J. Math. Kyoto Univ. (JMKYAZ)
39-3 (1999), 493–527

1-cocycles on the group of diffeomorphisms II

By

Hiroaki SHIMOMURA

§1. Introduction

In this paper we consider 1-cocycles θ over finite or infinite configuration spaces on C^{∞}-manifolds M and natural representations connected with θ, which is exactly a continuation of the previous author's work [22]. Here the 1-cocycle is as a definition a $U(H)$-valued function on $X\times \mathrm{Diff}_0^*(M)$, which fulfills the so called cocycle equality, and $U(H)$ is the unitary group on a finite dimensional Hilbert space over C, $\mathrm{Diff}_0^*(M)$ is the connected component of the identity id in the group of diffeomorphisms with compact supports on M, and X is a collection B_M^n of all n-point subsets in M, or X is a space Γ_M of all infinite configurations on M. Historically in the first paper of Ismagilov [7] it is described that every irreducible unitary representation of the group $\mathrm{Diff}_0^*(T^1)$ with some additional properties is characterized as the natural representation $U_{\mu,\theta}$ with a suitable measure μ and a 1-cocycle θ on a configuration space or on an analogous one. After this natural representations frequently appeared in order to analyse or to construct unitary representations of $\mathrm{Diff}_0^*(M)$. (cf. [5], [6], [9], [25]) But the study of 1-cocycles have been rather neglected. Recently the author found that when the configuration space is M itself, the form of 1-cocycles is closely connected with a geometrical structure of M. (cf.[22]) That is, under an assumption that M is simply connected, every continuous[1] 1-cocycle θ has a canonical form consisting of only 1-coboundary and Jacobian term which are also the standard examples of 1-cocycles. Besides θ takes locally the canonical form without any assumption on M. Thus it is thought that a glueing of pieces actually determines the form of 1-cocycles on M. Combining these results with [7], we are led to a motivation of the present paper. That is: *Is the situation for a general configuration space similar with the previous one*?, and the answer is affirmative.

Let us explain our results in more detail. The next section begins with five definitions for regularity of 1-cocycles. Among them a notion of precontinuity is most fundamental. The principal part of this section is devoted to the study of precontinuous 1-cocycles $\bar{\theta}(\bar{P},g)$ on $B_M^n\times \mathrm{Diff}_0^*(M)$. Since B_M^n is a quotient space of $\hat{M}^n:=\{\hat{P}=(P_1,\cdots,P_n)\in M^n \mid {}^{\forall}i\neq j,\ P_i\neq P_j\}$ by an equivalence relation, we can always lift $\bar{\theta}$ to $\hat{M}^n\times \mathrm{Diff}_0^*(M)$ as a symmetric one. So it is reasonable to start our study at precontinuous, however not necessarily symmetric, 1-cocycles on $\hat{M}^n\times \mathrm{Diff}_0^*(M)$.

---

[1] The result is still true for precontinuous 1-cocycles as will be seen in the present paper.

Communicated by Prof. T.Hirai, January 20, 1999

The analysis consists of three steps. The first one is a preparation for our tools used in the analysis of $\mathrm{Diff}_0^*(M)$ from the theory on infinite dimensional Lie groups. The most important one among them are the primitive Campbell-Hausdorff formula and a theorem for denseness which assures that diffeomorphisms of exponential mappings generated by smooth vector fields with compact supports generate the whole group. These were already obtained in [22], and as a byproduct we are naturally led to a simple proof of the fact that the whole group $\mathrm{Diff}_0^*(M)$ is generated by local diffeomorphisms. In the second step, the analysis of 1-cocycles is turned to a linear one with a help of these theorems, and a partition of unity reduces it to a local consideration of Lie algebras. Finally in Theorem 2.3 a linear representation of a Lie algebra of smooth vector fields will be examined. In the last step a local behaviour of 1-cocycles is first given by these results without any assumption on M, and next it is observed in order to patch up them globally. It is the time to need a geometrical condition for M or more directly for $\hat{M}^n$. Simply speaking, we have a situation closely resembling to analytic continuation. The 1-coboundary term appeared at this stage defines as a rule a many valued function. In the case of analytic continuation, Principle of monodoromy works so effectively that this kind of ambiguity is clearly resolved, and we will find that it is also useful for our case. This is the reason why we impose the simply connected condition on $\hat{M}^n$. That is to say, whenever $\hat{M}^n$ is simply connected, every precontinuous $\hat{\theta}$ takes a canonical form (Theorem 2.8). In addition a more general and precise statement for cocycle form is given in Theorem 2.5 for our later discussions. According to [3] for the simply connectedness on $\hat{M}^n$ for every $n \in N$, it is sufficient that M is simply connected and $\dim(M) \geq 3$.

The rest of this subsection is devoted to the study of exceptional cases, that is to say $M = \boldsymbol{R}^1, \boldsymbol{R}^2$ and $\boldsymbol{T}^1$. In the first case, B_M^n is itself simply connected, and $\hat{M}^n$ has $n!$ connected components which are all isomorphic to B_M^n. Taking a such isomorphic section we can describe the cocycle form. Of course there are non canonical 1-cocycles on $\hat{M}^n \times \mathrm{Diff}_0^*(M)$, and a later theorem implies that natural representations corresponding to these 1-cocycles are never irreducible, unless $\dim(H) = 1$. So in this case, a class of irreducible natural representations are something narrow. Next, if $M = \boldsymbol{R}^2$, we will see that there is also an example of non canonical 1-cocycle being closely resemble to the one described in the cylindrical case. (cf. [22]) Finally the last case $M = \boldsymbol{T}^1 \equiv \boldsymbol{T}$ is more interesting. $B_{\boldsymbol{T}}^n$ and $\hat{\boldsymbol{T}}^n$ are non simply connected, but the general form of precontinuous 1-cocycles and an example of non canonical one are also given.

In the latter half of this section, we take up natural representations $U_{\bar{\theta}}$ of $\mathrm{Diff}_0^*(M)$ corresponding to a standard measure $\bar{\mu}$ on B_M^n derived from a locally Euclidean finite smooth measure μ on M and to measurable 1-cocycles $\bar{\theta}$. According to [7], a definition for irreducibility of 1-cocycles is given and a criterion for the irreducibility of 1-cocycles is obtained in Theorem 2.12. Further it is assured in Theorem 2.15 that for the irreducibility of $U_{\bar{\theta}}$ it is necessary and sufficient that $\bar{\theta}$ is irreducible. It is noteworthy to remark that in the above theorem we assume that

$\tilde{\theta}$ is strongly Borelian but it is unnecessary to require that $\tilde{\theta}$ is canonical. In the proof we use only the local form of $\tilde{\theta}$ which was already established in Theorem 2.4. Equivalence of natural representations corresponding to 1-cocycles is stated in the last Theorem 2.16 in terms of the cohomologous relation.

In section 3 we assume that M is non compact, and first consider precontinuous 1-cocycles $\tilde{\theta}$ over the infinite configuration space Γ_M. As before we identify $\tilde{\theta}$ with a symmetric precontinuous one $\hat{\theta}$ over $\hat{M}^{\infty}:=\{\hat{P}=(P_1,\cdots,P_n,\cdots)\,|\,{}^{\forall}i\neq j,\ P_i\neq P_j,$ $\{P_n\}_n$ has no accumulation points$\}$. $\mathrm{Diff}_0^*(M)$ acts on $\hat{M}^{\infty}$ diagonally and each orbit $[P]$ containig $\hat{P}$ cosists of all $\hat{Q}\in\hat{M}^{\infty}$ whose components are all equal to that of $\hat{P}$ except for finite numbers of n, of course under an additional but natural assumption on M. So it is reasonable to first restrict our 1-cocycle $\hat{\theta}$ to each $[A]$, $\hat{A}\in\hat{M}^{\infty}$. Then the problem is reduced to the one on finite configuration spaces, and we gather all the resuts in Theorem 2.5 and patch up them by an inductive limit method. In particular If M is simply connected and $\dim(M)\geq 3$, every precontinuous 1-cocycle on $\hat{M}^{\infty}\times\mathrm{Diff}_0^*(M)$ takes a canonical form. However the canonical form obtained here is someting different from that one obtained in section 2. The Jacobian term is the difference between these formulas and it depends, in the present case, on not only $\hat{P}$ of course but also the residue class $[P]$. (cf. Theorem 3.2) As before we consider natural representations of $\mathrm{Diff}_0^*(M)$ over Γ_M which are alike to the one over the finite configuration space. This time, however $\mathrm{Diff}_0^*(M)$-quasi-invariant measure on $(\Gamma_M,\mathfrak{B})$, $\mathfrak{B}$ is the natural Borel field, is not uniquely determined (up to equivalence), so we must consider also a factor of such probabitiy measures $\tilde{\nu}$ on $(\Gamma_M,\mathfrak{B})$. Hence a natural representation is a function of two variables, measure and 1-cocycle, and also a definition of irreducibility of measurable 1-cocycles must be given in terms of $\tilde{\nu}$, which we call it $\tilde{\nu}$-irreducibility. However the resuts are almost parallel to the finite dimensional case.

Finally we wish to say a few words about the dimension of the Hilbert space H. As it was pointed out earlier, throughout this paper $\dim(H)$ is assumed to be finite, unless otherwise stated. However most of the results obtained here seems to be still (or under some additional conditions) true for the infinite dimensional space, especially when M is compact, though I have no proofs yet for them. Perhaps more profound study for the differential representation dU of a given infinite dimensional representation will derive the proper proof.

§2. 1-cocycles on the finite configuration space

2.1. Five definitions of 1-cocycles. Throughout this paper, M stands for d-dimensional paracompact C^{∞}-manifold, $\mathrm{Diff}_0(M)$ is a set of all C^{∞}-diffeomorphisms g on M with compact supports. $\mathrm{Diff}_0(M)$ is equipped with the inductive limit topology τ of τ_K on $\mathrm{Diff}(K)$, where K runs through all compact sets of M, $\mathrm{Diff}(K):=\{g\in\mathrm{Diff}_0(M)\,|\,\mathrm{supp}\,g\subseteq K\}$ and τ_K is the natural C^{∞}-topology on it. The connected component of the unit element id of $\mathrm{Diff}_0(M)$ will be denoted by $\mathrm{Diff}_0^*(M)$, and it is noteworthy to remark that $\mathrm{Diff}_0^*(M)$ is also arcwise connected. (cf. [22])

Hereafter we will work on $\mathrm{Diff}_0^*(M)$, and in a little while we denote $\mathrm{Diff}_0^*(M)$ or its subgroup by G. Suppose that G acts on a measurable space $(X, \mathfrak{B})$ from left as a measurable transformation, gx.

Then we consider a $U(H)$-valued function $\theta(x,g)$ on $X \times G$, called 1-cocycle, which satisfies the following relation.

$$\forall g_1, g_2 \in G, \quad \theta(x, g_1)\theta(g_1^{-1}x, g_2) = \theta(x, g_1 g_2), \tag{2.1}$$

for all $x \in X$, where H is a complex finite or infinite dimensional Hilbert space, and $U(H)$ is the unitary group. We give as below five definitions for regularity of 1-cocycles.

Definition 2.1. (1) *θ is said to be precontinuous, if for any fixed $x_0 \in X$, $\theta(x_0, g)$ is continuous on $G(x_0) := \{g \in G \mid gx_0 = x_0\}$ as a function of g.*
(Of course if G acts transitively, the word "any" can be replaced by "some".)
(2) *θ is said to be continuous, if for any fixed $x_0 \in X$, $\theta(x_0, g)$ is continuous on G as a function of g.*
(3) *θ is said to be Borelian, if it is precontinuous and for any fixed $g \in G$, $\theta(x,g)$ is $\mathfrak{B}$-measurable.*
(4) *θ is said to be strongly Borelian, if it is precontinuous and $\theta(x,g)$ is jointly measurable of both variables.*
(5) *θ is said to be measurable, if for any fixed $g \in G$, $\theta(x,g)$ is $\mathfrak{B}$-measurable.*

In addition it is sometimes expected that the following condition, a kind of continuity, is imposed, whenever μ is G-quasi-invariant, in order that the natural representation corresponding to θ is continuous.

(6) $\forall h_1, h_2 \in H, \quad \langle \theta(x,g)h_1, h_2 \rangle_H \to \langle h_1, h_2 \rangle_H$ in μ, whenever $g \to \mathrm{id}$.

Anyway the relation between these five notions are as follows.
"Strong Borel" implies "Borel", "Borel" implies "Measurability" and "Precontinuity".' Also "Continuity" implies "Precontinuity", and sometimes it implies "Strong Borel", for example Theorem 2.13. Of course "Continuity plus Measurabilty" implies "Strong Borel".

2.2. Local form of precontinuous 1-cocycles. In this subsection we consider precontinuous 1-cocycles $\bar{\theta}$ on $B_M^n \times \mathrm{Diff}_0^*(M)$, where B_M^n $(n = 1, 2, \cdots)$ is a collection of all n-point subsets $\bar{P} = \{P_1, \cdots, P_n\}$ in M with a natural action $\bar{g}$, $g \in \mathrm{Diff}_0^*(M)$ from left.

In order to observe such cocycles, it is convenient to lift them as symmetric 1-cocycles on $\hat{M}^n := \{\hat{P} = (P_1, \cdots, P_n) \in M^n \mid \forall i \neq j,\ P_i \neq P_j\}$ on which $\mathrm{Diff}_0^*(M)$ acts diagonally as $\hat{g}(\hat{P}) := (g(P_1), \cdots, g(P_n))$. Moreover also in order to prepare for the discussions in section 3 we will begin more generally with a study of precontinuous 1-cocycles $\hat{\theta}$ on $\hat{M}_A^n \times \mathrm{Diff}_{0,A}^*(M)$, where $A := \{A_{n+1}, A_{n+2}, \cdots\}$ is an arbitrary set (may be empty set) of M which has no accumulation points, $\hat{M}_A^n := \{\hat{P} \in \hat{M}^n \mid \bar{P} \cap A = \emptyset\}$, and $\mathrm{Diff}_{0,A}^*(M) := \{g \in \mathrm{Diff}_0^*(M) \mid$ there exists a continuous path $\{g_t\}_{0 \le t \le 1}$ in

$\mathrm{Diff}_0^*(M)$ such that $g_0=\mathrm{id}$, $g_1=g$, and $g_t(A_i)=A_i$ for $^\forall i\geq n+1$ and $^\forall t\in[0,1]\}$. Throughout this section $\hat{M}_A^n$ will be sometimes denoted by $\hat{M}_A$ for simplicity.

