without using the homotopy $H : I \times G \times G \to BLG$.

Remark 1.4. The method of the proof of [7, Proposition 3.3] also works for our case where G is a simply connected finite loop space and $p = 2$. Hence we see that (i) implies (ii) in Theorem 1.3. Moreover, since the classifying space BLG and the loop space LBG are of the same homotopy type, it follows from the argument in [7, §4] that (i) implies (iii) in Theorem 1.3. The result [5, Theorem 1.1] of Kane allows us to conclude that (v) and (vi) in Theorem 1.2 are equivalent for any prime p. Since G is totally non-homologous to zero in BLG with respect to $\mathbf{Z}/p$ if and only if the Leray-Serre spectral sequence for the fibration $G \xrightarrow{j} BLG \to BG$ collapses at the E_2-term, it is clear that the conditions (iv) and (vii) in Theorem 1.2 are equivalent for any prime p. In [3, Section 6], Iwase proves that (ii) implies (vi). The method of the proof also works for any prime p. Let $\{E_r, d_r\}$ be the Eilenberg-Moore spectral sequence converging to $H^*(BLG; \mathbf{Z}/p)$ whose E_2-term is isomorphic to $Cotor_{H^*(G;\mathbf{Z}/p)}(\mathbf{Z}/p, H^*(G; \mathbf{Z}/p))$ as an algebra. Here the left comodule structure of $H^*(G; \mathbf{Z}/p)$ is given by the adjoint action ad of G on itself. By considering the cobar complex of $H^*(G; \mathbf{Z}/p)$, we see that $ad^* = d_1 : E_1^{0,*} = H^*(G; \mathbf{Z}/p) \to E_1^{1,*} = H^*(G; \mathbf{Z}/p) \otimes \bar{H}^*(G; \mathbf{Z}/p)$, where $\bar{H}^*(G; \mathbf{Z}/p)$ denotes $\bigotimes_{i \geq 1} H^i(G; \mathbf{Z}/p)$. Moreover, by using the fact that the edge homomorphism

$$H^*(BLG; \mathbf{Z}/p) \to E_\infty^{0,*} \hookrightarrow \cdots \hookrightarrow E_2^{0,*} \hookrightarrow H^*(G; \mathbf{Z}/p)$$

is the homomorphism $j^* : H^*(BLG; \mathbf{Z}/p) \to H^*(G; \mathbf{Z}/p)$, we can see that (iv) implies (v) for any prime p. (See also [4, Section 4].) We stress that the homotopy $H : I \times G \times G \to BLG$ mentioned above is not need in this proof though the original proof [3] relies on the homotopy. The result [1, Theorem 1] of

Browder states that the condition (vi) implies (i) unless $p = 2$. Thus we can have Theorem 1.2.

For the case $p = 2$, the condition (v) does not imply (i) in general. As such an example, we can give the case where G is the exceptional Lie group G_2. Since the conditions (iv)–(vii) hold in the case $(G, p) = (G_2, 2)$, one may conjecture that, for $p = 2$, the conditions (i)–(vii) are separated into two equivalence classes $\mathscr{A}$ consisting of (i)–(iii) and $\mathscr{B}$ of (iv)–(vii). Of course the existence of the class $\mathscr{A}$ is guaranteed by Theorem 1.3. However such an equivalence class $\mathscr{B}$ does not exist. By the following theorem, we see that the condition (iv)–(vii) are separated into two equivalence classes. One class consists of (iv) and (vii) and the other consists of (v) and (vi).

Theorem 1.5. *The induced homomorphism* $j^* : H^*(BL\,Spin(17); \mathbf{Z}/2) \to H^*(Spin(17); \mathbf{Z}/2)$ *is not surjective.*

The third aim of this manuscript is to explain that the concept of module derivations is not only useful for general theory of the adjoint action of finite loop spaces but also for some explicit calculation of the adjoint actions. The calculation is based on the fact that BLG has the homotopy type of LBG and the following theorem.

Theorem 1.6. *Let x be a simply connected space whose mod 2 cohomology is isomorphic to the polynomial algebra* $\mathbf{Z}/2[y_1, y_2, \dots, y_n]$. *Then*

$$H^*(LX; \mathbf{Z}/2) \cong \mathbf{Z}/2[\bar{y}_1, \bar{y}_2, \dots, \bar{y}_n] \otimes \mathbf{Z}/2[y_1, y_2, \dots, y_n]/(\bar{y}_i^2 + \mathscr{D}Sq^{\deg y_i - 1} y_i; i = 1, 2, \dots, n)$$

as an $H^*(X; \mathbf{Z}/2)$*-algebra, where* $\deg \bar{y}_i = \deg y_i - 1$ *and* $\mathscr{D}$ *is the module derivation defined by* $\mathscr{D} y_i = \bar{y}_i$.

To prove Theorem 1.6, we use the Eilenberg-Moore spectral sequence $\{{}_{EM}E_r^{*,*}(X), d_r\}$ converging to $H^*(LX; \mathbf{Z}/2)$ whose E_2-term is isomorphic to the Hochschild homology $HH(H^*(X; \mathbf{Z}/2))$. In the proof, the module derivation $\mathscr{D} : H^*(X; \mathbf{Z}/2) \to HH(H^*(X; \mathbf{Z}/2))$ plays an important role in order to solve extension problems in the spectral sequence.

Applying Theorem 1.6 to the case $X = BG_2$, we determine completely the algebra structure of $H^*(LBG_2; \mathbf{Z}/2)$, which is isomorphic to $H^*(BLG_2; \mathbf{Z}/2)$ as an algebra.

Theorem 1.7. $H^*(BLG_2; \mathbf{Z}/2) \cong H^*(LBG_2; \mathbf{Z}/2) \cong$

$$\mathbf{Z}/2[x_3, x_5] \otimes \mathbf{Z}/2[y_4, y_6, y_7] \Big/ \begin{pmatrix} x_3^4 + x_5 y_7 + y_6 x_3^2 \\ x_5^2 + x_3 y_7 + y_4 x_3^2 \end{pmatrix}.$$

The structure of the E_2-term of the Rothenberg-Steenrod spectral sequence $\{{}_{RS}E_r^{*,*}(BLG), d_r\}$ and a relation between indecomposable elements of $H^*(LBG_2;$

$\mathbf{Z}/2)$ enable us to deduce the non-triviality of some adjoint action of G_2 on ΩG_2. In consequence, we can obtain a certain important calculation of adjoint action of G_2 on ΩG_2 due to Kono and Kozima [7] and Hamanaka [11]. Our calculation of the adjoint action is simple and algebraic. Therefore, from the consideration of the Hopf algebra structure of the homology $H_*(LG_2; \mathbf{Z}/2)$ due to Hamanaka [11], we have

Theorem 1.8. *Let X be a loop space whose mod 2 cohomology is isomorphic to that of G_2 as an algebra over the Steenrod algebra. Then $H_*(\Omega X; \mathbf{Z}/2) \cong H_*(\Omega G_2; \mathbf{Z}/2)$ as a Hopf algebra over the Steenrod algebra. Moreover the adjoint action $Ad_* : H_*(X; \mathbf{Z}/2) \otimes H_*(\Omega X; \mathbf{Z}/2) \to H_*(\Omega X; \mathbf{Z}/2)$ coincides with that of G_2. Thus $H_*(LX; \mathbf{Z}/2) \cong H_*(LG_2; \mathbf{Z}/2)$ as a Hopf algebra.*

Let $B\mathrm{DI}(4)$ be the complex constructed by Dwyer and Wilkerson [2] and put $G = \Omega B\mathrm{DI}(4)$. Since $H^*(B\mathrm{DI}(4); \mathbf{Z}/2)$ is a polynomial algebra, by virtue of Theorem 1.5, we can determine the algebra structure of $H^*(LBDI(4); \mathbf{Z}/2)$.

Theorem 1.9. $H^*(BLG; \mathbf{Z}/2) \cong H^*(LBG; \mathbf{Z}/2) \cong H^*(LBDI(4); \mathbf{Z}/2) \cong$

$$\mathbf{Z}/2[x_7, x_{11}, x_{13}] \otimes \mathbf{Z}/2[y_8, y_{12}, y_{14}, y_{15}] \Big/ \begin{pmatrix} x_{11}^2 + x_7 y_{15} + y_8 x_7^2 \\ x_{13}^2 + x_{11} y_{15} + y_{12} x_7^2 \\ x_7^4 + x_{13} y_{15} + y_{14} x_7^2 \end{pmatrix}.$$

This result also allows us to deduce simply the main theorem [8, Theorem] in which non-triviality of the adjoint action $Ad^* : H^*(\Omega G; \mathbf{Z}/2) \to H^*(G; \mathbf{Z}/2) \otimes H^*(\Omega G; \mathbf{Z}/2)$ is clarified.