The following theorem and its proof is quite similar with theorem 2.2 in [22]. We omit its proof.

Theorem 2.1. *Put* $\Gamma_{0,A}(M):=\{X\in\Gamma_0(M)\,|\,X(Q)=0$ *for all* $Q\in A\}$, *where* $\Gamma_0(M)$ *is the set of all* C^∞*-vector fields on* M *with compact supports, and let* $\{\mathrm{Exp}(tX)\}_{t\in\boldsymbol{R}}$ *be a* 1*-parameter subgroup of diffeomorphisms generated by* $X\in\Gamma_{0,A}(M)$. *Then a group generated by* $\mathrm{Exp}(X)$, $X\in\Gamma_{0,A}(M)$ *is dense in* $\mathrm{Diff}_{0,A}^*(M)$.

Remark 2.1. *Actually this theorem is true not only for the above set* A *but also for any general subset of* M.

As an immediate consequence of Remark 2.1 and primitive Campbell-Hausdorff formula in [22],

Theorem 2.2. *Let* $\{V_\lambda\}_{\lambda\in\Lambda}$ *be any open covering of* M. *Then a group generated by all subgroups of local diffeomorphisms,* $\mathrm{Diff}_{0,A}^*(V_\lambda)$, $\lambda\in\Lambda$, *is dense in* $\mathrm{Diff}_{0,A}^*(M)$.

Proof. Take any g from $\mathrm{Diff}_{0,A}^*(M)$. Then it is approximated by a finite product of $\mathrm{Exp}(X)$, $X\in\Gamma_{0,A}(M)$ by Remark 2.1. Next decompose X into finitely many $X_i\in\Gamma_{0,A}(M)$, using a partition of unity subordinate to a locally finite refinement of the above covering. Thus each $\mathrm{Exp}\left(\dfrac{X_i}{n}\right)$, $n\in\boldsymbol{N}$ belongs to our local diffeomorphism groups. Finally applying the primitive Campbell-Hausdorff formula to them repeatedly. This completes the proof.

In particular in the case of $A=\emptyset$, Theorem 2.2 assures that a group generated by all local diffeomorphisms is dense in $\mathrm{Diff}_0^*(M)$. It is somewhat well known, but the proof stated here rather simple. Next we go to a key theorem in this section.

Theorem 2.3. *Let* H *be a Hilbert space over* $\boldsymbol{C}$ *whose dimension may be infinite. For* $\alpha>0$ *put*

$$U_\alpha:=\{x\in\boldsymbol{R}^d\,|\,\,|x_k|<\alpha\ (k=1,\cdots,d)\},$$

$$\mathscr{G}_\alpha^0:=\{F=(f_k(x))_{1\leq k\leq d}\,|\,f_k\in C_0^\infty(U_\alpha)\ (k=1,\cdots,d)\quad \textit{and}\quad F(0)=0\},$$

where $C_0^\infty(U_\alpha)$ *is the test function space in* U_α, *and*

$$[F,G]:=\sum_{k=1}^{d}\{f_k(x)\frac{\partial G}{\partial x_k}(x)-g_k(x)\frac{\partial F}{\partial x_k}(x)\}\quad \text{for all } F,G\in\mathscr{G}_\alpha^0.$$

Then for any continuous linear representation dU *from* $\mathscr{G}_\alpha^0$ *to* $S(H):=\{T:\text{bdd.op. on } H\,|\,T^*=-T\}$, (*the toplogy on* $\mathscr{G}_\alpha^0$ *is the natural one derived from the test function space and* $S(H)$ *is equipped with the weak operator topology*), *there exists a* $S\in S(H)$ *such that*

$$dU(F)=\left(\sum_{k=1}^{d}\frac{\partial f_k}{\partial x_k}(0)\right)S.$$

Proof. Before begining the proof let us write down explicitly the assumption on preservation of the Lie brackets;

$$dU([F,G])=-[dU(F),dU(G)]:=dU(G)dU(F)-dU(F)dU(G). \tag{2.2}$$

Now take any point $x\neq 0$ in U_α and consider a cubic neighbourhood $U(x)$ not containig 0. Evidently $C_0^\infty(U(x))\subset\mathscr{G}_\alpha^0$, so it follows from lemma 3.1 in [22] that

$$dU\mid C_0^\infty(U(x))\equiv 0. \tag{2.3}$$

In other words, $F\in\mathscr{G}_\alpha^0$ and $0\notin\operatorname{supp}F$ imply $dU(F)=0$. Henceforth we take and fix a $\phi\in C_0^\infty(U_\alpha)$ satisfying $\phi\equiv 1$ on a neighbourhood of 0. Then

$$dU(F)=dU(\phi F) \tag{2.4}$$

for any $F\in\mathscr{G}_\alpha^0$. Here we break off the proof for a little while, since we need the following lemma.

Lemma 2.1. *Let A be a bounded operator on H and $B\in S(H)$, and suppose that $AB-BA=cB$ holds for some non zero $c\in\boldsymbol{C}$. Then we must have $B=0$.*

Proof. From the assumption and mathematical induction on n,

$$AB^n-B^nA=cnB^n \quad (n=1,2,\cdots)$$

follows easily. Take the operator norm of both sides in the above equality. Then,

$$|c|n\|B^n\|\leq 2\|A\|\|B^n\|.$$

Thus either it holds $A=0$, which gives $B=0$ or it holds $B^n=0$ for a sufficiently large n, which also gives $B=0$ due to the assumption.

Let us return to the proof of Theorem 2.3. Put

$$C_{0,0}^\infty(U_\alpha):=\{f\in C_0^\infty(U_\alpha)\mid f(0)=0\}$$

and

$$dU_k(f):=dU(f\hat{1}_k) \quad (1\leq k\leq d),$$

where $\hat{1}_k$ is an $\boldsymbol{R}^d$-valued constant map whose kth component is equal to 1 and the other components are all equal to 0. Further put

$$[f,g]_k:=f\frac{\partial g}{\partial x_k}-g\frac{\partial f}{\partial x_k} \quad (f,g\in C_{0,0}^\infty(U_\alpha)).$$

Then

(2.5) $$dU_k([f,g]_k)=-[dU_k(f),dU_k(g)],$$

and for a polynomial with parameter $t\in \boldsymbol{R}^{d-1}$; $P_t(x)\equiv P_{t,k}(x):=t_1x_1+\cdots+t_{k-1}x_{k-1}+x_k+t_{k+1}x_{k+1}+\cdots+t_dx_d$ and for all $m\in \boldsymbol{N}$,

(2.6) $$[P_t\phi, P_t^m\phi]_k=(m-1)P_t^m\phi^2.$$

It follows from the definition of ϕ that

(2.7) $$[dU_k(P_t\phi), dU_k(P_t^m\phi)]=-(m-1)dU_k(P_t^m\phi).$$

Therefore for $m\geq 2$ we find that

(2.8) $$dU_k(P_t^m\phi)=0$$

by virtue of Lemma 2.1. Consequently

(2.9) $$dU_k(x_1^{\alpha_1}\cdots x_d^{\alpha_d}\phi)=0$$

holds for all d-tuple $(\alpha_1,\cdots,\alpha_d)$ of non negative integers satisfying $\alpha_1+\cdots+\alpha_d\geq 2$.

On the other hand take any $g\in C_0^\infty(U_\alpha)$ and set

$$h(x):=\phi(x)\int_{-\alpha}^{x_k} g(x_1,\cdots,x_{k-1},u,x_{k+1},\cdots,x_d)du.$$

Then,

$$\frac{\partial h}{\partial x_k}(x)=\phi(x)g(x)+g_1(x),$$

where $g_1\in C_0^\infty(U_\alpha)$ is a function vanishing on a neighbourhood of 0. Thus for $m\geq 2$ an equality

$$[P_t^m h, P_t^m\phi]_k(x)=P_t^{2m}(x)(h(x)\frac{\partial\phi}{\partial x_k}(x)-\phi(x)\frac{\partial h}{\partial x_k}(x))$$

and (2.8) lead to

(2.10) $$dU_k(P_t^{2m}g)=0.$$

In particular for all $(\alpha_1,\cdots,\alpha_d)$ with $\alpha_1+\cdots+\alpha_d=4$, we find that

(2.11) $$dU_k(x_1^{\alpha_1}\cdots x_d^{\alpha_d}g)=0.$$

Now take any $f\in C_{0,0}^\infty(U_\alpha)$ and choose the above ϕ so that it is equal to 1 on a neighbourhood of $\mathrm{supp} f\cup\{0\}$. Then it follows from Taylor's expansion of f at 0,

$$f(x)=\phi(x)\sum_{i=1}^{3}\frac{1}{i!}\sum_{|\alpha|=i}\frac{\partial^i f}{\partial x_1^{\alpha_1}\cdots\partial x_d^{\alpha_d}}(0)x_1^{\alpha_1}\cdots x_d^{\alpha_d}+\frac{1}{3!}\phi(x)\int_0^1\frac{\partial^4 f}{\partial t^4}(tx)(1-t)^3dt,$$

and from (2.9), (2.11) that

$$dU_k(f)=\sum_{i=1}^{d}\frac{\partial f}{\partial x_i}(0)dU_k(x_i\phi).$$

By the way for $i\neq k$, it is straightforward to check that

$$[dU_k(x_k\phi), dU_k(x_i\phi)]_k=-dU_k([x_k\phi, x_i\phi]_k)=dU_k(x_i\phi),$$

so we find again that

$$dU_k(x_i\phi)=0 \tag{2.12}$$

by virtue of Lemma 2.1. Hence

$$dU_k(f)=\frac{\partial f}{\partial x_k}(0)S_k,$$

in other words for all $F\in\mathscr{G}_\alpha^0$

$$dU(F)=\sum_{k=1}^{d}\frac{\partial f_k}{\partial x_k}(0)S_k, \tag{2.13}$$

where $S_k:=dU_k(x_k\phi)$ does not depend on a particular choice of ϕ.

Lastly we show that S_k is the same one for all k. Let $A:=(a_{i,j})$, $B:=(b_{i,j})$ be any $d\times d$ matrices, and take $F=(f_k)_{1\leq k\leq d}$, $G=(g_k)_{1\leq k\leq d}\in\mathscr{G}_\alpha^0$ such that

$$\frac{\partial f_i}{\partial x_j}(0)=a_{i,j},\quad\text{and}\quad\frac{\partial g_i}{\partial x_j}(0)=b_{i,j}$$

for all $1\leq i,j\leq d$. It follows from (2.2) and (2.13) that

$$\sum_{i,j=1}^{d}a_{i,j}b_{j,i}(S_i-S_j)=\sum_{i,j=1}^{d}a_{i,i}b_{j,j}[S_i, S_j]. \tag{2.14}$$

Thus $a_{i,j}:=\alpha_i\delta_{i,j}$ and $b_{i,j}:=\beta_i\delta_{i,j}$, $(\alpha_i, \beta_i\in \boldsymbol{C})$ give $[S_i, S_j]=0$, while $a_{i,j}=b_{i,j}:=\delta_{i,i_0}\delta_{j,j_0}$ give that $S_{i_0}=S_{j_0}$ for all $1\leq i_0, j_0\leq d$.

With a help of Theorem 2.1, Theorem 2.3 and primitive Campbell-Hausdorff formula in [22] it enables for us to decide a local form of precontinuous 1-cocycle $\hat{\theta}$ on $\hat{M}_A\times\mathrm{Diff}_{0,A}^*(M)$. Take any $\hat{Q}=(Q_1,\cdots,Q_n)\in\hat{M}_A$ and fix it, Further put

$$\Gamma_{0,A,Q}(M):=\{X\in\Gamma_{0,A}(M)\mid X(Q_i)=0 \text{ for } 1\leq{}^\forall i\leq n\},\quad\text{and}$$

$\mathrm{Diff}_{0,A,Q}^*(M):=\{g\in\mathrm{Diff}_{0,A}^*(M)\mid$ there exists a continuous path $\{g_t\}_{0\leq t\leq 1}$ in $\mathrm{Diff}_0^*(M)$ such that $g_0=\mathrm{id}$, $g_1=g$, and $\hat{g}_t(\hat{Q})=\hat{Q}$, $g_t(A_{n+i})=A_{n+i}$ for ${}^\forall i\in\boldsymbol{N}$ and ${}^\forall t\in(0,1]\}$. Then Stone's theorem and the primitive Campbell-Hausdorff formula lead to a weakly continuous linear representation dU from $\Gamma_{0,A,Q}$ to $S(H)$ such that

$$\hat{\theta}(Q, \mathrm{Exp}(X)) = \exp(dU(X)) \tag{2.15}$$

for all $X \in \Gamma_{0,A,Q}$. Decompose X into finitely many X_i's using a partition of unity so that $\operatorname{supp} X_i$ is contained in a cubic neighbourhood, and next apply the result in Theorem 2.3. Finally calculating in the same way as in [22], we find that

$$dU(X) = \sqrt{-1} \sum_{i=1}^{n} \log J_{\mathrm{Exp}(X)}(Q_i) H_i + \sqrt{-1} \sum_{i=n+1}^{\infty} \log J_{\mathrm{Exp}(X)}(A_i) H_i, \tag{2.16}$$

where $\{H_i\}_{1 \le i < \infty}$ is a commutative system of self-adjoint operators on H and $J_{\mathrm{Exp}(X)}(Q_i)$ is the Jacobian matrix of $\mathrm{Exp}(X)$ at Q_i.

Now take open neighbourhoods $V^0(Q_i)$, $V^1(Q_i)$ of Q_i $(i = 1, \cdots, n)$ which fulfill the following conditions.

(a) $V^0(Q_i)$ is diffeomorphic to $\boldsymbol{R}^d$,

(b) $V^0(Q_i) \cap V^0(Q_j) = \emptyset$ for $^\forall i \neq j$,

(c) $V^0(Q_i) \cap \{A_{n+1}, A_{n+2}, \cdots\} = \emptyset$ for $1 \le {}^\forall i \le n$,

(d) $\overline{V^1(Q_i)}$ is compact and it is contained in $V^0(Q_i)$ for $1 \le {}^\forall i \le n$.

Put

$$V^0(\hat{Q}) := V^0(Q_1) \times \cdots \times V^0(Q_n) \quad \text{and} \quad V^1(\hat{Q}) := V^1(Q_1) \times \cdots \times V^1(Q_n).$$

It is not hard to see that there exists a continuous section $s_{P_i} \in \mathrm{Diff}_0^*(V^0(Q_i))$ on $V^1(Q_i)$. That is, $s_{P_i}(Q_i) = P_i$ and a map $P_i \in V^1(Q_i) \mapsto s_{P_i} \in \mathrm{Diff}_0^*(V^0(Q_i))$ is continuous. Thus

$$s_{\hat{P}} := s_{P_1} \circ s_{P_2} \circ \cdots \circ s_{P_n}$$

verifies the following conditions

$$s_{\hat{P}}(Q_i) = P_i \ (1 \le {}^\forall i \le n),\ s_{\hat{P}}(A_i) = A_i \ ({}^\forall i \ge n+1), \text{ and}$$

a map; $\hat{P} \in V^1(\hat{Q}) \mapsto s_{\hat{P}} \in \mathrm{Diff}_0^*(V^0(Q_1) \cup \cdots \cup V^0(Q_n))$ is continuous.

Here let us impose the following fundamental condition (*) on $(\hat{P}, g)$.

(*) For $(\hat{P}, g) \in \hat{M}_A^n \times \mathrm{Diff}_{0,A}^*(M)$, there exists a continuous path $\{g_t\}_{0 \le t \le 1} \subset \mathrm{Diff}_{0,A}^*(M)$ such that $g_0 = \mathrm{id}$, $g_1 = g$, and $\hat{g}_t^{-1}(\hat{P}) \in V^1(\hat{Q})$ for $0 \le {}^\forall t \le 1$.

$s_{\hat{P}}^{-1} \circ g_t \circ s_{\hat{g}_t^{-1}(\hat{P})}$ is defined for such a pair $(\hat{P}, g)$ and it gives a continuous path in $\mathrm{Diff}_{0,A,Q}^*(M)$ connecting id and $s_{\hat{P}}^{-1} \circ g \circ s_{\hat{g}^{-1}(\hat{P})}$. It follows from Theorem 2.1 and from (2.16) that

$$\hat{\theta}(\hat{Q}, \phi) = \prod_{k=1}^{n} J_\phi(Q_k)^{\sqrt{-1} H_k} \prod_{k=n+1}^{\infty} J_\phi(A_k)^{\sqrt{-1} H_k} \tag{2.17}$$

holds for $\phi := s_{\hat{P}}^{-1} \circ g \circ s_{\hat{g}^{-1}(\hat{P})}$, whenever $(\hat{P}, g)$ satisfies the condition (*). Let us introduce any σ-finite smooth locally Euclidean measure μ on M. Note that for $1\leq{}^{\forall}i\leq n$

$$(2.18)\qquad J_{s_{P_i}^{-1}\circ g\circ s_{g^{-1}(P_i)}}(Q_i)^{-1}=\frac{d\mu_g}{d\mu}(P_i)\frac{d\mu_{s_{g^{-1}(P_i)}}}{d\mu}(g^{-1}(P_i))\left(\frac{d\mu_{s_{P_i}}}{d\mu}(P_i)\right)^{-1},$$

and that for $1\leq{}^{\forall}i\leq n$, ${}^{\forall}j\geq n+1$

$$(2.19)\qquad J_{s_{P_i}^{-1}\circ g\circ s_{g^{-1}(P_i)}}(A_j)^{-1}=\frac{d\mu_g}{d\mu}(A_j).$$

Hence from the cocycle equality,

$$\hat{\theta}(\hat{P},g)=\hat{\theta}(\hat{Q},s_{\hat{P}}^{-1})^{-1}\hat{\theta}(\hat{Q},s_{\hat{P}}^{-1}\circ g\circ s_{g^{-1}(\hat{P})})\hat{\theta}(\hat{Q},s_{\hat{g}^{-1}(\hat{P})}^{-1}),$$

we have the following theorem with $C(\hat{P})$ defined by

$$(2.20)\qquad C(\hat{P}):=\prod_{k=1}^{n}\left(\frac{d\mu_{s_{P_k}}}{d\mu}(P_k)\right)^{-\sqrt{-1}H_k}\hat{\theta}(\hat{Q},s_{\hat{P}}^{-1}).$$

Theorem 2.4 (Local form of precontinuous 1-cocycle). *Let $\hat{\theta}$ be a $U(H)$-valued precontinuous 1-cocycle on $\hat{M}_A^n\times\mathrm{Diff}_{0,A}^*(M)$. Then for any $\hat{Q}\in\hat{M}_A^n$ there exist a relatively compact open neighbourhood of $V^1(\hat{Q})\subset\hat{M}_A^n$, a $U(H)$-valued map C defined on $V^1(\hat{Q})$ and a commutative system of self-adjoint operators $\{H_k\}_{1\leq k<\infty}$ on H such that*

$$(2.21)\quad \hat{\theta}(\hat{P},g)=C(\hat{P})^{-1}\prod_{k=1}^{n}\left(\frac{d\mu_g}{d\mu}(P_k)\right)^{\sqrt{-1}H_k}\prod_{k=n+1}^{\infty}\left(\frac{d\mu_g}{d\mu}(A_k)\right)^{\sqrt{-1}H_k}C(\hat{g}^{-1}(\hat{P})),$$

provided that $(\hat{P},g)$ satisfies the condition $(*)$.

If moreover $\hat{\theta}$ is continuous, then so is the map C.

2.3. Canonical form of precontinuous 1-cocycles. Next we shall observe a behaviour of the pair $(C,\{H_k\}_k)$ varying the basic point $\hat{Q}$. Take an open covering $\{V_i^2\}_i$, a locally finite refinement of the covering $\{V^1(\hat{Q})\}_{\hat{Q}}$, so that any intersection of any two sets in this collection is connected unless it is not empty. For example it is enough to take a simple covering. Moreover take open coverings $\{V_i^3\}_i$, $\{V_i^4\}_i$, $\{V_i^5\}_i$ and $\{V_i^6\}_i$ which satisfy,

$$\overline{V_i^6}\subset V_i^5,\ \overline{V_i^5}\subset V_i^4,\ \overline{V_i^4}\subset V_i^3,\text{ and }\overline{V_i^3}\subset V_i^2.$$

Finally take any one of $V^1(\hat{Q})$ containing V_i^2 and denote it by V_i^1. We may assume that $\overline{V_i^2}\subset V_i^1$. Let $(C_i,\{H_{k,i}\}_k)$ be a pair corresponding to the relatively compact open set V_i^1, and restrict C_i to V_i^2, which will be again denoted by the same letter C_i. It is easy to see that for any point $\hat{P}=(P_1,\cdots,P_n)\in V_i^2\cap V_j^2$, for any k and for any $a_k>0$ $(k=1,\cdots,n)$, there exists a continuous path $\{g_t\}_{0\leq t\leq 1}$ in

$\mathrm{Diff}^*_{0,A}(M)$ connecting id and g such that $\frac{d\mu_g}{d\mu}(P_k)=a_k$, $\operatorname{supp} g_t \cap (A \cup \{P_1, \cdots, P_{k-1}, P_{k+1}, \cdots, P_n\}) = \emptyset$ and $\hat{g}_t^{-1}(\hat{P})=\hat{P}$ for all $t \in [0,1]$. It together with (2.21) lead us to

$$C_i(\hat{P})^{-1} H_{k,i} C_i(\hat{P}) = C_j(\hat{P})^{-1} H_{k,j} C_j(\hat{P}) \tag{2.22}$$

for all $\hat{P} \in V_i^2 \cap V_j^2$. Thus (2.21) and (2.22) imply that whenever $(\hat{P}, g)$ satisfies the condition (*) in which $V^1(\hat{Q})$ is now replaced by V_i^2 or V_j^2, and further satisfies $\operatorname{supp} g \cap A = \emptyset$, then

$$C_i(\hat{P}) C_j(\hat{P})^{-1} = C_i(\hat{g}^{-1}(\hat{P})) C_j(\hat{g}^{-1}(\hat{P}))^{-1}. \tag{2.23}$$

Hence if $V_i^2 \cap V_j^2 \neq \emptyset$, using local diffeomorphisms repeatedly, we find that $C_i(\hat{P})C_j(\hat{P})^{-1}$ is constant on $V_i^2 \cap V_j^2$. Set

$$K_{i,j} := C_i(\hat{P}) C_j(\hat{P})^{-1}. \tag{2.24}$$

Immediately (2.22) shows that

$$H_{k,i} = K_{i,j} H_{k,j} K_{i,j}^{-1} \tag{2.25}$$

holds for $1 \leq {}^\forall k \leq n$. Moreover taking diffeomorphisms with small supports which are alike to similar transformations at A_k, we find that (2.25) is also valid for all $k \geq n+1$.

The third step is to patch up these results. Put

$$K_m := \mathrm{Cl}(V_1^3 \cup \cdots \cup V_m^3), \qquad I_m := \{i \in N \mid K_m \cap V_i^3 \neq \emptyset\},$$

and

$\mathscr{U}_m := \{g \in \mathrm{Diff}^*_{0,A}(M) \mid$ there exists a continuous path $\{g_t\}_{0 \leq t \leq 1} \subset \mathrm{Diff}^*_{0,A}(M)$ connecting id and g such that $\operatorname{supp} g_t \subseteq K_m$, $\hat{g}_t^{-1}(\overline{V_i^6}) \subset V_i^5$, $\hat{g}_t(\overline{V_i^5}) \subset V_i^4$, $\hat{g}_t^{-1}(\overline{V_i^4}) \subset V_i^3$, and $\hat{g}_t(\overline{V_i^3}) \subset V_i^2$ for ${}^\forall i \in I_m$ and ${}^\forall t \in [0,1]\}$.

Note that $\operatorname{supp} g_t \cap A = \emptyset$ for all $t \in [0,1]$ and that $m \leq m'$ implies $\mathscr{U}_m \subseteq \mathscr{U}_{m'}$. Finally put

$$m(\hat{P}) := \inf\{m \in N \mid \hat{P} \in V_1^3 \cup \cdots \cup V_m^3\}.$$

It is not hard to see that $\{\hat{g}(\hat{P}) \mid g \in \mathscr{U}_{m(\hat{P})}\}$ contains a connected open neighbourhood $O_{\hat{P}}$ of $\hat{P}$ in some V_i^6.

From now on we assume that $\hat{M}_A^n$ *is simply connected*, and use the Principle of monodoromy. Put

$$D := \cup_{\hat{P} \in \hat{M}_A} O_{\hat{P}} \times O_{\hat{P}},$$

and define a map $\varphi_{\hat{P},\hat{Q}}$, $((\hat{P},\hat{Q}) \in D)$ on $U(H)$ by

$$\varphi_{\hat{P},\hat{Q}}(U) = C_i(\hat{Q})^{-1} C_i(\hat{P}) U, \quad \text{if} \quad (\hat{P},\hat{Q}) \in V_i^6 \times V_i^6. \tag{2.26}$$

This definition does not depend on i by virtue of (2.24). Now in order to check the relation,

$$\varphi_{\hat{P},\hat{R}}=\varphi_{\hat{Q},\hat{R}}\circ\varphi_{\hat{P},\hat{Q}},$$

let $(\hat{P},\hat{Q})\in O_{\hat{X}}\times O_{\hat{X}}$, $(\hat{Q},\hat{R})\in O_{\hat{Y}}\times O_{\hat{Y}}$, and $(\hat{P},\hat{R})\in O_{\hat{Z}}\times O_{\hat{Z}}$. Then there exist $g_1, g_2\in\mathscr{U}_{m(\hat{X})}$, $g_3, g_4\in\mathscr{U}_{m(\hat{Y})}$ and $i,j,k\in N$ so that

$$\hat{P}=\hat{g}_1(\hat{X}),\ \hat{Q}=\hat{g}_2(\hat{X}),\ \hat{Q}=\hat{g}_3(\hat{Y}),\ \hat{R}=\hat{g}_4(\hat{Y}),$$
$$O_{\hat{X}}\subseteq V_i^6,\ O_{\hat{Y}}\subseteq V_j^6,\quad \text{and } O_{\hat{Z}}\subseteq V_k^6.$$

Put $m:=\max(m(\hat{X}),\ m(\hat{Y}))$. Evidently,

$$g_s\in\mathscr{U}_m\ (s=1,2,3,4),\ \hat{X},\hat{Y}\in K_m,\ \text{and } i,j\in I_m.$$

So taking a continuous path $\{g_{s,t}\}_{0\le t\le 1}$ corresponding to g_s appeared in the definition $\mathscr{U}_m$, we find that

$$\hat{g}_{4,t}\circ\hat{g}_{3,t}^{-1}\circ\hat{g}_{2,t}\circ\hat{g}_{1,t}^{-1}(\overline{V_i^6})\subset V_i^2,$$
$$\hat{g}_{2,t}\circ\hat{g}_{1,t}^{-1}(\overline{V_i^6})\subset V_i^4,\quad \hat{g}_{4,t}\circ\hat{g}_{3,t}^{-1}(\overline{V_j^6})\subset V_j^4,$$

and it follows from Theorem 2.4 that

$$\hat{\theta}(\hat{P},g)=C_i(\hat{P})^{-1}\prod_{k=1}^{n}\left(\frac{d\mu_g}{d\mu}(P_k)\right)^{\sqrt{-1}H_{k,i}}C_i(\hat{g}^{-1}(\hat{P})),$$

$$\hat{\theta}(\hat{P},g_1\circ g_2^{-1})=C_i(\hat{P})^{-1}\prod_{k=1}^{n}\left(\frac{d\mu_g}{d\mu}(P_k)\right)^{\sqrt{-1}H_{k,i}}C_i(\hat{g}_2\circ\hat{g}_1^{-1}(\hat{P})),$$

$$\hat{\theta}(\hat{P},g_3\circ g_4^{-1})=C_j(\hat{Q})^{-1}\prod_{k=1}^{n}\left(\frac{d\mu_g}{d\mu}(Q_k)\right)^{\sqrt{-1}H_{k,j}}C_j(\hat{g}_4\circ\hat{g}_3^{-1}(\hat{P})),$$

where $g:=g_1\circ g_2^{-1}\circ g_3\circ g_4^{-1}$. Note that $\hat{Q}\in V_i^6\cap V_j^6$, which together with (2.22) lead us to

$$C_i(\hat{Q})^{-1}H_{k,i}C_i(\hat{Q})=C_j(\hat{Q})^{-1}H_{k,j}C_j(\hat{Q}).$$

Hence from the cocycle equality we get

$$C_i(\hat{R})=C_i(\hat{Q})C_j(\hat{Q})^{-1}C_j(\hat{R}). \tag{2.27}$$

In other words,

$$C_k(\hat{R})^{-1}C_k(\hat{P})=C_i(\hat{R})^{-1}C_i(\hat{P})=C_j(\hat{R})^{-1}C_j(\hat{Q})C_i(\hat{Q})^{-1}C_i(\hat{P}),$$

and the desired result follows. (The first equality is an immediate consequence of

$\hat{P}, \hat{R} \in V_i^2 \cap V_k^2$ and (2.24).)