Throughout this manuscript, let X be a simply connected space. Moreover a graded commutative algebra A over a field $\mathbf{k}$ is assumed to be 1-connected, that is, $A^0 = \mathbf{k}$ and $A^i = 0$ if $i < 0$ or $i = 1$. We denote by $\bar{A}$ the vector space $\bigotimes_{i \geq 2} A^i$.

The rest of this manuscript organized as follows: In Section 2, we prove Theorem 1.4. In Section 3, we define important module derivations. Section 4 devotes to prove that (iii) implies (i) and that (ii) implies (i) in the case where G is a finite loop space and $p = 2$. Theorem 1.6, 1.7 and 1.8 are proved in Section 5. Consequently, we can calculate the adjoint actions of G_2 and $\Omega B\mathrm{DI}(4)$ mentioned above in a simpler and more algebraic manner.

2. Proof of Theorem 1.5

We will prove Theorem 1.5 by reducing to the problem of whether the loop space $\Omega B\,Spin(17)$ is totally non homologous to zero in the free loop space $LB\,Spin(17)$ with respect to $\mathbf{Z}/2$.

Define the map φ_L by $\varphi_L(t_1\gamma_1 \oplus \cdots \oplus t_n\gamma_n)(s) = t_1\gamma_1(s) \oplus \cdots \oplus t_n\gamma_n(s)$ and φ_Ω as the restriction to $B\Omega G$ of φ_L. Then we can obtain the commutative diagram (2.1):

$$\begin{array}{ccccc}
G & \xleftarrow[\cong]{} & B\Omega G & \xrightarrow[\varphi_\Omega]{} & \Omega BG \\
{\scriptstyle j}\downarrow & & \downarrow & & \downarrow{\scriptstyle j} \\
EG \times_G G & \xleftarrow[\cong]{} & BLG & \xrightarrow[\varphi_L]{} & LBG \\
\downarrow & & \downarrow & & \downarrow \\
BG & \xleftarrow[=]{} & BG & \xrightarrow[=]{} & BG.
\end{array}$$

Notice that the right and left verticals are fibrations. Since the map φ_Ω induces the map of fibrations

$$\begin{array}{ccc}
\Omega G & \xrightarrow[=]{} & \Omega G \\
\downarrow & & \downarrow \\
E\Omega G & \longrightarrow & \Omega EG \\
\downarrow & & \downarrow \\
B\Omega G & \xrightarrow[\varphi_\Omega]{} & \Omega BG,
\end{array}$$

we see that φ_Ω is a homotopy equivalence and hence so is φ_L. Thus classifying spaces of loop groups can be considered from the view point of loop spaces.

Our proof of Theorem 1.5 is based on the result [9, Proposition 1.7 (1)] and depends on the algebra structure of $H^*(BSpin(17);\mathbf{Z}/2)$ which has been determined by Quillen. From the diagram (2.1), in order to prove Theorem 1.5, it suffices to show that $\Omega Spin(17)$ is not totally non-homologous to zero in $LBSpin(17)$ with respect to the field $\mathbf{Z}/2$. Let $\{\hat{E}_r(X), \hat{d}_r\}$ and $\{\bar{E}_r(X), \bar{d}_r\}$ denote the Eilenberg-Moore spectral sequence of the path loop fibration $\Omega X \to PX \to X$ and the Leray-Serre spectral sequence of the free loop fibration $\Omega X \to LX \to X$ respectively. We assume that

$$H^*(X;\mathbf{Z}/p) \cong \Lambda(y_1, \dots, y_l) \otimes \mathbf{Z}/p[x_1, \dots, x_n]/(\rho_1, \dots, \rho_m)$$

for $* \le N$, where $\rho_1, \dots, \rho_m$ is a regular sequence and each ρ_i is decomposable. Then we have

Proposition 2.1 ([9], Proposition 1.7 (1)]. *Suppose that there exist integers i $(1 \le i \le m)$ and j $(1 \le j \le n)$ such that $\partial\rho_i/\partial x_j \ne 0$ in $\mathbf{Z}/p[x_1, \dots, x_n]/(\rho_1, \dots, \rho_m)$ and $\hat{d}_r^{s,t} = 0$ for any $r \ge 2$, s and t; $s + t \le \deg\rho_i - 2$. Then there exist integers r, s and t such that $\bar{d}_r^{s,t} \ne 0$ and $s + t \le \deg\rho_i - 2$.*

Remark 2.2. For any algebra A and B, $Tor_A^{*,*}(\mathbf{Z}/2, \mathbf{Z}/2) \cong Tor_B^{*,*}(\mathbf{Z}/2, \mathbf{Z}/2)$ for the total degrees less than $N - 2$ if $A \cong B$ for the degrees less than N. Therefore we can apply the same argument as the proof [9, Proposition 1.7 (1)] though the condition such that $N = \infty$ is assumed in the original proof.

By virtue of Quillen's result [17, Theorem 6.5] concerning the algebra structure of $H^*(BSpin(17); \mathbf{Z}/2)$, we have

$$H^*(BSpin(17); \mathbf{Z}/2) \cong \mathbf{Z}/2[w_4, w_6, w_7, w_8, w_{10}, w_{11}, w_{12}, w_{13}, w_{14}, w_{15}, w_{16}]/(\rho)$$

for $* \leq 33$. Here $\rho = w_{16}w_{13}w_4 + w_{16}w_{11}w_6 + w_{16}w_{10}w_u + \alpha$ for some $\alpha \in I$, where I is the ideal generated by the elements $w_4^2, w_6^2, w_7^2, w_8^2, w_{10}^2, \dots, w_{15}^2$ of $\mathbf{Z}[w_4, w_6, w_7, w_8, w_{10}w_{11}, \dots, w_{15}, w_{16}]$. Since $\partial\alpha/\partial w_{16}$ belongs to the ideal I, it follows that $\partial\rho/w_{16} = w_{13}w_{14} + w_{11}w_6 + w_{10}w_7 + \partial\alpha/\partial w_{16} \neq 0$ in $H^*(BSpin(17); \mathbf{Z}/2)$. By using the Koszul type resolution ([9], [15]), we can get

$$\hat{E}_2^{*,*} \cong \Lambda(s^{-1}w_4, s^{-1}w_6, s^{-1}w_7, s^{-1}w_8, s^{-1}w_{10}, s^{-1}w_{11}, \dots, s^{-1}w_{16}) \otimes \Gamma[\tau\rho]$$

for total degrees ≤ 31, where bideg $s^{-1}w_i = (-1, i)$ and bideg $\tau\rho = (-2, 33)$. This fact enables us to conclude that $\hat{d}_r^{s,t} = 0$ for any r if $s + t \leq 31$. From Proposition 2.1, we have Theorem 1.5.

Remark 2.3. As for the case of the cohomology with the rational coefficient, by the main theorem of Smith [16], we see that $j^* : H^*(BL\,Spin(n); \mathbf{Q}) \to H^*(Spin(n); \mathbf{Q})$ is surjective for any $n \geq 3$. Let $\mathbf{k}$ be a field whose characteristic is a prime p. From the proof of [16, Theorem] due to Smith, it follows that the Eilenberg-Moore spectral sequence $\{{}_{EM}E_r^{*,*}(X), d_r\}$ over $\mathbf{k}$ collapses at the E_2-term if $H^*(X; \mathbf{k})$ is a polynomial algebra generated by elements with even degree. Therefore we can also conclude that $j^* : H^*(BL\,Spin(n); \mathbf{Z}/p) \to H^*(Spin(n); \mathbf{Z}/p)$ is surjective unless $p = 2$.

3. Module derivations

We begin with the definition of an algebraic module derivation.

Definition 3.1. *Let A be a graded commutative algebra over a field $\mathbf{k}$ and M a left A-module. A module derivation of A with values in M is a $\mathbf{k}$-linear map $\mathscr{D} : A \to M$ with degree -1 such that*

$$\mathscr{D}(ab) = (-1)^{(\deg a+1)\deg b} b\mathscr{D}(a) + (-1)^{\deg a} a\mathscr{D}(b)$$

for any $a, b \in A$.

Let $HH(A)$ be the Hochschild homology of a graded algebra A. We regard $HH(A)$ as the homology of the bar complex $(A \otimes \bar{B}(A \otimes A) \otimes A, \partial)$ constructed from the bar resolution of A as an $A \otimes A$-modules. For details of the bar complex see [12].

The following module derivation is used in order to solve some extension problems of the Eilenberg-Moore spectral sequence $\{{}_{EM}E_r^{*,*}(LBG), d_r\}$ (see §5).

Proposition 3.2. *Define the map*

$$\mathscr{D} : A \to HH(A) = Tor_{A \otimes A}(A, A) = H(A \otimes \bar{B}(A \otimes A) \otimes A, \partial)$$

by $x \mapsto [x \otimes 1 - 1 \otimes x]$. *Then* $\mathscr{D}$ *is a module derivation of* A *with values in* $HH(A)$.