In conclusion we have a $U(H)$-valued map C on the whole of $\hat{M}_A$ so that $(\hat{P}, \hat{Q}) \in D \cap (V_i^6 \times V_i^6)$ implies

$$C_i(\hat{Q})^{-1} C_i(\hat{P}) C(\hat{P})^{-1} = C(\hat{Q})^{-1}. \tag{2.28}$$

It is straightforward to check that the continuity of C follows from that of C_i.

Let us go to the next step succesively. For $1 \leq k < \infty$ define H_k on $\hat{M}_A$ by

$$H_k(\hat{P}) := C(\hat{P}) C_i(\hat{P})^{-1} H_{k,i} C_i(\hat{P}) C(\hat{P})^{-1}, \quad \text{if } \hat{P} \in V_i^2. \tag{2.29}$$

It is well-defined by virtue of (2.25) and

$$H_k(\hat{P}) = H_k(\hat{Q}) \tag{2.30}$$

for all $(\hat{P}, \hat{Q}) \in D \cap (V_i^6 \times V_i^6)$. Hence $H_k(\hat{P})$ is locally and therefore globally constant, say H_k, due to the connectedness of $\hat{M}_A$. Consequently (2.21) in Theorem 2.4 now becomes

$$\hat{\theta}(\hat{P}, g) = C(\hat{P})^{-1} \prod_{k=1}^{n} \left(\frac{d\mu_g}{d\mu}(P_k) \right)^{\sqrt{-1}H_k} \prod_{k=n+1}^{\infty} \left(\frac{d\mu_g}{d\mu}(A_k) \right)^{\sqrt{-1}H_k} C(\hat{g}^{-1}(\hat{P})), \tag{2.31}$$

provided that $(\hat{P}, g) \in \hat{M}_A \times \mathrm{Diff}_{0,A}^*(M)$ fulfills the following condition (**).

(**) For $(\hat{P}, g)$ there exists a continuous path $\{g_t\}_{0 \leq t \leq 1} \subset \mathrm{Diff}_{0,A}^*(M)$ connecting id and g and an $\hat{X} \in \hat{M}_A$ such that

$$\hat{g}_t^{-1}(\hat{P}) \in O_{\hat{X}} \quad \text{for} \quad {}^\forall t \in [0, 1].$$

Put

$$\hat{\zeta}(\hat{P}, g) := \text{the right hand side of (2.31).}$$

We claim that $\hat{\theta} = \hat{\zeta}$ at the final stage. Let K be any compact set of M and set $\mathrm{Diff}^*(K) := \{g \in \mathrm{Diff}_0^*(M) |$ there exists a continuous path $\{g_t\}_{0 \leq t \leq 1}$ connecting id and g such that $\operatorname{supp} g_t \subseteq K$ for $\forall t \in [0, 1]\}$.
Then for any $\hat{P} \in \hat{M}_A$ there exists a neighbourhood $\mathscr{U}_{K, \hat{P}}$ of id in $\mathrm{Diff}^*(K)$ so that $g \in \mathscr{U}_{K, \hat{P}}$ implies $\hat{g}^{-1}(\hat{P}) \in O_{\hat{P}}$. Now take an arbitrary $g \in \mathrm{Diff}_{0,A}^*(M)$ with the corresponding path $\{g_t\}_{0 \leq t \leq 1} \subset \mathrm{Diff}_{0,A}^*(M)$. By a property of inductive limit topology there exist a compact set K of M such that $\operatorname{supp} g_t \subseteq K$ for all $t \in [0, 1]$. We claim that

$$T := \{t \in [0, 1] \mid \hat{\theta}(\hat{P}, g_t) = \hat{\zeta}(\hat{P}, g_t)\}$$

coincides with $[0, 1]$. First we show that it is open. Let $t \in T$ and put $\hat{P}_t := \hat{g}_t^{-1}(\hat{P})$. Take a $\delta > 0$ such that $|s - t| < \delta$ implies $q_s := g_t^{-1} g_s \in \mathscr{U}_{K, \hat{P}_t}$. Thus $(\hat{P}_t, q_s)$ satisfies the condition (**) and it follows that

$$\hat{\theta}(\hat{P}_t, q_s) = \hat{\zeta}(\hat{P}_t, q_s).$$

In other words,

$$\begin{aligned}\hat{\theta}(\hat{P}, g_s) &= \hat{\theta}(\hat{P}, g_t)\hat{\theta}(\hat{P}_t, q_s)\\ &= \hat{\zeta}(\hat{P}, g_t)\hat{\zeta}(\hat{P}_t, q_s)\\ &= \hat{\zeta}(\hat{P}, g_s).\end{aligned}$$

for all $|s-t|<\delta$. The closedness of T is similary proved, so the conclusion follows, since $T \ni 0$. By the above we have the following theorem.

Theorem 2.5 (Global form of precontinuous 1-cocycle). (1) *Suppose that $\hat{M}_A^n$ is simply connected. Then for any precontinuous $U(H)$-valued 1-cocycle $\hat{\theta}$ on $\hat{M}_A^n \times \mathrm{Diff}_{0,A}^*(M)$, there exists a $U(H)$-valued map C on $\hat{M}_A^n$ and a commutative system of self-adjoint operators $\{H_k\}_k$ on H such that*

$$(2.32)\qquad \hat{\theta}(\hat{P}, g) = C(\hat{P})^{-1} \prod_{k=1}^{n} \left(\frac{d\mu_g}{d\mu}(P_k)\right)^{\sqrt{-1}H_k} \prod_{k=n+1}^{\infty} \left(\frac{d\mu_g}{d\mu}(A_k)\right)^{\sqrt{-1}H_k} C(\hat{g}^{-1}(\hat{P}))$$

for all $(\hat{P}, g) \in \hat{M}_A^n \times \mathrm{Diff}_{0,A}^*(M)$.

If $\hat{\theta}$ is continuous, then so is the map C.

(2) *Assume that $\hat{M}_A^n$ is connected. Let $\hat{\theta}$ be given by* (2.32) *with $(C, \{H_k\}_k)$ and let $(C', \{H_k'\}_k)$ be another such pair. Then there exists some $T \in U(H)$ such that*

$$C'(\hat{P}) = TC(\hat{P}) \quad \textit{for} \quad {}^{\forall}\hat{P} \in \hat{M}_A^n \quad \textit{and} \quad H_k' = TH_kT^{-1} \quad \textit{for} \quad 1 \leq {}^{\forall}k < \infty.$$

Proof. We need only prove the uniqueness part. To this end we again take diffeomorphisms which were used in the just behind of (2.25). It follows from them that

$$(2.33)\qquad C(\hat{P})^{-1}H_kC(\hat{P}) = C'(\hat{P})^{-1}H_k'C'(\hat{P})$$

for ${}^{\forall}k \geq 1$ and ${}^{\forall}\hat{P} \in \hat{M}_A$, and further

$$(2.34)\qquad C'(\hat{g}^{-1}(\hat{P})) = C'(\hat{P})C(\hat{P})^{-1}C(\hat{g}^{-1}(\hat{P}))$$

for all $g \in \mathrm{Diff}_{0,A}^*(M)$. Thus the assertion follows from the transitivity of $\mathrm{Diff}_{0,A}^*(M)$ on $\hat{M}_A$.

Corollary 2.6. *If some connected component W of $\hat{M}_A^n$ is simply connected, though $\hat{M}_A^n$ itself is not so, we have the same assertions in Theorem 2.5 as for 1-cocycles on $W \times \mathrm{Diff}_{0,A}^*(M)$.*

Proof. It is carried out in a quite similar way with the above one, only changing the open coverings $\{V_i^s\}_i$ of $\hat{M}_A$ to those of W $(s=1, \cdots, 6)$.

Now it is important to search for sufficient conditions to assure the simply connectedness of $\hat{M}_A$, and accoding to [3], thanks to Dimension theory, such a condition, for example, is given as follows.

Theorem 2.7. *Under the assumption that A has no accumulation points,*
(1) *if* $\dim(M)\geq 2$ *and M is connected, then so is $\hat{M}^n_A$ for every $n\in N$.*
(2) *if* $\dim(M)\geq 3$ *and M is simply connected, then so is $\hat{M}^n_A$ for every $n\in N$.*

Proof. (1) is derived from corollary 12.5 in [3] and from mathematical induction on n. (2) is also assured from them, however using lately proposition 12.6 and its proof in [3].

Hereafter till the end of this section we assume that $A=\emptyset$. In this case, we have

$$\hat{M}^n_A\equiv\hat{M}_A=\hat{M}^n \quad\text{and}\quad \mathrm{Diff}^*_{0,A}(M)=\mathrm{Diff}^*_0(M),$$

and (2.32) becomes

$$\hat{\theta}(\hat{P},g)=C(\hat{P})^{-1}\prod_{k=1}^{n}\left(\frac{d\mu_g}{d\mu}(P_k)\right)^{\sqrt{-1}H_k}C(\hat{g}^{-1}(\hat{P})). \tag{2.35}$$

Definition 2.2. *It is said that* 1*-cocycle $\hat{\theta}$ has canonical form, or $\hat{\theta}$ is canonical, if* (2.35) *holds for all* $(\hat{P},g)\in\hat{M}^n\times\mathrm{Diff}^*_0(M)$. (*If it is so, then sometimes θ will be explicitly denoted by* $\hat{\theta}(C,H_k)$)

Thus as a special case of Theorem 2.5,

Theorem 2.8. *Suppose that $\hat{M}^n$ is simply connected.* (*For example it holds good, if so is M and* $\dim(M)\geq 3$) *Then every precontinuous* 1*-cocycle on* $\hat{M}^n\times\mathrm{Diff}^*_0(M)$ *is canonical.*

At the end of this subsection we consider 1-cocycles $\bar{\theta}$ on $B^n_M\times\mathrm{Diff}^*_0(M)$ to which there corresponds a symmetric 1-cocycles $\hat{\theta}$ on $\hat{M}^n\times\mathrm{Diff}^*_0(M)$. Of course, $\hat{\theta}$ is said to be symmetric if and only if,

$$\hat{\theta}(\hat{P}_\sigma,g)=\hat{\theta}(\hat{P},g) \tag{2.36}$$

for all $(\hat{P},g)\in\hat{M}^n\times\mathrm{Diff}^*_0(M)$, where

$$\hat{P}_\sigma:=(P_{\sigma(1)},\cdots,P_{\sigma(n)}) \quad (\sigma\in\mathfrak{S}_n),$$

and $\mathfrak{S}_n$ is the group of permutatins on $\{1,\cdots,n\}$.

Theorem 2.9. *Suppose that $\hat{M}^n$ is connected. Then to every precontinuous* 1*-cocycle $\bar{\theta}$ on* $B^n_M\times\mathrm{Diff}^*_0(M)$ *the corresponding symmetric* 1*-cocycle is canonical, that* *is,* $\hat{\theta}\equiv\hat{\theta}(C,H_k)$ *and the pair* (C,H_k) *fulfills the condition below with a unitary representation* (T,H) *of* $\mathfrak{S}_n$,

$$^{\forall}\sigma\in\mathfrak{S}_n,\quad C(\hat{P})=T(\sigma)C(\hat{P}_\sigma)\quad \textit{for}\quad {}^{\forall}\hat{P}\in\hat{M}^n,\qquad \textit{and}$$
$$H_k=T(\sigma)^{-1}H_{\sigma(k)}T(\sigma)\quad \textit{for}\quad 1\leq{}^{\forall}k\leq n.$$

Proof. It is straightforward from the uniqueness in Theorem 2.5.

2.4. Further study of 1-cocycles in the exceptional case. In this subsection we consider 1-cocycles when the manifold M is $\boldsymbol{R}^1$, $\boldsymbol{R}^2$ or $\boldsymbol{T}^1$. First let $M=\boldsymbol{R}^1$. Then B_M^n is simply connected with the natural topology, and $\hat{M}^n$ consists of $n!$ connected components which are all isomorphic to B_M^n. So applying Corollary 2.6, we have the following result.

Theorem 2.10. *Let* $M=\boldsymbol{R}^1$ *and take an isomorphic section* τ *from* B_M^n *to* $\hat{M}^n$. *Then the general form of precontinuous* 1*-cocycles on* $B_M^n\times\mathrm{Diff}_0^*(M)$ *is as follows.*

$$\bar{\theta}(\bar{P},g)=C(\bar{P})^{-1}\prod_{k=1}^{n}\left(\frac{d\mu_g}{d\mu}((\tau(\bar{P}))_k)\right)^{\sqrt{-1}H_k}C(\bar{g}^{-1}(\bar{P})),\tag{2.37}$$

where C *is a* $U(H)$*-valued map and* $\{H_k\}_k$ *is a commutative system of self-adjoint operators on* H.

If the 1*-cocycle* $\bar{\theta}$ *is continuous, then so is map* C.

Remark 2.2. *Of course even in the case* $H=\boldsymbol{C}$, *there exists a non canonical* 1*-cocycle on* $\hat{M}^n\times\mathrm{Diff}_0^*(M)$ *corresponding to the one given by* (2.37).

The second case is that $M=\boldsymbol{R}^2$. Here $\hat{M}^n$ is connected contrary to the previous case, however it is not simply connected for $n\geq 2$, and there exists a non canonical but symmetric 1-cocycle as will be seen in the following example, $n=2$ and $H=\boldsymbol{C}$.

For any $(\hat{P},g)\in\hat{M}^2\times\mathrm{Diff}_0^*(M)$, take an continuous angular function $\varphi(t,\hat{P})$ of a path, $t\in[0,1]\mapsto g_t^{-1}(P_1)-g_t^{-1}(P_2)\in\boldsymbol{R}^2\backslash\{0\}$, where $\{g_t\}_{0\leq t\leq 1}$ is a continuous path in $\mathrm{Diff}_0^*(M)$ connecting id and g, and put

$$\Phi(\hat{P},g):=\varphi(1,\hat{P})-\varphi(0,\hat{P}).\tag{2.38}$$

Actually this definition does not depend on a particular choice of $\{g_t\}_{0\leq t\leq 1}$ and Φ is a continuous function of g for each fixed $\hat{P}$. Set for $\Omega\in\boldsymbol{R}$,

$$\zeta_\Omega(\hat{P},g):=\exp(\sqrt{-1}\Omega\Phi(\hat{P},g)).\tag{2.39}$$

Then the continuity, symmetry and cocycle equality are easily checked, however ζ_Ω is not canonical unless $\Omega\in\boldsymbol{N}$. For, take any point $\hat{P}=(P_1,P_2)\in\hat{M}^2$ and take an open disk $U_r(0)$ centered at the origin which contains P_1 and P_2. Integrating a smooth vector field with compact support, we have a $g\in\mathrm{Diff}_0^*(\boldsymbol{R}^2)$ which gives 1 rotation around 0 on $U_r(0)$. Therefore g acts identically near at P_i $(i=1,2)$. Nevertheless, $\zeta_\Omega(\hat{P},g)\neq 1$, since we have $\Phi(\hat{P},g)=-2\pi$.

The last case $M=\boldsymbol{T}^1\equiv\boldsymbol{T}$ is more interesting. Let $\bar{\theta}$ be a precontinuous 1-cocycle on $B^n_{\boldsymbol{T}}\times\mathrm{Diff}^*_0(\boldsymbol{T})$, and $\hat{\theta}$ be the corresponding symmetric one on $\hat{\boldsymbol{T}}^n\times\mathrm{Diff}^*_0(\boldsymbol{T})$. $B^n_{\boldsymbol{T}}$ and $\hat{\boldsymbol{T}}^n$ are non simply connected, but they are connected. Now consider a set

$$I:=\{(z_1,\cdots,z_n)\in\hat{\boldsymbol{T}}^n \mid \arg z_1^{-1}z_k<\arg z_1^{-1}z_{k+1}\ (k=1,\cdots,n-1)\},$$

where the value of the argument is taken so as to be in $[0,2\pi)$. I is connected open and $\mathrm{Diff}^*_0(\boldsymbol{T})$-invariant. Take any point $\hat{A}=(a_1,\cdots,a_n)\in I$ and fix it. Then the following lemma gives a continuous section $s_{(z_1,\cdots,z_n)}\in\mathrm{Diff}^*_0(\boldsymbol{T})=\mathrm{Diff}^*(\boldsymbol{T})$ on I. That is, $s_{(z_1,\cdots,z_n)}(a_k)=z_k$ for $1\leq{}^\forall k\leq n$, and s is continuous from I to $\mathrm{Diff}^*(\boldsymbol{T})$.

Lemma 2.2. *Let* $0<a_1<a_2<\cdots<a_n<1$ *and* $0<b_1<b_2<\cdots<b_n<1$. *Then there exists a* $\phi_{a,b}\in\mathrm{Diff}^*_0(\boldsymbol{R}^1)$ *which satisfies*

(1) $\phi_{a,b}(x)=x$, *if* $x\leq 0$ *or* $x\geq 1$,

(2) $\phi_{a,b}(a_k)=b_k$ $(k=1,\cdots,n)$, *and*

(3) *a map*, $(a,b)\equiv(a_1,\cdots,a_n,b_1,\cdots,b_n)\mapsto\phi_{a,b}\in\mathrm{Diff}^*_0(\boldsymbol{R}^1)$, *is continuous.*

Proof. Take a C^∞-function $\rho_0(x)$ on $\boldsymbol{R}^1$ such that $\rho_0\geq 0$, $\rho_0\equiv 1$ on $(-\infty,0]$, $\rho_0\equiv 0$ on $[1,\infty)$, and put

$$\rho_{\alpha,\beta,\gamma,\delta}(x):=\rho_0\left(\frac{x-\beta}{\alpha-\beta}\right)\rho_0\left(\frac{x-\gamma}{\delta-\gamma}\right)\qquad(\alpha<\beta<\gamma<\delta).$$

Clearly we have $\rho_{\alpha,\beta,\gamma,\delta}\equiv 0$ on $(-\infty,\alpha]\cup[\delta,\infty)$, and $\rho_{\alpha,\beta,\gamma,\delta}\equiv 1$ on $[\beta,\gamma]$. Now consider a diffeomorphism

$$g_n(x)\equiv\mathrm{Exp}(X_n)(x),\qquad X_n(x):=(b_n-a_n)\rho_{0,m(a_n,b_n),M(a_n,b_n),1}(x)\frac{d}{dx},$$

where

$$M(a_n,b_n):=\mathrm{Exp}(Y_n)(b_n),\quad m(a_n,b_n):=\mathrm{Exp}(Z_n)(b_n)\ \text{and}$$

$$Y_n(x):=a_n\rho_{-1,-a_n,a_n,1}(x)\frac{d}{dx},\qquad Z_n(x):=(a_n-1)\rho_{0,a_n,1,a_n+1}(x)\frac{d}{dx}.$$

It is easy to see that $0<m(a_n,b_n)<a_n,b_n\leq M(a_n,b_n)<1$ and that g_n satisfies

$$\begin{cases} g_n(x)=x, & \text{if } x\leq 0 \text{ or } x\geq 1,\\ g_n(a_n)=b_n. \end{cases}\tag{2.40}$$

Moreover $b'_{n-1}:=g_n(a_{n-1})<b_n$, since g_n is monotone increasing. Next take a diffeomorphism

$$g_{n-1}(x)\equiv \mathrm{Exp}(X_{n-1})(x),$$

$$X_{n-1}(x):=(b_{n-1}-b'_{n-1})\rho_{0,m_1(b'_{n-1},b_{n-1}:b_n),M_1(b'_{n-1},b_{n-1}:b_n),b_n}(x)\frac{d}{dx},$$

where

$$m_1(b'_{n-1},b_{n-1}:b_n):=m\left(\frac{b'_{n-1}}{b_n},\frac{b_{n-1}}{b_n}\right)b_n,\quad \text{and } M_1(b'_{n-1},b_{n-1}:b_n):=M\left(\frac{b'_{n-1}}{b_n},\frac{b_{n-1}}{b_n}\right)b_n.$$

Then g_{n-1} satisfies

$$\begin{cases} g_{n-1}(x)=x, & \text{if } x\leq 0 \text{ or } x\geq b_n, \\ g_{n-1}(b'_{n-1})=b_{n-1}. \end{cases} \tag{2.41}$$

Thus we have

$$\begin{cases} g_{n-1}\circ g_n(x)=x, & \text{if } x\leq 0 \text{ or } x\geq 1, \\ g_{n-1}\circ g_n(a_n)=b_n, & g_{n-1}\circ g_n(a_{n-1})=b_{n-1}, \end{cases} \tag{2.42}$$

and the proof follows from the above procedures repeated $(n-1)$ times.

By virtue of the discussions on the local form of 1-cocycle and of using this global section we find that

$$\hat{\theta}(\hat{P},g)=C(\hat{P})^{-1}\prod_{k=1}^{n}\left(\frac{d\mu_g}{d\mu}(z_k)\right)^{\sqrt{-1}H_k}C(\hat{g}^{-1}(\hat{P})), \tag{2.43}$$

for all $\hat{P}=(z_1,\cdots,z_n)\in I$ and $g\in \mathrm{Diff}^*(\boldsymbol{T})$, where C is a $U(H)$-valued map on I, $\{H_k\}_k$ is a commutative system of self-adjoint operators on H and μ is a Haar measure on $\boldsymbol{T}$. Put

$$\tau:=\begin{pmatrix} 1 & 2 & \cdots & n \\ 2 & 3 & \cdots & 1\end{pmatrix}.$$

Since I is invariant under τ and $\hat{\theta}$ is symmetric,

$$\hat{\theta}(\hat{P},g)=C(\hat{P}_\tau)^{-1}\prod_{k=1}^{n}\left(\frac{d\mu_g}{d\mu}(z_{k+1})\right)^{\sqrt{-1}H_k}C(\hat{g}^{-1}(\hat{P}_\tau)), \tag{2.44}$$

where of course $z_{n+1}:=z_1$. Taking a diffeomorphism which acts as a translation near at each z_k $(k=1,\cdots,n)$, we see that $C(z_2,z_3,\cdots,z_n,z_1)C(z_1,z_2,\cdots,z_n)^{-1}$ is locally, hence globally constant. Put

$$C(z_2,z_3,\cdots,z_n,z_1)=TC(z_1,z_2,\cdots,z_n)\qquad ((z_1,z_2,\cdots,z_n)\in I). \tag{2.45}$$

It follows that

$$T^n=\mathrm{id}, \tag{2.46}$$

and

$$H_k = T^{-(k-1)} H_1 T^{(k-1)} \tag{2.47}$$

for $1 \leq {}^{\forall}k \leq n$.

Conversely, suppose that a $U(H)$-valued map C on I, a commutative system $\{H_k\}_k$ of self-adjoint operators on H and a $T \in U(H)$ are given so that they satisfy (2.45), (2.46) and (2.47). Then for $\bar{P} \in B_{\boldsymbol{T}}^n$ we oder its elements z_k $(k=1, \cdots, n)$ in such a way that $\hat{P} := (z_1, \cdots, z_n)$ belongs to I and define

$$\tilde{\theta}(\bar{P}, g) := C(\hat{P})^{-1} \prod_{k=1}^{n} \left(\frac{d\mu_g}{d\mu}(z_k) \right)^{\sqrt{-1} H_k} C(\hat{g}^{-1}(\hat{P})). \tag{2.48}$$

Although there are many, exactly n, ways of this ordering, the definition does not depend on them, and actually it gives a precontinuous 1-cocycle on $B_{\boldsymbol{T}}^n \times \mathrm{Diff}^*(\boldsymbol{T})$. Thus,

Theorem 2.11. *The general form of precontinuous* 1*-cocycles on* $B_{\boldsymbol{T}}^n \times \mathrm{Diff}^*(\boldsymbol{T})$ *is given by* (2.48).

Now a question arises: Is every symmetric precontinuous 1-cocycle on $\hat{\boldsymbol{T}}^n \times \mathrm{Diff}^*(\boldsymbol{T})$ canonical?

The following example gives us a negative answer.

Let $n=4$ and $H = \boldsymbol{C}^2$ and put

$$T := \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix},$$

$$H_1 = H_3 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad H_2 = H_4 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Finally for any point $(z_1, z_2, z_3, z_4) \in I$, put

$$C(z_1, z_2, z_3, z_4) := \frac{1}{\sqrt{(|z_1 - z_3|^2 + |z_2 - z_4|^2)}} \begin{pmatrix} z_1 - z_3 & \overline{z_2 - z_4} \\ -(z_2 - z_4) & \overline{z_1 - z_3} \end{pmatrix} \qquad ((z_1, \cdots, z_n) \in I).$$

The triplet $(C, \{H_k\}_k, T)$ satisfies the above conditions, so they define a 1-cocycle. However it is not canonical, as is easily seen.

2.5. Natural representations I. As before we fix a smooth locally Euclidean measure μ on M, and additionally assume that it is finite. Let $\hat{\mu} \equiv \hat{\mu}^n$ be the product measure on $\hat{M}^n$ and $\bar{\mu}$ be the image measure of $\hat{\mu}$ by the map π, $\hat{P} \in \hat{M}^n \mapsto \bar{P} \in B_M^n$. Up to the equivalence $\bar{\mu}$ is the unique measure on B_M^n being $\mathrm{Diff}_0^*(M)$-quasi-invariant under a natural assumption that M is connected. In this section, we consider natural representations of $\mathrm{Diff}_0^*(M)$ associated with μ,

$$\bar{U}_{\bar{\theta}}(g) : f(\bar{P}) \in \mathrm{L}_{\bar{\mu}}^2(B_M^n, H) \mapsto \sqrt{\frac{d\bar{\mu}_{\bar{g}}}{d\bar{\mu}}}(\bar{P}) \bar{\theta}(\bar{P}, g) f(\bar{g}^{-1}(\bar{P})) \in \mathrm{L}_{\bar{\mu}}^2(B_M^n, H), \tag{2.49}$$

or

$$(2.50)\qquad \hat{U}_{\hat{\theta}}(g): f(\hat{P}) \in \hat{\mathrm{L}}^2_{\hat{\mu}}(\hat{M}^n, H) \mapsto \sqrt{\frac{d\hat{\mu}_{\hat{g}}}{d\hat{\mu}}}(\hat{P})\hat{\theta}(\hat{P}, g) f(\hat{g}^{-1}(\hat{P})) \in \hat{\mathrm{L}}^2_{\hat{\mu}}(\hat{M}^n, H),$$

where $\bar{\theta}\,(\hat{\theta})$ is a $U(H)$-valued measurable (symmetric measurable) 1-cocycle, respectively, and $\hat{\mathrm{L}}^2_{\hat{\mu}}(\hat{M}^n, H)$ is the set of all square summable H-valued symmetric functions on $\hat{M}^n$. Of course the representations $(\bar{U}_{\bar{\theta}}, \mathrm{L}^2_{\bar{\mu}}(B^n_M, H))$ and $(\hat{U}_{\hat{\theta}}, \hat{\mathrm{L}}^2_{\hat{\mu}}(\hat{M}^n, H))$ are mutually equivalent, if $\bar{\theta}$ and $\hat{\theta}$ correspondes to each other. We will use more convenient form of (2.49) or (2.50) alternatively. First of all let us introduce the following definition according to [7].

Definition 2.3. (1) *A measurable* 1*-cocycle $\bar{\theta}$ is said to be irreducible, if for any $U(H)$-valued measurable map $V(\bar{P})$ there exists some complex constant k such that*

$$V(\bar{P}) = k\mathrm{Id}$$

for $\bar{\mu}$-a.e.$\bar{P}$, povided that

$$(2.51)\qquad V(\bar{P})\bar{\theta}(\bar{P}, g) = \bar{\theta}(\bar{P}, g) V(\bar{g}^{-1}(\bar{P}))$$

for $\bar{\mu}$-a.e.$\bar{P}$.
(2) *A parallel definition for a symmetric measurable* 1*-cocycle $\hat{\theta}$ is given, in which $V(\bar{P})$ is replaced by a symmetric measurable map $V(\hat{P})$.*

Theorem 2.12. *Assume that $\hat{M}^n$ is connected and that a strongly Borelian symmetric* 1*-cocycle $\hat{\theta}(C, H_k)$ has the canonical form* (2.35). *Then in order that $\hat{\theta}$ is irreducible, it is necessary and sufficient that the representation (T, H) appeared in Theorem* 2.9 *and $\{H_k\}_k$ satisfy the following condition* (c.1).