Proof. It suffices to show that $[xy \otimes 1 - 1 \otimes xy] = (-1)^{(a+1)b} \cdot y[x \otimes 1 - 1 \otimes x] + (-1)^a x[y \otimes 1 - 1 \otimes y]$ in $HH(A)$, where $\deg x = a$ and $\deg y = b$. We choose an element

$$u = [y \otimes 1 | x \otimes 1 - 1 \otimes x] - (-1)^{(a+1)(b+1)}[1 \otimes x \mid y \otimes 1 - 1 \otimes y]$$

from $A \otimes \bar{B}^2(A \otimes A) \otimes A = A \otimes \overline{A^{\otimes 2}} \otimes \overline{A^{\otimes 2}} \otimes A$. Then we have

$$\begin{aligned} d(u) &= y[x \otimes 1 - 1 \otimes x] + (-1)^{b+1}[yx \otimes 1 - y \otimes x] \\ &\quad - (-1)^{(a+1)(b+1)}\{x[y \otimes 1 - 1 \otimes y] + (-1)^{a+1}[(-1)^{ab} yx \otimes 1 - 1 \otimes xy]\} \\ &= y[x \otimes 1 - 1 \otimes x] - (-1)^{(a+1)(b+1)} x[y \otimes 1 - 1 \otimes y] \\ &\quad + (-1)^{b+1+ab}[xy \otimes 1] + (-1)^{b(a+1)}[1 \otimes xy] \\ &= y[x \otimes 1 - 1 \otimes x] - (-1)^{(a+1)(b+1)} x[y \otimes 1 - 1 \otimes y] \\ &\quad - (-1)^{b(a+1)}[xy \otimes 1 - 1 \otimes xy] \end{aligned}$$

This completes the proof.

To prove that (iii) implies (i), we use the module derivation defined below.

Proposition 3.3. *Let* $ev : S^1 \times LX \to X$ *be the evaluation map and* $\int_{S^1} : H^*(S^1 \times LX; \mathbf{Z}/p) \to H^{*-1}(LX; \mathbf{Z}/p)$ *the integration along* S^1. *Then the composition*

$$\mathscr{D}_X := \int_{S^1} \circ\, ev^* : H^*(X; \mathbf{Z}/p) \to H^{*-1}(LX; \mathbf{Z}/p)$$

is a module derivation of $H^*(X; \mathbf{Z}/p)$ *which is compatible with the action of Steenrod operations.*

Proof. Let $\Omega X \to LX \xrightarrow{p} X$ be the free loop fibration. We define a map $i : LX \to S^1 \times LX$ by $i(\gamma) = (1, \gamma)$. Then $ev \circ i = p$. Therefore we can write that $ev^*(u) = 1 \otimes p^*(u) + g \otimes w$, where g is the generator of $H^*(S^1; \mathbf{Z}/2)$. Since the integration along S^1 is defined by $\int_{S^1}(g \otimes w) = w$, it follows that $\mathscr{D}_X(uv) = (-1)^{\deg u} p^*(u)\mathscr{D}_X(v) + \mathscr{D}_X(u)p^*(v)$. So $\mathscr{D}_X$ is module derivation with values in $H^*(LX; \mathbf{Z}/2)$. The action of the Steenrod operations on $H^*(S^1; \mathbf{Z}/2)$ is trivial. Hence $\mathscr{D}_X$ is compatible with the action.

Let $\{_{EM}E_r(LX), d_r\}$ be the Eilenberg-Moore spectral sequence converging to $H^*(LX; \mathbf{Z}/p)$ and $\{F^pH^*(LX; \mathbf{Z}/p)\}_{p \leq 0}$ the filtration of $H^*(LX; \mathbf{Z}/p)$ given by the spectral sequence.

The following proposition says that the cohomology suspension

$\sigma^* : H^*(G;\mathbf{Z}/p) \to H^{*-1}(\Omega X;\mathbf{Z}/p)$ factors through a module derivation. Let $\{f_r\} : \{{}_{EM}E_r^{*,*}(LX), d_r\} \to \{{}_{EM}E_r^{*,*}(\Omega X), d_r\}$ be the morphism of spectral sequence defined in [9, Lemma 1.3].

Proposition 3.4. (1) *The image of $H^*(LX;\mathbf{Z}/p)$ by the module derivation $\mathscr{D}_X$ is contained in the filter $F^{-1}H^*(LX;\mathbf{Z}/p)$.*

(2) *Define maps η_{EM} and η_F by*

$$H^*(X;\mathbf{Z}/p) \xrightarrow{\mathscr{D}} HH^{-1,*}(H^*(X;\mathbf{Z}/p)) \cong {}_{EM}E_2^{-1,*}(LX) \xrightarrow{r} {}_{EM}E_\infty^{-1,*}(LX)$$

$$\xrightarrow{f_\infty} {}_{EM}E_\infty^{-1,*}(\Omega X) = F^{-1}H^*(\Omega X;\mathbf{Z}/p) \hookrightarrow H^{*-1}(\Omega X;\mathbf{Z}/p)$$

and

$$H^*(X;\mathbf{Z}/p) \xrightarrow{\mathscr{D}_X} Im\{\mathscr{D}_X : H^*(X;\mathbf{Z}/p) \to H^{*-1}(LX;\mathbf{Z}/p)\} \hookrightarrow HF^{-1}(LX;\mathbf{Z}/p)$$

$$\xrightarrow{j^*} F^{-1}H^*(\Omega X;\mathbf{Z}/p) \hookrightarrow H^{*-1}(\Omega X;\mathbf{Z}/p)$$

respectively, where $r : {}_{EM}E_2^{-1,*}(LX) \to {}_{EM}E_\infty^{-1,*}(LX)$ is the natural projection. Then $\eta_{EM} = \sigma^*$ and $\eta_F = \sigma^*$.

(3) *Let π denote the projection $F^{-1}H^*(LX;\mathbf{Z}/p) \to {}_{EM}E_\infty^{-1,*}(LX)$. Then $\pi \circ \mathscr{D}_X = r \circ \mathscr{D}$.*

Proof. From [10, Lemma 4.3], we have (1) and (3). The result [9, Lemma 1.3] allows us to deduce that $f_\infty r\mathscr{D}(x) = r'f_2[x \otimes 1 - 1 \otimes x] = r'[x]$, where $r' : {}_{EM}E_2^{-1,*}(\Omega X) \to {}_{EM}E_\infty^{-1,*}(\Omega X)$ is the natural projection. Hence, from [15, Proposition 4.5], we can deduce that $\eta_{EM} = \sigma^*$. Since $f_\infty\pi = j^*$, it follows that $\eta_F = j^*\mathscr{D}_X = f_\infty\pi\mathscr{D}_X = f_\infty r\mathscr{D} = \eta_{EM}$. This completes the proof.

We define a map $\widetilde{ev} : S^1 \times BLG \to BG$ by $\widetilde{ev}(s, t_1\gamma_1 \oplus \cdots \oplus t_n\gamma_n) = t_1\gamma_1(s) \oplus \cdots \oplus t_n\gamma_n(s)$. Applying the same argument as the proof of Proposition 3.3, we see that the map $\tilde{\mathscr{D}}_{BG} : H^*(BG;\mathbf{Z}/p) \to H^{*-1}(BLG;\mathbf{Z}/p)$ defined by $\int_{S^1} \circ\, \widetilde{ev}^*$ is a module derivation with the values in $H^*(BLG;\mathbf{Z}/2)$. We mention here that the module derivation $\tilde{\mathscr{D}}_{BG}$ coincides with $\mathscr{D}_{BG}$ up to the induced isomorphism $\varphi^* : H^*(LBG;\mathbf{Z}/p) \xrightarrow{\cong} H^*(BLG;\mathbf{Z}/p)$. The following theorem states that the module derivation $\tilde{\mathscr{D}}_{BG}$ can be decomposed into a morphism of spectral sequences $\{g_r\} : \{{}_{RS}E_r^{*,*}(BG), d_r\} \to \{{}_{RS}E_r^{*,*}(BLG), d_r\}$.

Proposition 3.5. *There exists a morphism of spectral sequences $\{g_r\} : \{{}_{RS}E_r(BG), d_r\} \to \{{}_{RS}E_r(BLG), d_r\}$ with bidegree $(0, -1)$ such that each map g_r is a module derivation and $g_\infty = \tilde{\mathscr{D}}_{BG} : {}_{RS}E_\infty^{*,*}(BG) \to {}_{RS}E_\infty^{*,*-1}(BLG)$.*

Proof. We denote the singular chain complex with the coefficient $\mathbf{Z}/p$ of a space X by $C_*(X)$. Let $P_\cdot \xrightarrow{\varepsilon} C_*(*) \to 0$ and $Q_\cdot \xrightarrow{\varepsilon} C_*(*) \to 0$ be the bar resolutions of $C_*(*)$ as $C_*(LG)$-modules and as $C_*(G)$-modules respectively. Define a map ${}_{LG}\eta_{k,i} : \underbrace{LG \times \cdots \times LG}_{k \text{ times}} \to \underbrace{LG \times \cdots \times LG}_{k-1 \text{ times}}$ by ${}_G\eta_{k,i}(\gamma_1, \ldots, \gamma_k) = (\gamma_1, \ldots, \gamma_{i-1},$

$\gamma_i\gamma_{i+1}, \gamma_{i+1}, \dots, \gamma_k)$ and ${}_G\eta_{k,i} : \underbrace{G \times \cdots \times G}_{k \text{ times}} \to \underbrace{G \times \cdots \times G}_{k-1 \text{ times}}$ by the similar fashion. Moreover we define a map $ev_k : S^1 \times LG \times \cdots \times LG \to G \times \cdots \times G$ by $ev_k(s, \gamma_1, \dots, \gamma_k) = (\gamma_1(s), \dots, \gamma_k(s))$ and, for any $\mathbf{Z}/2$-module A, a map $t\wedge : A \to C_*(S^1) \otimes A$ by $t \wedge (a) = t \otimes a$, where t is a representative element of the generator of $H^*(S^1; \mathbf{Z}/2)$. Since $ev_{k-1} \circ (1 \times_{LG} \eta_{k,i}) = {}_G\eta_{k,i} \circ ev_k$, we have commutative diagrams of chain complexes:

$$\begin{array}{ccc}
C_*(ELG) \otimes_{C_*(LG)} Tot(P_\cdot) & \xrightarrow{1 \otimes \varepsilon} & C_*(ELG) \otimes_{C_*(LG)} C_*(*) \\
\Big\downarrow{\scriptstyle t\wedge} & & \Big\downarrow{\scriptstyle t\wedge} \\
C_*(S^1) \otimes C_*(ELG) \otimes_{C_*(LG)} Tot(P_\cdot) & \xrightarrow{1 \otimes 1 \otimes \varepsilon} & C_*(S^1) \otimes C_*(ELG) \otimes_{C_*(LG)} C_*(*) \\
\Big\downarrow{\scriptstyle \widetilde{ev}_* \otimes (ev_*)_*} & & \Big\downarrow{\scriptstyle \widetilde{ev}_* \otimes 1} \\
C_*(EG) \otimes_{C_*(G)} Tot(Q_\cdot) & \xrightarrow{1 \otimes \varepsilon} & C_*(EG) \otimes_{C_*(G)} C_*(*)
\end{array}$$

$$\begin{array}{cc}
\xrightarrow{\bar{\alpha}_{1*}} & C_*(ELG \times_{LG} *) \\
& \Big\downarrow{\scriptstyle t\wedge} \\
\xrightarrow{1 \otimes \bar{\alpha}_{1*}} & C_*(S^1) \otimes C_*(ELG \times_{LG} *) \\
& \Big\downarrow{\scriptstyle \widetilde{ev}_*} \\
\xrightarrow{\bar{\alpha}_{2*}} & G_*(EG \times_G *)
\end{array}$$

Here $\bar{\alpha}_{1*}$ and $\bar{\alpha}_{2*}$ are the maps induced from the projections $\alpha_1 : ELG \times * \to ELG \times_{LG} *$ and $\alpha_2 : EG \times * \to EG \times_G *$ respectively. We note that the composition of $1 \times \varepsilon$ and α_{i*} induces an isomorphism on homologies, which is called the Eilenberg-Moore map ([12], [15]). Consider the cohomologies of cochain complexes obtained from the above chain complexes via the dual. Then we see

$$(3.1) \qquad \tilde{\mathscr{D}}_{BG} = H(dual(\widetilde{ev}_* \otimes (ev_*)_* \circ t\wedge))$$

up to the Eilenberg-Moore maps. Let $F(BLG)$ and $F(BG)$ be the filtered differential graded modules which give the Rothenberg-Steenrod spectral sequence $\{{}_{RS}E_r^{*,*}(BLG), d_r\}$ and $\{{}_{RS}E_r^{*,*}(BG), d_r\}$ respectively. It is clear, from the definitions of the map ev_k $(k \geq 1)$, that the map $dual(\widetilde{ev}_* \otimes (ev_*)_* \circ t\wedge)$ is a morphism of filtered differential graded modules with degree -1 from $F(BG)$ to $F(BLG)$. Thus $dual(\widetilde{ev}_* \otimes (ev_*)_* \circ t\wedge)$ derives a morphism of spectral sequences $\{g_r\} : \{{}_{RS}E_r^{*,*}(BG), d_r\} \to \{{}_{RS}E_r^{*,*}(BLG), d_r\}$ with bidegree $(0, -1)$. The algebra structure of the E_r-term is induced from that of the E_{r-1}-terms. Therefore g_r is a module derivation of $E_r^{*,*}(BG)$ if g_{r-1} is that of $E_{r-1}^{*,*}(BG)$. Applying the same argument as the proof of Proposition 3.3, we see that the map $g_1 =$

$\int_{S^1} \circ \widetilde{ev}^* \otimes (ev_*)^* : \Omega(H^*(G;\mathbf{Z}/2)) \to \Omega(H^*(LG;\mathbf{Z}/2))$ is a module derivation of $\Omega(H^*(G;\mathbf{Z}/2)) = E_1^{*,*}(BG)$, where $\Omega(C)$ denotes the cobar construction for a coalgebra C. Hence each map g_r is a module derivation of $E_r^{*,*}(BG)$. The fact (3.1) enables us to conclude that $g_\infty = \tilde{\mathscr{D}}_{BG}$.

In the following section, we will prove Theorem 1.3 by making use of the module derivations $\mathscr{D}_{BG} : H^*(BG;\mathbf{Z}/p) \to H^{*-1}(LBG;\mathbf{Z}/p)$ and $g_2 : {}_{RS}E_2^{*,*}(BG) \to {}_{RS}E_2^{*,*-1}(BLG)$.

4. Proof of Theorem 1.3

As mentioned in Remark 1.4, in order to prove Theorem 1.3, it suffices to show the following propositions in the case where G is a simply connected finite loop space and $p = 2$.

Proposition 4.1. *The condition* (iii) *implies* (i).

Proposition 4.2. *The condition* (ii) *implies* (i).

In order to prove Proposition 4.1, we need a normalized 2-simple system of generators of a Hopf algebra.

Lemma 4.3. *Let H be a Hopf algebra over a field $\mathbf{Z}/2$. Suppose that H is primitively generated and $H \cong \bigotimes_{i \in I}\mathbf{Z}/2[x_i]/(x_i^{2^{s_i}})$ as an algebra. Then there exists a 2-simple system of generators $\{\alpha_i, \alpha_i^2, \dots, \alpha_i^{2^{s_i}}\}_{i\in I}$ such that each α_i is primitive and $\pi(\alpha_i) = x_i$, where $\pi : PH \to QH$ is the natural projection.*

Proof. Since π is surjective, for any i, we can choose a primitive element α_i so that $\pi\alpha_i = x_i$, that is, $\alpha_i = x_i + d_i$ for some decomposable element d_i. It is clear that $\{\alpha_i, \alpha_i^2, \dots, \alpha_i^{2^{2_i}}\}_{i\in I}$ is a set of generators. Moreover, the set is a 2-simple system of generators because the elements of the set are linearly independent and primitive elements.

Let Q be the module of indecomposable elements of $H^*(BG;\mathbf{Z}/2)$. We denote by I_s the ideal of $H^*(BG;\mathbf{Z}/2)$ generated by the s-fold product of Q.

Proof of Proposition 4.1. The condition (iii) implies (vii) and hence (v), that is, the Hopf algebra $H^*(G;\mathbf{Z}/p)$ is primitively generated. Assume that $H^*(G;\mathbf{Z})$ has 2-torsion. We choose a 2-simple system of generators $S := \{x_1, x_2, \dots, x_n\}$ of $H^*(G;\mathbf{Z}/2)$ so as to satisfy the condition in Lemma 4.3: $H^*(G;\mathbf{Z}/2) \cong \Delta(x_1, x_2, \dots, x_n)$. We order the elements so that $\deg x_1 \le \cdots \le \deg x_n$. Note that the degree of some x_i is even since $H^*(G;\mathbf{Z})$ as 2-torsion. Let $\{{}_{RS}E_r^{*,*}(BG), d_r\}$ be the Rothenberg-Steenrod spectral sequence converging to $H^*(BG;\mathbf{Z}/2)$. Then we have

$${}_{RS}E_2^{*,*}(BG) \cong Cotor^{*,*}_{H^*(G;\mathbf{Z}/2)}(\mathbf{Z}/2, \mathbf{Z}/2) = \mathbf{Z}/2[sr_1, sx_2, \dots, sx_n]$$

as a bigraded algebra, where bideg $sx_i = (1, \deg x_i)$.