(c.1) *A unitary operator A on H is a scalar one, provided that*

$$(2.52)\qquad AT(\sigma) = T(\sigma)A \quad \text{for} \quad {}^{\forall}\sigma \in \mathfrak{S}_n \quad \text{and}$$

$$(2.53)\qquad AH_k = H_k A \quad \text{for} \quad 1 \leq {}^{\forall}k \leq n.$$

Proof. First we prove the necessity without the assumption on connectedness. To this end take on operator A satisfying (2.52) and (2.53) and put

$$V(\hat{P}) := C(\hat{P})^{-1} A^{-1} C(\hat{P}).$$

If follows from Theorem 2.9 that for any $\sigma \in \mathfrak{S}_n$

$$V(\hat{P}_\sigma) = C(\hat{P})^{-1} T(\sigma) A^{-1} T(\sigma)^{-1} C(\hat{P}) = V(\hat{P}),$$

and (2.51) is a direct consequence ot (2.53). The measurabilty of V follows from that of C, which is assured by the strongly Borelian assumption. Hence by the assumption we have $V(\hat{P}) = {}^{\exists}k\mathrm{Id} \mod \hat{\mu}$, in other words A is a scalar operator.

Next we prove the sufficiency. Let $V(\hat{P})$ be a symmetric $U(H)$-valued measurable map satisfying the relation being parallel to (2.51). Take any $\hat{P} = (P_1, \cdots, P_n) \in \hat{M}^n$ and take an open neighbourhood $U_k(P_k)$ of P_k for each $1 \leq k \leq n$ such that

(a) $U_k(P_k) \cap U_{k'}(P_{k'}) = \emptyset$ whenever $k \neq k'$,
(b) $U_k(P_k)$ is diffeomorphic to a connected open set G_k in $\boldsymbol{R}^d$ via a map ϕ_k, and
(c) the image measure $\mu | U_k(P_k) \circ \phi_k^{-1}$ coincides with the Lebesgue measure dx_k on G_k.

Then for any $h_k \in \mathrm{Diff}_0^*(G_k)$ $(1 \leq k \leq n)$ the equation (2.51) now becomes

$$(2.54) \qquad \tilde{V}(x)\tilde{C}(x)^{-1} \prod_{k=1}^{n} (J_{h_k}(x_k))^{\sqrt{-1}H_k} \tilde{C}(h(x)) \tilde{V}(h(x))^{-1}$$
$$= \tilde{C}(x)^{-1} \prod_{k=1}^{n} (J_{h_k}(x_k))^{\sqrt{-1}H_k} \tilde{C}(h(x))$$

for $dx_1 \times \cdots \times dx_n$-a.e.$x = (x_1, \cdots, x_n)$, where $h(x) := (h_1(x_1), \cdots, h_n(x_n))$, $\tilde{C}(x) := C(\phi_1^{-1}(x_1), \cdots, \phi_n^{-1}(x_n))$ and $\tilde{V}(x) := V(\phi_1^{-1}(x_1), \cdots, \phi_n^{-1}(x_n))$. Taking parallel displacements with small lengths as the diffeomorphisms h_k and using Fubini's theorem, we find that $\tilde{C}(x)\tilde{V}(x)\tilde{C}(x)^{-1}$ is almost all equal to a constant $A \in U(H)$ on $G_1 \times G_2 \times \cdots \times G_n$. Hence globally

$$(2.55) \qquad C(\hat{P})V(\hat{P})C(\hat{P})^{-1} = A$$

holds for $\hat{\mu}$-a.e.$\hat{P}$ by the assumption on connectedness. (2.52) and (2.53) are direct consequences of the symmetry of V and (2.54). Thus the condition (c.1) is fulfilled, and the rest of the proof follows from (2.55).

Here let us see the following property of 1-cocycles.

Theorem 2.13. *If a 1-cocycle $\hat{\theta}$ on $\hat{M}^n \times \mathrm{Diff}_0^*(M)$ is continuous, then it is strongly Borelian, and the same assertion holds for a 1-cocycle on $B_M^n \times \mathrm{Diff}_0^*(M)$.*

Proof. To this end it is enough to show that $\hat{\theta}$ is measurable. Moreover using cocycle equality, the continuous assumtption and Remark 2.1, we need only check that $\hat{\theta}(\cdot, g)$ is measurable, whenever $g = \mathrm{Exp}(X)$, $X \in \Gamma_0(M)$. Put $g_t := \mathrm{Exp}(tX)$, and take a set $V^1(\hat{Q})$ appeared in the arguments on the local form of 1-cocycles. $V^1(\hat{Q})$ is approximated by an increasing sequence of compact sets. Let K be any one of these compact sets and fix it. Then there exists a $\delta > 0$ such that

$$\hat{g}_t^{-1}(K) \subset V^1(\hat{Q}) \quad \text{for} \quad 0 \leq {}^{\forall}t \leq \delta.$$

Hence $(\hat{P}, g_\delta)$, $\hat{P} \in K$ satisfies the condition (*) and it follows from (2.21) that $\hat{\theta}(\cdot, g_\delta)$

is measurable on K. Now we claim that $\hat{\theta}(\cdot, g_{k\delta})$ is measurable on K for all $k \in \boldsymbol{N}$ by mathematical induction. Assume that it is O.K. for all $1, \cdots, k$. Since we have

$$\hat{\theta}(\hat{P}, g_{(k+1)\delta}) = \hat{\theta}(\hat{P}, g_{k\delta})\hat{\theta}(\hat{g}_{k\delta}^{-1}(\hat{P}), g_\delta),$$

it is enough to show that $\hat{\theta}(\cdot, g_\delta)$ is measurable on $\hat{g}_{k\delta}^{-1}(K)$. To this end note that

$$\hat{g}_t^{-1}(\hat{g}_{k\delta}^{-1}(K)) \subset \hat{g}_{k\delta}^{-1}(V^1(\hat{Q})) \quad \text{for} \quad 0 \leq {}^\forall t \leq \delta,$$

and that there is a continuous section on the whole set $\hat{g}_{k\delta}^{-1}(V^1(\hat{Q}))$ transformed from the section s on $V^1(\hat{Q})$. Thus the similar proof works on with the above one and the claim has been proved. In other words, $\hat{\theta}(\hat{P}, g)$ is measurable on K. As K and $V^1(\hat{Q})$ are arbitrary, the measurability is proved.

A criterion for irreducibility is obtained for 1-cocycles of another type, for example given by the formula (2.37).

Lemma 2.3. *Assume that $M = \boldsymbol{R}^1$. Then a strongly Borelian* 1*-cocycle $\bar{\theta}(C, H_k)$ on $B_M^n \times \mathrm{Diff}_0^*(M)$ given by the formula* (2.37) *is irreducible if and only if the $\{H_k\}_k$ satisfies the following condition* (c.2).

(c.2) *An $A \in U(H)$ is a scalar operator, provided that*

$$AH_k = H_k A \quad \textit{for} \quad 1 \leq {}^\forall k \leq n.$$

Proof. The proof of necessity is similar with it in Theorem 2.12.
The converse also goes in the same way, and we find an $A \in U(H)$ which satisfies

$$C(\bar{P})V(\bar{P}) = AC(\bar{P}) \quad \text{for} \quad \bar{\mu}\text{-a.e.} \bar{P}, \text{ and hence}$$

$$AH_k = H_k A \quad \text{for} \quad 1 \leq {}^\forall k \leq n.$$

Thus $A = {}^\exists k\mathrm{Id}$ and the conclusion immediately follows.

Theorem 2.14. *Under the same assumption of Lemma* 2.3, *$\bar{\theta}$ is irreducible if and only if* $\dim(H) = 1$.

Proof. Since $\{H_k\}_k$ is commutative, the condition (c.2) implies that each H_k is a scalar operator, hence so is every bounded operator.

Theorem 2.15 (Irreducibility). *Let $\bar{\theta}$ be a strongly Borelian* 1*-cocycle on $B_M^n \times \mathrm{Diff}_0^*(M)$ and $(\bar{U}_{\bar{\theta}}(g), \mathrm{L}_{\bar{\mu}}^2(B_M^n, H))$ be the corresponding natural representation given by* (2.49). *Then $(\bar{U}_{\bar{\theta}}(g), \mathrm{L}_{\bar{\mu}}^2(B_M^n, H))$ is irreducible, if and only if so is $\bar{\theta}$.*

Proof. The necessity is obvious, so let us assume that $\bar{\theta}$ is irreducible, and consider $(\hat{U}_{\hat{\theta}}(g), \hat{\mathrm{L}}_{\hat{\mu}}^2(\hat{M}, H))$ given by (2.50). Take an open set G of M. Put

$$H_G(\hat{U}_{\hat{\theta}}) := \{f \in \hat{\mathrm{L}}^2_{\hat{\mu}}(\hat{M}, H) \mid \hat{U}_{\hat{\theta}}(g)f = f \quad \text{for} \quad {}^\forall g \in \mathrm{Diff}^*_0(G)\},$$

and

$$\Delta_G := \{\bar{P} \in B^n_M|^\#| \bar{P} \cap G| = 0\}.$$

Throughout the proof we identify $f \in \hat{\mathrm{L}}^2_{\hat{\mu}}(\hat{M}, H)$ with the corresponding function on $\mathrm{L}^2_{\bar{\mu}}(B^n_M, H)$. Now Theorem 2.4 leads us to that f belongs to $H_G(\hat{U}_{\hat{\theta}})$, whenever $f(\bar{P}) = 0$ for $\bar{\mu}$-a.e.$\bar{P} \in \Delta^C_G$.

Let us prove the converse, so suppose that $f \in H_G(\hat{U}_{\hat{\theta}})$. Take any relatively compact open subset Y of G and put for each $1 \leq k \leq n$

$$Y_k := \{\hat{P} \in \hat{M}|^\#| \bar{P} \cap \bar{Y}| = 1 \text{ and } P_k \in Y\}.$$

Further take any point $\hat{Q} \in Y_k$ and a neighbourhood $V^1(\hat{Q})$, a $U(H)$-valued map C defined on $V^1(\hat{Q})$ and a commutative system $\{H_k\}_k$ of self-adjoint operators on H assured by Theorem 2.4. If necessary, taking a smaller neighbouhood, we may assume that $V^1(\hat{Q}) = V_1(Q_1) \times \cdots \times V_k(Q_k) \times \cdots \times V_n(Q_n)$, $V_k(Q_k)$ being diffeomorphic to a connected open subset of $\boldsymbol{R}^d$, such one as in (c) of the proof of Theorem 2.12, $V_k(Q_k) \subseteq Y$ and $V_i(Q_i) \subset \bar{Y}^c$ for all $i \neq k$. Then by virtue of Theorem 2.4 we have for any $g \in \mathrm{Diff}^*_0(V_k(Q_k))$

(2.56)
$$\sqrt{\frac{d\mu_g}{d\mu}}(P_k) C(\hat{P})^{-1} \left(\frac{d\mu_g}{d\mu}(P_k)\right)^{\sqrt{-1}H_k} C(P_1, \cdots, g^{-1}(P_k), \cdots, P_n) \cdot$$
$$f(P_1, \cdots, g^{-1}(P_k), \cdots, P_n) = f(\hat{P})$$

for $\hat{\mu}$-a.e.$\hat{P} \in V^1(\hat{Q})$. As before considering local translations, we find that

(2.57)
$$C(\hat{P})^{-1} C(P_1, \cdots, P_{k-1}, R_k, P_{k+1}, \cdots, P_n) f(P_1, \cdots, P_{k-1}, R_k, P_{k+1}, \cdots, P_n) = f(\hat{P})$$

for $\mu \times \hat{\mu}$-a.e.$(R_k, \hat{P}) \in V_k(Q_k) \times V^1(\hat{Q})$, and the left hand side of (2.57) is jointly measurable. So by virtue of Fubini's theorem there exists an $A_k \in V_k(Q_k)$ such that

(2.58)
$$C(\hat{P})^{-1} C(P_1, \cdots, P_{k-1}, A_k, P_{k+1}, \cdots, P_n) f(P_1, \cdots, P_{k-1}, A_k, P_{k+1}, \cdots, P_n) = f(\hat{P})$$

for $\hat{\mu}$-a.e.$\hat{P} \in V^1(\hat{Q})$. Of course (2.58) is also valid for $g^{-1}(P_k)$ in place of P_k due to the $\mathrm{Diff}^*_0(V_k(Q_k))$-invariance. Thus, substituting (2.58) to (2.56) we get for any $g \in \mathrm{Diff}^*_0(V_k(Q_k))$

(2.59)
$$\sqrt{\frac{d\mu_g}{d\mu}}(P_k) \left(\frac{d\mu_g}{d\mu}(P_k)\right)^{\sqrt{-1}H_k} h(\hat{P}, A_k) = h(\hat{P}, A_k)$$

for $\hat{\mu}$-a.e.$\hat{P} \in V^1(\hat{Q})$, where

$$h(\hat{P}, A_k) := C(P_1, \cdots, P_{k-1}, A_k, P_{k+1}, \cdots, P_n) f(P_1, \cdots, P_{k-1}, A_k, P_{k+1}, \cdots, P_n).$$

Moreover the left hand side of (2.59) is continuous as a function of P_k for each fixed $(P_1, \cdots, P_{k-1}, P_{k+1}, \cdots, P_n, A_k)$, so (2.59) holds for every P_k. In particular taking a diffeomorphism which is alike to a similar transformation at P_k and taking the norm of both sides of (2.59), we find that $h(\hat{P}, A_k)=0$ and it follows from (2.58) that

$$f(\hat{P})=0 \tag{2.60}$$

mod $\hat{\mu}$ on $V^1(\hat{Q})$ and hence on Y_k. Now

$$\text{“}\hat{P}\in\Delta_G^c\text{” implies that “}{}^{\exists}k, Y, \hat{P}\in Y_k\text{”},$$

and we can take such many Y_k from a countable collection, so the desired result follows from (2.60).