First we consider the case where $H^*(BG;\mathbf{Z}/2)$ is not a polynomial algebra. Let sx_l be the lowest dimensional generator which has non trivial image by some differential d_r. Suppose that $\deg x_l$ is even. Then, from the fact that $Q^{\text{even}}H^*(G;\mathbf{Z}/2)=0$ due to Kane [6, Theorem, §40–1], it follows that there exists a primitive element x_i with odd degree in S such that $x_l=(x_i)^{2^u}$ for some u. By making use of the Steenrod operations in the Rothenberg-Steenrod spectral sequence [14], we deduce that sx_l is a permanent cycle, which is a contradiction. So $\deg x_l$ is odd. Put $N=\deg x_l$. Then we have $H^*(BLG;\mathbf{Z}/p)\cong\mathbf{Z}/2[y_1,\dots,y_k]/(\rho_1,\dots,\rho_t)$ for $*\le N+2$, where $\deg\rho_i=N+2$ and $\deg y_i=\deg x_i+1$. Moreover, we can write

$$\rho_i=\sum y_{i_1}^{k_{i_1}}\cdots y_{i_l}^{k_{i_l}}+W=0 \tag{4.1}$$

in $H^*(BG;\mathbf{Z}/2)$, where $1\le k_{i_j}$, $3\le k_{i_1}+\cdots+k_{i_l}=s$, y_{i_j} is in $\{y_1,\dots,y_k\}$ and W belongs to I_{s+1} for some integer s. Since N is odd, it follows that there is an odd integer k_{i_m}. Without loss of generality we may assume that such an integer is k_{i_1}. The equality (4.1) can be written as follows:

$$0=Y_1y_{i_1}^{k_1}+Y_2y_{i_1}^{k_2}+\cdots+Y_my_{i_1}^{k_m}+Z+W,$$

where the elements $Y_1,\dots Y_n$ and Z have no term which contains the element y_{i_1}, k_1 is odd and $k_i\neq k_j$ if $i\neq j$. Applying the module derivation $\mathscr{D}_{BG}$ to the above equality, we have

$$\begin{aligned}0=\ &Y_1y_{i_1}^{k_1-1}\mathscr{D}_{BG}y_{i_1}+k_2Y_2y_{i_1}^{k_2-1}\mathscr{D}_{BG}y_{i_1}+\cdots+k_mY_my_{i_1}^{k_m-1}\mathscr{D}_{BG}y_{i_1}\\&+y_{i_1}^{k_1}\mathscr{D}_{BG}Y_1+\cdots+y_{i_1}^{k_m}\mathscr{D}_{BG}Y_m+\mathscr{D}_{BG}Z+\mathscr{D}_{BG}W.\end{aligned}\tag{4.2}$$

Since $H^*(BG;\mathbf{Z}/2)$ is isomorphic to $\mathbf{Z}/2[y_1,\dots,y_k]$ for $*\le N-1$, by using the usual argument on the Eilenberg-Moore spectral sequence, it follows that $H^*(\Omega BG;\mathbf{Z}/2)\cong\Lambda(s^{-1}y_1,\dots,s^{-1}y_k)$ for $*\le N-3$. By virtue of Proposition 3.4, we see that $j^*\mathscr{D}_{BG}y_{i_1}=s^{-1}y_{i_1}$, where j is the inclusion $\Omega BG\to LBG$. The diagram (2.1) and the assumption (iii) yield an isomorphism $\Psi: H^*(LBG;\mathbf{Z}/2)\xrightarrow{\cong}H^*(BG;\mathbf{Z}/2)\otimes H^*(\Omega BG;\mathbf{Z}/2)$ which satisfies $inc_2^*\circ\Psi=j^*$. Thus we see that $\Psi(\mathscr{D}_{BG}y_i)=s^{-1}y_i+w$ for some element w in the ideal generated by $\overline{H}^*(BG;\mathbf{Z}/2)$ of the algebra $H^*(BG;\mathbf{Z}/2)\otimes H^*(\Omega BG;\mathbf{Z}/2)$. Let Ξ be the vector space $H^*(BG;\mathbf{Z}/2)/I_s\otimes\Lambda(s^{-1}y_1,\dots,s^{-1}y_k)$. Then the equality (4.2) induces an equality in Ξ:

$$\begin{aligned}0=\ &Y_1y_{i_1}^{k_1-1}s^{-1}y_{i_1}+k_2Y_2y_{i_1}^{k_2-1}s^{-1}y_{i_1}+\cdots+k_mY_my_{i_1}^{k_m-1}s^{-1}y_{i_1}\\&+y_{i_1}^{k_1}\mathscr{D}'Y_1+\cdots+y_{i_1}^{k_m}\mathscr{D}'Y_m+\mathscr{D}'Z.\end{aligned}$$

Here $\mathscr{D}'$ is the derivation defined by $y_i\mapsto s^{-1}y_i$. This equality contradicts the structure of the vector space Ξ.

Let us consider the case where $H^*(BG;\mathbf{Z}/2)$ is isomorphic to the polynomial algebra $\mathbf{Z}/2[y_1,y_2,\dots,y_n]$. Since $H^*(G;\mathbf{Z})$ has 2-torsion, there exists an element with odd degree in the set $\{y_1, y_2, \dots, y_n\}$ of indecomposable elements of $H^*(BG;\mathbf{Z}/2)$. Let y_i be such an element with the highest odd degree and put $2l+1 = \deg y_i$. Then we can write

$$Sq^{2l} y_i = Yy_i + \sum y_i' y_j' + W, \tag{4.3}$$

where Y is indecomposable or zero, the element y_i' is in the set $\{y_1, y_2, \dots, y_{i-1}, y_{i+1}, \dots, y_n\}$ and W belongs to the ideal I_3 of $H^*(BG;\mathbf{Z}/2)$. Since $Sq^1 Sq^{2l} y_i = Sq^{2l+1} y_i = y_i^2 \neq 0$, it follows that Y is non zero in $H^*(BG;\mathbf{Z}/2)$. By applying the module derivation $\mathscr{D}_{BG}$ to the equality (4.3), we have

$$\begin{aligned}(\mathscr{D}_{BG} y_i)^2 = \mathscr{D}_{BG}(Sq^{2i} y_i) &= y_i \mathscr{D}_{BG} Y + Y \mathscr{D}_{BG} y_i \\ &+ \sum y_i' \mathscr{D}_{BG} y_j' + y_j' \mathscr{D}_{BG} y_i' + \mathscr{D}_{BG} W.\end{aligned} \tag{4.4}$$

Using the same argument as the case where $H^*(BG;\mathbf{Z}/2)$ is not isomorphic to the polynomial algebra, we have an isomorphism $\Psi : H^*(LBG;\mathbf{Z}/2) \xrightarrow{\cong} H^*(BG;\mathbf{Z}/2) \otimes H^*(\Omega BG;\mathbf{Z}/2) = \mathbf{Z}/2[y_1, \dots, y_n] \otimes \Delta(s^{-1}y_1, \dots, s^{-1}y_n)$ which satisfies $\Psi(\mathscr{D}_{BG} y_u) = s^{-1} y_u + w$ for some element w in the ideal generated by $\overline{H}^*(BG;\mathbf{Z}/2)$ of $H^*(BG;\mathbf{Z}/2) \otimes H^*(\Omega BG;\mathbf{Z}/2)$. Let Ξ' be the vector space $QH^*(BG;\mathbf{Z}/2) \otimes \Delta(s^{-1}y_1, \dots, s^{-1}y_n)$. We can write $Y = \sum y_l$ in $QH^*(BG;\mathbf{Z}/2)$, where $\deg y_l = 2l$. Therefore the equality (4.4) enables us to deduce that

$$0 = (s^{-1}y_i)^2 + \sum_l y_i s^{-1} y_l + \sum_l y_l s^{-1} y_i + \sum (y_j' s^{-1} y_i' + y_i' s^{-1} y_j')$$

in Ξ', which contradicts the structure of the vector space Ξ'.

In order to prove that (ii) implies (i), we find some relations between generators in the E_2-term of the Rothenberg-Steenrod spectral sequence $\{{}_{RS}E_r^{*,*}(BG), d_r\}$ converging to $H^*(BG;\mathbf{Z}/2)$. To this end, we need a good 2-simple system of generators of $H^*(G;\mathbf{Z}/2)$. Assume that $H^*(G;\mathbf{Z})$ has 2-torsion. Let S be a 2-simple system of generators which satisfies the condition in Lemma 4.3. Then S has a primitive element with even degree. Let x_{even} be such an element with the lowest even degree. From the fact that $Q^{\text{even}} H^*(G;\mathbf{Z}/2) = 0$, we see that there exists a primitive element x_1 with odd degree in S such that $x_1^2 = x_{\text{even}}$. Since $x_{\text{even}} = x_1^2 = Sq^{\deg x_1} x_1 = Sq^1 Sq^{\deg x_1 - 1} x_1$, it follows that $Sq^{\deg x_1 - 1} x_1$ is non zero. Put $x_2 = Sq^{\deg x_1 - 1} x_1$. If $x_{\text{even}}^2 \neq 0$, then $x_2^2 \neq 0$ because $Sq^2 x_2^2 = (Sq^1 x_2)^2 = x_1^4 \neq 0$. Thus we can get a finite sequence $x_1, x_2, \dots, x_N$ consisting of primitive elements with odd degree such that $x_{i+1} = Sq^{\deg x_i - 1} x_i$ for $i \le N-1$, $x_i^4 \neq 0$ if $i < N-1$, $x_{N-1}^2 \neq 0$ and $x_{N-1}^4 = 0$. The finiteness of the sequence is deduced from that of G. By extending the set $\{x_1, \dots, x_N, x_1^2, \dots, x_N^2\}$, we can construct another 2-simple system of generators B of $H^*(G;\mathbf{Z}/2)$ which consists of primitive elements. If there exists an element y of B such that $Sq^1 y = x_{N-1}^2 + \sum x_j'$, where x_j' is in $B \backslash \{x_{N-1}^2\}$, then we replace the element y with the element $y + x_N$. The set of $H^*(G;\mathbf{Z}/2)$ which is obtained form B with this replacement can also be a 2-simple system of generators. Thus we have

Lemma 4.4. *Suppose that $H^*(G;\mathbf{Z})$ has 2-torsion and $H^*(G;\mathbf{Z}/2)$ is primitively generated. Let $x_1,\dots,x_N$ be the sequence of primitive elements which is mentioned above. Then there exists a 2-simple system of generators B consisting of primitive elements such that $B\supset\{x_1,\dots,x_N,x_1^2,\dots,x_N^2\}$ and, for any element $y\in B$, if Sq^1y can be written as $x_{N-1}^2+\sum x_j'$ with $x_j'\in B\backslash\{x_{N-1}^2\}$, then $y=x_N$.*

Proof of Proposition 4.2. From the fact mentioned before Theorem 1.4, it follows that $H^*(G;\mathbf{Z}/2)$ is primitively generated under the condition (ii). Let us consider the Rothenberg-Steenrod spectral sequence $\{{}_{RS}E_r^{*,*}(BG),d_r\}$ whose E_2-term is isomorphic as bigraded algebra to $Cotor_{H^*(G;\mathbf{Z}/2)}^{*,*}(\mathbf{Z}/2,\mathbf{Z}/2)$. We regard $Cotor_{H^*(G;\mathbf{Z}/2)}^{*,*}(\mathbf{Z}/2,\mathbf{Z}/2)$ as the homology obtained from the cobar complex for $H^*(G;\mathbf{Z}/2)\cong\Delta(y_i;y\in B)$, that is, $Cotor_{H^*(G;\mathbf{Z}/2)}^{*,*}(\mathbf{Z}/2,\mathbf{Z}/2)\cong\mathbf{Z}/2[[y_i];y_i\in B]$ as a bigraded algebra, where bideg $[y_i]=(1,\deg y_i)$ and B is the 2-simple system of generators described in Lemma 4.4. Since the element x_{even} has the lowest even degree, it follows that the indecomposable element $[y_i]$ is a permanent cycle if $\deg y_i<\deg x_{\text{even}}$. In particular $[x_1]$ is a permanent cycle. Using the Steenrod operation on the Rothenberg-Steenrod spectral sequence, we see that $[x_N]$ and $[x_{N-1}^2]$ survive to the E_∞-term because $x_N=Sq^{\deg x_{N-1}-1}Sq^{\deg x_{N-2}-1}\dots Sq^{\deg x_1-1}x_1$ and $x_{N-1}^2=Sq^1x_N$. since $Sq^{\deg x_{N-1}^2}[x_{N-1}^2]=[Sq^{\deg x_{N-1}^2}x_{N-1}]=[x_{N-1}^4]=0$ in ${}_{RS}E_0^{1,*}(BG)$, it follows that $Sq^{\deg x_{N-1}^2}[x_{N-1}^2]$ belongs to the filter ${}_{\text{RS}}F^2H^*(BG;\mathbf{Z}/2)$ which is given by the spectral sequence. Therefore we can write

$$Sq^{\deg x_{N-1}^2}[x_{N-1}^2]=\sum w_j+\sum[x_l'\,|\,x_l'']$$

in ${}_{RS}E_0^{2,*}(BG)$, where $[x_l'\,|\,x_l'']$ is decomposable in ${}_{RS}E_0^{*,*}(BG):[x_l'\,|\,x_l'']=[x_l']\cdot[x_l'']$, $x_l',x_i''\in B$, and w_j does not have a term consisting of decomposable elements, that is, $w_j=\sum[a_{jk}'\,|\,a_{jk}'']$ for some elements $a_{jk}',a_{kj}''\in B$ and $[a_{jk}'\,|\,a_{jk}'']$ is indecomposable element in ${}_{RS}E_0^{*,*}(BG)$. Applying the squaring operation Sq^1 to the above equality, we have

$$\begin{aligned}[x_{N-1}^2\,|\,x_{N-1}^2]&=Sq^{\deg x_{N-1}^2+1}[x_{N-1}^2]=Sq^1Sq^{\deg x_{N-1}^2}[x_{N-1}]\\&=\sum Sq^1w_j+\sum([Sq^1x_l'\,|\,x_l'']+[x_l'\,|\,Sq^1x_l'']).\end{aligned}$$

We can regard the vector space ${}_{RS}E_\infty^{2,*}$ as the subspace of ${}_{RS}E_2^{2,*}$. Therefore, in ${}_{RS}E_2^{2,*}$,

$$\begin{aligned}[x_{N-1}^2\,|\,x_{N-1}^2]&=\sum([Sq^1a_{jk}'\,|\,a_{jk}'']+[a_{jk}'\,|\,Sq^1a_{jk}''])\\&\quad+\sum([Sq^1x_l'\,|\,x_l'']+[x_l'\,|\,Sq^1x_l'']).\end{aligned}$$

The element $[x_N\,|\,x_{N-1}^2]$ is decomposable meanwhile the element $[a_{jk}'\,|\,a_{jk}'']$ is indecomposable in ${}_{RS}E_\infty^{2,*}$. Hence the property of the elements of B described in Lemma 4.4 enables us to deduce that there exists the only element $[x_m'\,|\,x_m'']$ which coincides with the element $[x_N\,|\,x_{N-1}^2]$ in ${}_{RS}E_2^{2,*}$. Thus we can find the relation

$$Sq^{\deg x_{N-1}^2}[x_{N-1}^2]=\sum_J w_j+[x_N\,|\,x_{N-1}^2]+\sum_{I\backslash\{m\}}[x_l'\,|\,x_l''] \tag{4.5}$$

in ${}_{RS}E_0^{*,*}$. From Proposition 3.5, we have the commutative diagram:

$$\begin{array}{ccccc}
{}_{RS}F^2H^*(BG;\mathbf{Z}/2) & \longrightarrow & {}_{RS}E_\infty^{2,*}(BG) & \hookrightarrow & {}_{RS}E_2^{2,*}(BG) \\
\Big\downarrow{\scriptstyle \tilde{\mathscr{D}}_{BG}} & & \Big\downarrow{\scriptstyle g_\infty} & & \Big\downarrow{\scriptstyle g_2} \\
{}_{RS}F^2H^{*-1}(BLG;\mathbf{Z}/2) & \longrightarrow & {}_{RS}E_\infty^{2,*-1}(BLG) & \hookrightarrow & {}_{RS}E_2^{2,*-1}(BLG).
\end{array}$$

Since $\tilde{\mathscr{D}}_{BG}$ is compatible with the Steenrod operations, it follows that

$$g_\infty(Sq^{\deg x_{N-1}^2+1}[x_{N-1}^2]) = (g_\infty[x_{N-1}^2])^2.$$

Hence we have, form (4.5), the relation in ${}_{RS}E_2^{2,*}$:

$$0 = \left(g_2[x_N] + \sum_{J'} g_w[a'_{jk}] + \sum_{I'} g_2[x'_l]\right)[x_{N-1}^2] + (g_2[x_{N-1}^2])^2 + W. \tag{4.6}$$