Let A be an intertwining unitary operator of the natural representation and put

$$(P_{\Delta_G}f)(\bar{P}) := \chi_{\Delta_G}(\bar{P})f(\bar{P}) \qquad (f\in \mathrm{L}^2_{\bar{\mu}}(B^n_M, H)),$$

where χ_{Δ_G} is the indicator function of the set Δ_G. Then by what we have seen,

$$AP_{\Delta_G}=P_{\Delta_G}A \tag{2.61}$$

for all open sets G of M. Besides a collection $\mathfrak{B} := \{\Omega:\text{ Borel set of } B^n_M \mid AP_\Omega = P_\Omega A\}$ forms a Borel field and $\{\Delta_G,\ G\text{: open set}\}$ generates the natural Borel field $\mathfrak{B}(B^n_M)$. (cf. p600–601 in [20]) Thus,

$$AP_\Omega=P_\Omega A \tag{2.62}$$

for every $\Omega\in\mathfrak{B}(B^n_M)$. It follows easily that there exists a $U(H)$-valued measurable map $V(\bar{P})$ defined on B^n_M such that

$$A(f)(\bar{P})=V(\bar{P})(f(\bar{P})) \tag{2.63}$$

for $\bar{\mu}$-a.e.$\bar{P}$, and it leads to

$$V(\bar{P})\bar{\theta}(\bar{P},g)=\bar{\theta}(\bar{P},g)V(\bar{g}^{-1}(\bar{P}))$$

holds for $\bar{\mu}$-a.e.$\bar{P}$. The rest of the proof is obvious.

Theorem 2.16 (Equivalence.). *Let $\bar{\theta}_i(\bar{P},g)$ $(i=1,2)$ be strongly Borelian* 1*-cocycles and $(\bar{U}_{\bar{\theta}_i}(g), \mathrm{L}^2_{\bar{\mu}}(B^n_M, H_i))$ $(i=1,2)$ be the corresponding natural representations. Then the representations are equivalent, if and only if the* 1*-cocycles are* 1*-cohomologous. That is, there exists a $U(H_1, H_2)$-valued measurable map $V(\bar{P})$ on B^n_M ($U(H_1, H_2)$ is the set of all unitary operators from H_1 to H_2) such that*

$$\bar{\theta}_1(\bar{P},g)=V^{-1}(\bar{P})\bar{\theta}_2(\bar{P},g)V(\bar{g}^{-1}(\bar{P})) \tag{2.64}$$

for $\bar{\mu}$-a.e.$\bar{P}$.

Proof. The sufficiency is obvious. For the necessity, we proceed in the same manner as before and first we get a relation correponding to (2.62), where A is an intertwinig unitary operator from $(\bar{U}_{\bar{\theta}_1}(g), L^2_{\bar{\mu}}(B^n_M, H_1))$ to $(\bar{U}_{\bar{\theta}_2}(g), L^2_{\bar{\mu}}(B^n_M, H_2))$. It follows that a $U(H_1, H_2)$-valued measurable map $V(\bar{P})$ on B^n_M is defined and it satisfies the relation (2.63) for $\bar{\mu}$-a.e.$\bar{P}$. This gives us the cohomologous relation.

§3. 1-cocycles on the infinite configuration space

3.1. Canonical form of precontinuous 1-cocycles. Throughout this section M is assumed to be non compact. Let

$\hat{M}^\infty := \{\hat{P} = (P_1, \cdots, P_n, \cdots) \in M^\infty \mid P_i \neq P_j$ for $^\forall i \neq j$, $\{P_k\}_k$ has no accumulation points$\}$,
and $\mathfrak{S}_\infty$ be the infinite permutation group on the set N. Define an equivalence relation $\sim$ on $\hat{M}^\infty$ by

$$\hat{P} \sim \hat{Q} \text{ if and only if } ^\exists\sigma \in \mathfrak{S}_\infty \quad \text{s.t., } \hat{Q} = \hat{P}_\sigma := (P_{\sigma(1)}, \cdots, P_{\sigma(n)}, \cdots).$$

The quotient space $\Gamma_M := \hat{M}^\infty/\sim$ is called infinite configuration space (over M), its element is generally denoted by $\bar{P} = \{P_1, \cdots, P_n, \cdots\}$, and the natural map, $\hat{P} \mapsto \bar{P}$ is denoted by π. As before $\mathrm{Diff}_0^*(M)$ acts on $\hat{M}^\infty$ or Γ_M as $\hat{g}$ or $\bar{g}$, respectively. Consequently 1-cocycles (plus continuity, measurability or etc.) on $\hat{M}^\infty \times \mathrm{Diff}_0^*(M)$ or $\Gamma_M \times \mathrm{Diff}_0^*(M)$ are defined similarly, and they are denoted by $\hat{\theta}$ or $\bar{\theta}$.

Next let us consider a new equivalence relation $\approx$ on $\hat{M}^\infty$ defined by,

$$\hat{P} \approx \hat{Q}, \quad \text{if and only if } P_n = Q_n \text{ for all } n \geq {}^\exists N.$$

Put for any $\hat{A} = (A_1, \cdots, A_n, \cdots) \in \hat{M}^\infty$,

$$\hat{M}^\infty_A := \{\hat{P} \in \hat{M}^\infty \mid \hat{P} \approx \hat{A}\}.$$

Clearly $\hat{M}^\infty_A$ is $\mathrm{Diff}_0^*(M)$-invariant. Using the notation in section 2, we find that

$$\hat{M}^\infty_A = \cup_{n=1}^\infty \hat{M}^n_{A^n} \times (A_{n+1}, A_{n+2}, \cdots), \tag{3.1}$$

where $A^n := \{A_{n+1}, A_{n+2}, \cdots\}$, and a sequence of the above sets is increasing. So the inductive limit topology τ^∞_A on $\hat{M}^\infty_A$ is given via the natural ones on $\hat{M}^n_{A^n} \times (A_{n+1}, A_{n+2}, \cdots)$, and it will be used ocasionally. Hereafter we will write $\hat{M}^n_A$ instead of $\hat{M}^n_{A^n}$ for simplicity.

Theorem 3.1. (1) *Suppose that $\hat{M}^n_A$ is simply connected for an $\hat{A} = (A_1, \cdots, A_n, \cdots) \in \hat{M}^\infty$ and for every $n \in N$. Let $\hat{\theta}$ be a $U(H)$-valued precontinuous 1-cocycle on $\hat{M}^\infty_A \times \mathrm{Diff}_0^*(M)$. Then there exists a $U(H)$-valued map $C \equiv C_A$ on $\hat{M}^\infty_A$ and a commutative system $\{H_k\}_k \equiv \{H^A_k\}_k$ of self-adjoint operators on H such that*

$$\hat{\theta}(\hat{P}, g) = C(\hat{P})^{-1} \prod_{k=1}^{\infty} \left(\frac{d\mu_g}{d\mu}(P_k)\right)^{\sqrt{-1}H_k} C(\hat{g}^{-1}(\hat{P})) \tag{3.2}$$

for all $\hat{P} \in \hat{M}_A^\infty$ *and* $g \in \mathrm{Diff}_0^*(M)$.

Moreover if $\hat{\theta}$ *is continuous, the map* C *is continuous on* $(\hat{M}_A^\infty, \tau_A^\infty)$.

(2) *For the uniqueness of the above pair* $(C, \{H_k^A\}_k)$, *we only assume that* $\hat{M}_A^n$ *is connected for every* $n \in N$. *Then for another pair* $(C', \{H_k'^A\}_k)$, *there exists a* $T \in U(H)$ *such that*

$$C'(\hat{P}) = TC(\hat{P}) \quad \text{for} \quad {}^\forall \hat{P} \in \hat{M}_A^\infty \quad \text{and} \tag{3.3}$$

$$H_k'^A = TH_k^A T^{-1} \quad \text{for} \quad 1 \leq {}^\forall k < \infty. \tag{3.4}$$

Proof. For each n, put

$$\hat{\theta}_n(\hat{P}_n, g) := \hat{\theta}(\hat{P}_n, A_{n+1}, A_{n+2}, \cdots, g) \quad ((\hat{P}_n, g) \in \hat{M}_A^n \times \mathrm{Diff}_{0,A,n}^*(M)),$$

where $\mathrm{Diff}_{0,A,n}^*(M) \equiv \mathrm{Diff}_{0,A^n}^*(M)$ which was already defined in section 2. Then $\hat{\theta}_n$ is precontinuous 1-cocycle and hence the assumption of simply connectedness and Theorem 2.5 yield a map C_n on $\hat{M}_A^n$ and a commutative system $\{H_k^n\}_k$ of self-adjoint operators on H so that

(3.5)
$$\hat{\theta}_n(\hat{P}_n, g) = C_n(\hat{P}_n)^{-1} \prod_{k=1}^{n} \left(\frac{d\mu_g}{d\mu}(P_k)\right)^{\sqrt{-1}H_k^n} \prod_{k=n+1}^{\infty} \left(\frac{d\mu_g}{d\mu}(A_k)\right)^{\sqrt{-1}H_k^n} C_n(\hat{g}^{-1}(\hat{P}_n))$$

for all $(\hat{P}_n, g) \in \hat{M}_A^n \times \mathrm{Diff}_{0,A,n}^*(M)$. Since $\mathrm{Diff}_{0,A,n}^*(M) \subset \mathrm{Diff}_{0,A,n+1}^*(M)$ and $\hat{\theta}_n(\hat{P}_n, g) \equiv \hat{\theta}_{n+1}(\hat{P}_n, A_{n+1}, g)$, so by virtue of the uniqueness in Theorem 2.5 there exists a $T_n \in U(H)$ such that

$$C_{n+1}(\hat{P}_n, A_{n+1}) = T_n C_n(\hat{P}_n) \quad \text{for} \quad {}^\forall \hat{P}_n \in \hat{M}_A^n \text{ and}$$
$$H_k^{n+1} = T_n H_k^n T_n^{-1} \quad \text{for} \quad 1 \leq {}^\forall k \leq n.$$

Thus changing C_{n+1} to $T_n^{-1} C_{n+1}$, it yields

$$C_{n+1}(\hat{P}_n, A_{n+1}) = C_n(\hat{P}_n) \tag{3.6}$$

for all $\hat{P}_n \in \hat{M}_A^n$, and

$$H_k^{n+1} = H_k^n \quad \text{for} \quad 1 \leq {}^\forall k \leq n.$$

Put

$$C(\hat{P}) := \lim_{n \to \infty} C_n(P_1, \cdots, P_n) \quad (\hat{P} \in \hat{M}_A^\infty), \tag{3.7}$$

and

$$H_k := H_k^k \quad (k \in N). \tag{3.8}$$

These are well-defined, and since for any $(\hat{P},g)\in\hat{M}_A^\infty\times\mathrm{Diff}_0^*(M)$ there exists an N such that $(\hat{P},g)\in\hat{M}_A^n\times(A_{n+1},A_{n+2},\cdots)\times\mathrm{Diff}_{0,A,n}^*(M)$, we find that

$$\hat{\theta}(\hat{P},g)=\lim_{n\to\infty}\hat{\theta}_n(P_1,\cdots,P_n,g)=C(\hat{P})^{-1}\prod_{k=1}^{\infty}\left(\frac{d\mu_g}{d\mu}(P_k)\right)^{\sqrt{-1}H_k}C(\hat{g}^{-1}(\hat{P})). \tag{3.9}$$

Moreover if $\hat{\theta}$ is continuous, C is continuous on $(\hat{M}_A^\infty,\tau_A^\infty)$, since $C(\hat{P}_n,A_{n+1},A_{n+2},\cdots)=C_n(\hat{P}_n)$ for all $\hat{P}_n\in\hat{M}_A^n$ by virtue of (3.6) and (3.7). The uniqueness follows from the relation,

$$C'(\hat{P})=T_nC(\hat{P})\quad\text{for all}\quad\hat{P}\in\hat{M}_A^n\times(A_{n+1},A_{n+2},\cdots),$$

where $T_n\in U(H)$ is some constant operator assured by Theorem 2.5.

Theorem 3.2. (1) *Suppose that M is simply connected and* $\dim(M)\geq 3$. *Then the general form of precontinuous $U(H)$-valued 1-cocycles on $\hat{M}^\infty\times\mathrm{Diff}_0^*(M)$ is as follows.*

$$\hat{\theta}(\hat{P},g)=C(\hat{P})^{-1}\prod_{k=1}^{\infty}\left(\frac{d\mu_g}{d\mu}(P_k)\right)^{\sqrt{-1}H_k^{[P]}}C(\hat{g}^{-1}(\hat{P})), \tag{3.10}$$

where C is a $U(H)$-valued map on $\hat{M}^\infty$, and $\{H_k^{[P]}\}_k$ is a commutative system of self-adjoint operators on H depending on the residue class $[P]\in\hat{M}^\infty/\approx$ to which $\hat{P}$ belongs. Moreover if $\hat{\theta}$ is continuous, C is continuous on $(\hat{M}_A^\infty,\tau_A^\infty)$ for each $\hat{A}\in\hat{M}^\infty$.

As before we call $\hat{\theta}$ given by (3.10) *canonical* 1*-cocycle.*

(2) *For the uniqueness of the above pair $(C,\{H_k^{[P]}\}_k)$ we assume that*

(†) *M is connected and* $\dim(M)\geq 2$.

Then for another pair $(C',\{H_k'^{[P]}\}_k)$, there exists a $U(H)$-valued map T on $\hat{M}^\infty/\approx$ such that

$$C'(\hat{P})=T([P])C(\hat{P}) \tag{3.11}$$

for all $\hat{P}\in\hat{M}^\infty$ and

$$H_k'^{[P]}=T([P])H_k^{[P]}T([P])^{-1} \tag{3.12}$$

for $1\leq{}^\forall k<\infty$ and ${}^\forall\hat{P}\in\hat{M}^\infty$.

Proof. It is a direct consequence of the above theorem.