Here W has no term with $[x_{N-1}^2]$ as factor and the sets J' and I' are appropriate subsets of J and $I\backslash\{m\}$ respectively. Notice that elements x_n, a'_{jk} and x'_l are different each other. Let $k : \Sigma LG \to BLG$ be the inclusion map. From the naturality of the Rothenberg-Steenrod spectral sequence and the definitions of the module derivations g_2 and $\mathscr{D}_G$, we have a commutative diagram:

$$\begin{array}{ccccc}
{}_{RS}E_2^{1,*}(BG) & = & \bar{H}^*(G;\mathbf{Z}/2) & & \\
\Big\downarrow{\scriptstyle g_2} & & \Big\downarrow{\scriptstyle \mathscr{D}_G} & & \\
{}_{RS}E_2^{1,*}(BLG) & \xrightarrow{k^*} & {}_{RS}E_2^{1,*}(\Sigma LG) & = & \bar{H}^*(LG;\mathbf{Z}/2) \\
\Big\downarrow{\scriptstyle j^*} & & \Big\downarrow{\scriptstyle j^*} & & \\
{}_{RS}E_2^{1,*}(B\Omega G) & \xrightarrow{k^*} & {}_{RS}E_2^{1,*}(\Sigma\Omega G) & = & \bar{H}^*(\Omega G;\mathbf{Z}/2).
\end{array}$$

Put $\alpha = g_2[x_N] + \sum_{J'} g_2[a'_{jk}] + \sum_{I'} g_2[x'_l]$. By virtue of Proposition 3.4(2), we see $j^* \circ \mathscr{D}_G = \sigma^*$ and therefore $k^*j^*(\alpha) = \sigma^*(x_N + \sum_{J'} a'_{jk} + \sum_{K'} x'_l)$. Since $\sigma^* : Q^{\mathrm{odd}}H^*(G;\mathbf{Z}/p) \to P^{\mathrm{even}}H^*(\Omega G;\mathbf{Z}/p)$ is injective, it follows that $j^*(\alpha) \neq 0$. Assume that $Ad^* \equiv pr_2^*$. Then, as algebras,

$${}_{RS}E_2^{*,*}(BLG) \cong {}_{RS}E_2^{*,*}(B\Omega G) \otimes {}_{RS}E_2^{*,*}(BG) \cong {}_{RS}E_2^{*,*}(B\Omega G) \otimes \mathbf{Z}/2[[y_i]; y_i \in B].$$

We see that α is non zero in ${}_{RS}E_2^{*,*}(BLG)/(\mathbf{Z}/2[[y_i]; y_i \in B])$ because $j^*(\alpha) \neq 0$. Thus the relation (4.6) contradicts the algebra structure of ${}_{RS}E_2^{*,*}(B\Omega G) \otimes \mathbf{Z}/2[[y_i]; y_i \in B]$. This completes the proof.

§5. Some calculation of adjoint action of finite loop spaces

In order to reconstruct the target of a spectral sequence from the E_∞-term, we need to solve extension problems which are awkward in general. Our proof of

Theorem 1.6 which is describe below shows that extension problems which exist in the first line of the Eilenberg-Moore spectral sequence $\{_{EM}E_r^{*,*}(LX), d_r\}$ can be simply solved by using the module derivation defined in Proposition 3.2.

Proof of Theorem 1.6. Let us consider the Eilenberg-Moore spectral sequence $\{_{EM}E_r^{*,*}(LX), d_r\}$ converging to $H^*(LX; \mathbf{Z}/2)$ for the fibration $\Omega X \to LX \xrightarrow{p} X$ by the assumption, we have

$$_{EM}E_2^{*,*}(LX) \cong HH(H^*(X; \mathbf{Z}/2)) \cong \mathbf{Z}/2[y_1, \dots, y_n] \otimes \Lambda(\tilde{y}_1, \dots, \tilde{y}_n)$$

as a bigraded algebra, where bideg $y_i = (0, \deg y_i)$ and bideg $\tilde{y}_i = (-1, \deg y_i)$. Note that the element $\tilde{y}_i$ corresponds to the elements $[y_i \otimes 1 - 1 \otimes y_i]$ in the Hochschild homology of $H^*(X; \mathbf{Z}/2)$ obtained by the bar complex. Since the indecomposable elements y_i and $\tilde{y}_j$ are permanent cycles, it follows that $E_2^{*,*} \cong E_\infty^{*,*} \cong E_0^{*,*}$ as bigraded algebras. Here $E_0^{p,q}$ denotes the associated bigraded vector space $F^pH^{p+q}(LX; \mathbf{Z}/2)/F^{p+1}H^{p+q}(LX; \mathbf{Z}/2)$. We must solve extension problems to determine the algebra structure of $H^*(LX; \mathbf{Z}/2)$. Let $s : X \to LX$ be the section of p defined by $s(a)(t) = a$ for $a \in X$ and $t \in S^1$. We can choose a representative element $\bar{y}_i$ of $\tilde{y}_i$ so that $s^*(\bar{y}_i) = 0$. Since $\bar{y}_i^2 = Sq^{\deg y_i - 1}\bar{y}_i$ in $H^*(LX; \mathbf{Z}/2)$, it follows that $Sq_{EM}^{\deg y_i - 1}\tilde{y}_i = Sq_{EM}^{\deg y_i - 1}[y_i \otimes 1 - 1 \otimes y_i] = [Sq_{EM}^{\deg y_i - 1} y_i \otimes 1 - 1 \otimes Sq_{EM}^{\deg y_i - 1} y_i]$ in $E_0^{*,*}$. By virtue of Proposition 3.2, we deduce $\bar{y}_i^2 = \mathscr{D}Sq_{EM}^{\deg y_i - 1} y_i + Q$, where $\mathscr{D}$ is a module derivation defined by $y_i \mapsto \bar{y}_i$ and Q is an appropriate element in $F^0H^*(LX; \mathbf{Z}/2)$. Since $s^*(\bar{y}_i) = 0$, we see that $Q = 0$. This completes the proof.

Proof of Theorem 1.7. We recall the algebra structure of $H^*(BG_2; \mathbf{Z}/2)$ over the Steenrod algebra: $H^*(BG_2; \mathbf{Z}/2) \cong \mathbf{Z}/2[y_4, y_6, y_7]$ and $Sq^2 y_4 = y_6$, $Sq^1 y_6 = y_7$. By the Adem relation $Sq^r Sq^4 = Sq^7 Sq^1 + Sq^6 Sq^2$, we see $Sq^4 Sq^4 y_6 = Sq^7 Sq^1 y_6 + Sq^6 Sq^2 y_6 = y_7^2 + Sq^6 Sq^3 Sq^1 y_4 = y_7^2 \neq 0$. This fact yields $Sq^4 y_6 \neq 0$ and hence $Sq^4 y_6 = y_4 y_6$. Thus we have $Sq^5 y_6 = Sq^1 Sq^4 y_6 = y_4 y_7$. From the Adem relation $Sq^3 Sq^4 = Sq^7$, it follows that $Sq^4 y_7 = y_4 y_7$. The relation $Sq^2 Sq^4 = Sq^6 + Sq^5 Sq^1$ enables us to conclude that $Sq^6 y_7 = y_6 y_7$. Applying Theorem 1.6 to these results, we can get Theorem 1.7.

The following proposition is useful for finding the first non-trivial adjoint action $Ad^* : H^*(\Omega G; \mathbf{Z}/2) \to H^*(G; \mathbf{Z}/2) \otimes H^*(\Omega G; \mathbf{Z}/2)$.

Proposition 5.1. *Suppose that*

(1) $H^*(BLG; \mathbf{Z}/2) \cong \mathbf{Z}/2[x_1, \dots, x_n] \otimes \mathbf{Z}/2[y_1, \dots, y_l]/(\rho_1, \dots, \rho_m)$ *for* $* \leq N$, *where each* ρ_j *is decomposable*, $\deg x_1 \geq 2$, $\deg y_1 \geq 2$, $\deg \rho_1 = N$ *and* $\rho_1 = x_1 y_1 +$ *other terms*;

(2) $j^*(x_1)$ *is indecomposable in* $H^*(G; \mathbf{Z}/2)$;

(3) y_1 *is the image by* p^* *of an element in* $H^*(BG; \mathbf{Z}/2)$ *which represents an element in* $_{RS}E_\infty^{1,*}(BG)$;

(4) $Ad^* \equiv pr_2^*$ *for* $* < N - 2$.

Then $Ad^* \not\equiv pr_2^*$ *on* $H^{N-2}(\Omega G; \mathbf{Z}/2)$.

Proof. Let $\{E_r^{*,*}(BLG), d_r\}$ and $\{E_r^{*,*}(B\Omega G), d_r\}$ be the Rothenberg-Steenrod spectral sequences converging to $H^*(BLG; \mathbf{Z}/p)$ and $H^*(B\Omega G; \mathbf{Z}/p)$ respectively. Assume that $Ad^* \equiv pr_2^*$ on $H^{N-2}(\Omega G; \mathbf{Z}/2)$. Then, since $H^*(LG; \mathbf{Z}/2) \cong H^*(\Omega G; \mathbf{Z}/2) \otimes H^*(G; \mathbf{Z}/2)$ as a coalgebra for $* \leq N-2$, it follows that $E_2^{*,*}(BLG) \cong E_2^{*,*}(B\Omega G) E_2^{*,*}(BG)$ as a bigraded algebra for $j \leq N - 2 + (i-1)$, where

$$E_2^{i,j}(BLG) \cong \sum_{i_1+i_2=i, j_1+j_2=j} E_2^{i_1,j_1}(B\Omega G) \otimes E_2^{i_2,j_2}(BG).$$

We show that the element x_1 represents an element which is non-zero in $E_0^{1,*}(BLG)/E_0^{1,*}(BLG) \cap E_2^{1,*}(BG)$. To see this, let us consider the spectral sequence $\{E_r^{*,*}(B\Omega G), d_r\}$. By [6, Corollary §43–1], the homology $H_*(\Omega G; \mathbf{Z}/2)$ is isomorphic to a tensor product of an exterior algebra and a polynomial algebra. Therefore we can write

$$\begin{aligned} E_2^{*,*}(B\Omega G) &\cong Cotor_{H^*(\Omega G; \mathbf{Z}/2)}(\mathbf{Z}/2, \mathbf{Z}/2) \\ &\cong \Lambda(a_1, \ldots, a_s) \otimes \mathbf{Z}/2[b_1, \ldots, b_l] \end{aligned}$$

with elements a_i and b_j in $E_2^{1,*}(B\Omega G)$. Since the spectral sequence $\{E_r^{*,*}(B\Omega G), d_r\}$ has a differential Hopf algebra structure, it follows that indecomposable elements in $H^*(B\Omega G; \mathbf{Z}/2)$ are represented by elements in $E_0^{1,*}(B\Omega G)$. Thus, from the assumption (2), we see that x_1 represents an element in $E_0^{1,*}(BLG)$ and $x_1 \neq 0$ in $E_0^{1,*}(BLG)/E_0^{1,*}(BLG) \cap E_2^{1,*}(BG)$. The assumption (3) enables us to conclude that y_1 represents an element in $E_0^{1,*}(BLG) \cap E_2^{1,*}(BG)$. Since $E_0^{2,*}$ is regarded as a subspace of $E_2^{2,*}$, we can find the relation $0 = \rho_1 = x_1 y_1 + \textit{other terms}$ in $E_2^{2,*}(BLG)$. This relation contradicts the algebra structure of $E_2^{*,*}(BLG)$. We have Proposition 5.1.

In order to explain some calculation of the adjoint action of G_2 on ΩG_2 due to Kono and Kozima [7] and Hamanaka [11], we recall the following results on algebra structure of the mod 2 cohomologies of the exceptional Lie group G_2 and its loop group ΩG_2.

$$H^*(G_2; \mathbf{Z}/2) \cong \mathbf{Z}/2[x_3]/(x_3^4) \otimes \Lambda(x_5),$$

$$H^*(\Omega G_2; \mathbf{Z}/2) \cong \mathbf{Z}/2[a_2]/(a_2^4) \otimes \Gamma[a_8, \omega_{10}].$$

By considering the Eilenberg-Moore spectral sequence $\{{}_{EM}E_r^{*,*}(\Omega BG_2), d_r\}$ we see that, for the element x_3 in Theorem 1.7, $j^*(x_3)$ is the indecomposable element x_3 in $H^3(\Omega BG_2; \mathbf{Z}/2) = H^3(G_2; \mathbf{Z}/2)$. Moreover the element y_4 in Theorem 1.7 is the image by p^* of the element which represents the element $[x_3]$ in ${}_{RS}E_0^{1,*}(BG_2)$. For dimensional reasons, it follows that $Ad^* \equiv pr_2^*$ for $* < 8$. We can write $Ad^*(a_8) = 1 \otimes a_8 + \varepsilon x_3^2 \otimes a_2$, where $\varepsilon = 0$ or 1. By virtue Proposition 5.1, we have

Theorem 5.2 ([7], [11]). $Ad^*(a_8) = 1 \otimes a_8 + x_3^2 \otimes a_2$.

The calculation of $Ad^*(a_8)$ due to Kono and Kozima [7] depends on the fact that the natural inclusion $G_2 \to E_6$ is mod 2 totally non-homologous to zero. The approach of Hamanaka for the calculation relies on some properties of inclusion $SO(3) \to G_2$ and of the 3-connected covers $\widetilde{SO(3)}$ and $\widetilde{G_2}$. Our assertion is that Theorem 5.2 can be proved without using other groups than the Lie group G_2. Since the consideration on the homology rings $H_*(\Omega G_2; \mathbf{Z}/2)$ and $H_*(LG_2; \mathbf{Z}/2)$ due to Hamanaka in [11] is algebraic except proof of Theorem 5.2, we can have Theorem 1.8.

Let $B\mathrm{DI}(4)$ be the Dwyer-Wilkerson complex, whose mode 2 cohomology is isomorphic as an algebra to $\mathbf{Z}/2[y_8, y_{12}, y_{14}, y_{15}]$. From the consideration concerning the Steenrod operation on $H^*(B\mathrm{DI}(4); \mathbf{Z}/2)$ in [8], we see that $Sq^7 y_8 = y_{15}, Sq^{11} y_{12} = y_8 y_{15}$, $Sq^{13} y_{14} = y_{12} y_{15}$ and $Sq^{14} y_{15} = y_{14} y_{15}$ in $H^*(B\mathrm{DI}(4); \mathbf{Z}/2)$. Applying Theorem 1.6, we have Theorem 1.9.

We recall the algebra structure of the mod 2 cohomologies of $\Omega B\mathrm{DI}(4) = G$ and ΩG:

$$H^*(G; \mathbf{Z}/2) \cong \mathbf{Z}/2[x_7]/(x_7^4) \otimes \Lambda(x_{11}, x_{13}),$$

$$H^*(\Omega G; \mathbf{Z}/2) \cong \mathbf{Z}/2[a_6]/(a_6^4) \otimes \Gamma[a_{10}, a_{24}, a_{26}].$$

Our method of the proof of Theorem 5.2 also works for the case where G is the space of based loops on $B\mathrm{DI}(4)$. To be exact, we see from Theorem 1.9 that there exists a relation with degree 22 in $H^*(BLG; \mathbf{Z}/2)$. Therefore, by Proposition 5.1, we can have

Theorem 5.3 ([8]). $Ad^*(\gamma_2(a_{10})) = 1 \otimes \gamma_2(a_{10}) + x_7^2 \otimes a_6$.

As considered in the proof of Theorem 5.1, for any finite loop space G, an influence of non-trivial adjoint action $Ad^* : H^*(\Omega G; \mathbf{Z}/p) \to H^*(G; \mathbf{Z}/2) \otimes H^*(\Omega G; \mathbf{Z}/2)$ appears in the second line of the E_2-term of the Rothenberg-Steenrod spectral sequence $\{{}_{RS}E_r^{*,*}(BLG), d_r\}$. Combining this fact with an explicit calculation of $H^*(BLG; \mathbf{Z}/2)$, we may be able to determine the adjoint action, which is no longer the first non-trivial one. In fact, for the case $G = \Omega B\mathrm{DI}(4)$, we have

Theorem 5.4. *$Ad^*(a_{24}) = 1 \otimes a_{24} + x_7^2 \otimes a_{10}$ and $Ad^*(a_{26}) = 1 \otimes a_{26} + x_7^2 \otimes a_6^2$.*

Proof. Since $Sq^4 \gamma_2(a_{10}) = a_{24}$ and $Sq^4 a_6 = a_{10}$, it follows from Theorem 5.5 that the first equality holds. For dimensional reasons, we can write $Ad^*(a_{26}) = 1 \otimes a_{26} + \varepsilon x_{13} x_7 \otimes a_6 + \eta x_7^2 \otimes a_6^2$, where ε and η are 0 or 1. Let ϕ be the coproduct of $H^*(G; \mathbf{Z}/2)$. Then, by [7, Proposition 2.4], $(\phi \otimes 1) \circ Ad^* = (1 \otimes Ad^*) \circ Ad^*$. This fact enables us to deduce that $\varepsilon = 0$. By Theorem 1.9, we see that $[x_7^2] \cdot [a_6^2] = [x_7^2 \mid a_6^2] = 0$ in ${}_{RS}E_2^{2,26}(BLG)$. It turns out that $\eta \neq 0$.

In general, we expect that the above method using module derivations paves the way for algebraic calculation of the adjoint action $Ad^* : H^*(\Omega G; \mathbf{Z}/p) \to$

$H^*(G; \mathbf{Z}/p) \otimes H^*(\Omega G; \mathbf{Z}/p)$ for an appropriate finite loop space G and any prime p.

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