Theorem 3.3. *Under the assumption* (†), *a canonical* 1*-cocycle $\hat{\theta}$ on $\hat{M}^\infty\times\mathrm{Diff}_0^*(M)$ is symmetric if and only if the pair $(C,\{H_k^{[P]}\}_k)$ satisfies the following two conditions.*

$$C(\hat{P})=R([P],\sigma)C(\hat{P}_\sigma) \tag{3.13}$$

for $^\forall\hat{P}\in\hat{M}^\infty$ *and* $^\forall\sigma\in\mathfrak{S}_\infty$, *where* R *is a* 1-*cocycle on* $\hat{M}^\infty/\approx\times\mathfrak{S}_\infty$. *Namely,*

$$^\forall[P], {}^\forall\sigma, \quad R([P],\sigma)R([P_\sigma],\tau)=R([P],\sigma\tau), \quad \textit{and}$$

$$H_k^{[P]}=R([P]),\sigma)H_{\sigma^{-1}(k)}^{[P_\sigma]}R([P],\sigma)^{-1} \tag{3.14}$$

for $1\leq{}^\forall k<\infty$, $^\forall[P]\in\hat{M}^\infty/\approx$ *and* $^\forall\sigma\in\mathfrak{S}_\infty$.

Proof. It is obvious.

3.2. Measurability of canonical 1-cocycle.

Theorem 3.4. *Let* $\hat{\theta}$ *be a canonical* 1-*cocycle given by* (3.10). *Then it is strongly Borelian if and only if*

$$C(\hat{P})^{-1}H_k^{[P]}C(\hat{P}) \textit{ is measurable for } 1\leq{}^\forall k<\infty, \tag{3.15}$$

and

$$C(\hat{P})^{-1}C(\hat{g}^{-1}(\hat{P})) \textit{ is jointly measurable on } \hat{M}^\infty\times\mathrm{Diff}_0^*(M). \tag{3.16}$$

Proof. The sufficiency is obvious. Let us prove the converse. To this end note that $\hat{M}^\infty$ is covered by at most countable sets of the form, $\hat{M}^\infty\cap(G\times G^c\times\cdots\times G^c\times\cdots)$, where G is an open set being diffeomorphic to $\boldsymbol{R}^d$. So for the measurability of $C(\hat{P})^{-1}H_1^{[P]}C(\hat{P})$ we need only assure it on these sets. Now take a compact sets $K_n\uparrow G$ such that

$$K_1\subset K_2^\circ\subset\cdots\subset K_n\subset K_{n+1}^\circ\subset\cdots.$$

Then for any $Q\in K_n$ and any $a\in\boldsymbol{R}^+$ there exists $g_{Q,a}\in\mathrm{Diff}_0^*(K_{n+1})$ such that

$$g_{Q,a}(Q)=Q, \quad \frac{d\mu_{g_{Q,a}}}{d\mu}(Q)=a,$$

and a map, $(Q,a)\in K_n\times\boldsymbol{R}^+\mapsto g_{Q,a}\in\mathrm{Diff}_0^*(K_{n+1})$ is continuous.
Thus $\hat{\theta}(\hat{P},g_{Q,a})$ is jointly measurable with respect to $(\hat{P},Q,a)$, and so is

$$\hat{\theta}(\hat{P},g_{P_1,a})=C(\hat{P})^{-1}a^{\sqrt{-1}H_1^{[P]}}C(\hat{P}),$$

with respect to $(\hat{P},a)$. It follows that $C(\hat{P})^{-1}H_1^{[P]}C(\hat{P})$ is measurable on $\hat{M}^\infty\cap(K_n\times G^c\times\cdots\times G^c\times\cdots)$ and the conclusion follows. The rest of the proof is straightforward.

3.3. Natural representation II. In this subsection we consider natural representations of $\mathrm{Diff}_0^*(M)$ on Γ_M which are alike to the one on the finite configuration space. However $\mathrm{Diff}_0^*(M)$-quasi-invariant measure on $(\Gamma_M,\mathfrak{B})$, $\mathfrak{B}$ is the natural Borel field, is not unique in this case, so we must consider also a factor of such probabitiy measures $\bar{\nu}$ on $(\Gamma_M, \mathfrak{B})$. It is known in [25] that to such a $\bar{\nu}$ there corresponds a $\mathrm{Diff}_0^*(M)$-quasi-invariant pobability measure $\hat{\nu}$ on $(\hat{M}^\infty, \mathscr{C})$, $\mathscr{C}$ is the natural Borel field on $\hat{M}^\infty$, such that

$$\hat{\nu}(E) = \sum_{\sigma \in \mathfrak{G}_\infty^\circ} c(\sigma)(s\bar{\nu})\sigma(E) \tag{3.17}$$

for all $E \in \mathscr{C}$, where $\mathfrak{G}_\infty^\circ := \cup_{n=1}^\infty \mathfrak{G}_n$, $c(\sigma) > 0$, $\Sigma_{\sigma \in \mathfrak{G}_\infty^\circ} c(\sigma) = 1$, s is a measurable section, and $(s\bar{\nu})\sigma$ is an image measure of $\bar{\nu}$ by a map, $\bar{P} \mapsto (s(\bar{P}))_\sigma$. Note that for any symmetric measurable function f on $\hat{M}^\infty$,

$$\int_{\hat{M}^\infty} f(\hat{P})\hat{\nu}(d\hat{P}) = \int_{\Gamma_M} f(\bar{P})\bar{\nu}(d\bar{P}),$$

where we use a natural identification f with the corresponding function on Γ_M.

Definition 3.1. (1) *A measurable* 1*-cocycle $\bar{\theta}$ on $\Gamma_M \times \mathrm{Diff}_0^*(M)$ is said to be $\bar{\nu}$-irreducible, if for any $U(H)$-valued measurable map V on Γ_M there exists a constant $k \in \boldsymbol{C}$ such that*

$$V(\bar{P}) = k\mathrm{Id}$$

for $\bar{\nu}$-a.e.$\bar{P}$, provided that

$$V(\bar{P})\bar{\theta}(\bar{P}, g) = \bar{\theta}(\bar{P}, g)V(\bar{g}^{-1}(\bar{P})) \tag{3.18}$$

for $\bar{\nu}$-a.e.$\bar{P}$.
(2) *A parallel definition for a symmetric measurable* 1*-cocycle $\hat{\theta}$ is given, in which $V(\bar{P})$ and $\bar{\nu}$ are replaced by a symmetric measurable map $V(\hat{P})$ and $\hat{\nu}$, respectively.*

Remark 3.1. (1) *Of course $\bar{\theta}$ is $\bar{\nu}$-irreducible, if the corresponding $\hat{\theta}$ is $\hat{\nu}$-irreducible and vice versa.*
(2) *If a $\bar{\nu}$-irreducible* 1*-cocycle exists at any rate, $\bar{\nu}$ must be $\mathrm{Diff}_0^*(M)$-ergodic.*

Theorem 3.5. *Let $\hat{\theta} = \hat{\theta}(C, H_k)$ be a canonical strongly Borelian symmetric* 1*-cocycle. Then in order that it is $\hat{\nu}$-irreducible, it is necessary and sufficient that for any $U(H)$-valued map $A([P])$ defined on $\hat{M}^\infty / \approx$ which satisfies the conditions* (3.19) *and* (3.20) *below, there exists a constant $k \in \boldsymbol{C}$ such that*

$$A([P]) = k\mathrm{Id}$$

for $\hat{\nu}$-a.e.$\hat{P}$.

(3.19) A map, $\hat{P} \mapsto C(\hat{P})^{-1}A([P])C(\hat{P})$ is measurable and it coincides with a symmetric measurable map $V(\hat{P})$ for $\hat{\nu}$-*a.e.*$\hat{P}$,

and

(3.20) $$\forall k \in N, \quad A([P])H_k^{[P]} = H_k^{[P]}A([P])$$

for $\hat{\nu}$-a.e.$\hat{P}$. As before, the necessity requires no condition on M but for the sufficiency we assume that M satisfies (†).

Proof. Necessity. Take a symmetric measurable map $V(\hat{P})$ which satisfies (3.19). Then,

$$V(\hat{P})\hat{\theta}(\hat{P}, g) = \hat{\theta}(\hat{P}, g)V(\hat{g}^{-1}(\hat{P}))$$

holds for $\hat{\nu}$-a.e.$\hat{P}$, as is easily seen. Thus V, hence A is almost all equal to a scalar operator.

Sufficiency. Let $V(\hat{P})$ be a symmetric measurable map satisfying the relation being parallel to (3.18). For each $n \in N$ let π_n, π^n be natural projections,

$$\begin{cases} \pi_n : \hat{P} = (P_1, \cdots, P_n, \cdots) \in \hat{M}^\infty \mapsto \hat{P}_n = (P_1, \cdots, P_n) \in \hat{M}^n \\ \pi^n : \hat{P} = (P_1, \cdots, P_n, \cdots) \in \hat{M}^\infty \mapsto \hat{P}^n = (P_{n+1}, \cdots) \in \hat{M}^\infty, \end{cases}$$

and put

$$\nu_n := \hat{\nu} \circ \pi_n^{-1} \qquad \nu^n := \hat{\nu} \circ (\pi^n)^{-1}.$$

Then we know that

(3.21) $$\nu \simeq \nu_n \times \nu^n.$$

(cf. [21] or [25]) It follows that there exists a Borel set Ω^n with $\nu^n(\Omega^n) = 1$ such that for all $\hat{P}^n \in \Omega^n$

(3.22) $$V(\hat{P}_n, \hat{P}^n)\hat{\theta}(\hat{P}_n, \hat{P}^n, g)V(\hat{g}^{-1}(\hat{P}_n, \hat{P}^n))^{-1} = \hat{\theta}(\hat{P}_n, \hat{P}^n, g)$$

for $\hat{\nu}_n$-a.e.$\hat{P}_n$. Let us fix a $\hat{P}^n \in \Omega^n$, and use again the discussions in the proof of Theorem 2.12, especially taking each neighbourhood $U_k(P_k)$ so as to be disjoint from the set $\overline{P^n}$. Then they give an $A_U^n(\hat{P}^n) \in U(H)$ such that

(3.23) $$C(\hat{P}_n, \hat{P}^n)V(\hat{P}_n, \hat{P}^n)C^{-1}(\hat{P}_n, \hat{P}^n) = A_U^n(\hat{P}^n)$$

for $\hat{\nu}_n$-a.e.$\hat{P}_n \in \hat{U} := U_1(P_1) \times \cdots \times U_n(P_n)$, and

(3.24) $$A_U^n(\hat{P}^n)H_k^{[P]} = H_k^{[P]}A_U^n(\hat{P}^n)$$

for $1 \leq {}^\forall k \leq n$. Since

$$\hat{M}_{P^n}^n := \{(P_1, \cdots, P_n) \in \hat{M}^n \mid \{P_1, \cdots, P_n\} \cap \overline{P^n} = \emptyset\}$$

is connected by the assumption (†), we can connect any two such sets $\hat{U}^1$, $\hat{U}^2$ of $\hat{M}_{P^n}^n$ by a finite chain of the $\hat{U}$'s. Thus $A_U^n(\hat{P}^n)$ does not depend on $\hat{U}$, denoted simply by $A^n(\hat{P}^n)$, which satisfies

$$A^n(\hat{P}^n)=\int_{\hat{M}^n_{\hat{P}^n}} C(\hat{z}_n,\hat{P}^n)V(\hat{z}_n,\hat{P}^n)C^{-1}(\hat{z}_n,\hat{P}^n)\nu_n(d\hat{z}_n). \tag{3.25}$$

Next let us observe the measurability of the function,

$$C(\hat{P})^{-1}A^n(\hat{P}^n)C(\hat{P}). \tag{3.26}$$

For it, it is enough to show that $C(\hat{P})^{-1}C(\hat{z}_n,\hat{P}^n)$ is jointly measurable with respect to $(\hat{P},\hat{z}_n)$. Take a countable open base $\{W_k\}_k$ in $\hat{M}^n$ so that each W_k is equal to a set of the form, $U_1\times\cdots\times U_n$, where U_i are all diffeomorphic to $\boldsymbol{R}^d$ and they are disjoint, and set

$$\Omega_{i_1,\dots,i_s}:=\{(\hat{P},\hat{z}_n)\,|\,\hat{P}_n\in W_{i_1},\ \hat{z}_n\in W_{i_s},\ W_{i_j}\cap W_{i_{j+1}}\neq\emptyset,\ \ \cup_{j=1}^s W_{i_j}\cap\overline{P^n}=\emptyset\}.$$

Then taking suitable diffeomorphisms on each W_{i_j} we find that

$$C(\hat{P}_n,\hat{P}^n)^{-1}C(\hat{Q}_1,\hat{P}^n),\quad C(\hat{Q}_1,\hat{P}^n)^{-1}C(\hat{Q}_2,\hat{P}^n),\cdots,C(\hat{Q}_{s-1},\hat{P}^n)^{-1}C(\hat{z}_n,\hat{P}^n)$$

are all jointly measurable on $\Omega_{i_1,\dots,i_s}\times(W_{i_1}\cap W_{i_2})\times\cdots\times(W_{i_{s-1}}\cap W_{i_s})$ with respect to the variables $(\hat{P},\hat{z}_n,\hat{Q}_1,\cdots,\hat{Q}_{s-1})$. Thus their product yields the measurability on $\Omega_{i_1,\dots,i_s}$ and hence on the whole space. It follows from (3.23) and Fubini's theorem that

$$V(\hat{P})=C(\hat{P})^{-1}A^n(\hat{P}^n)C(\hat{P})$$

for $\hat{\nu}$-a.e.$\hat{P}$. Finally put

$$A([P]):=\begin{cases}\lim\limits_{n\to\infty}A^n(\hat{P}^n), & \text{if the limit exists.}\\ \mathrm{Id}, & \text{otherwise.}\end{cases} \tag{3.27}$$

The well-definedness is obvious, $C(\hat{P})^{-1}A([P])C(\hat{P})$ is measurable, and

$$V(\hat{P})=C(\hat{P})^{-1}A([P])C(\hat{P}),\quad\text{and}\quad 1\leq{}^\forall k<\infty,\ \ A([P])H_k^{[P]}=H_k^{[P]}A([P])$$

holds for $\hat{\nu}$-a.e.$\hat{P}$. In other words the assumption (3.19) and (3.20) are fulfilled and we have

$$A([P])={}^\exists k\,\mathrm{Id}$$

for $\hat{\nu}$-a.e.$\hat{P}$. Hence the same holds for $V(\hat{P})$.

Let $\bar{\nu}$ be a $\mathrm{Diff}_0^*(M)$-quasi-invariant measure on $(\Gamma_M,\mathfrak{B})$ and $\bar{\theta}$ be a measurable 1-cocycle on $\Gamma_M\times\mathrm{Diff}_0^*(M)$. Hereafter we consider natural representation $(U_{\bar{\nu},\bar{\theta}},\ \mathrm{L}^2_{\bar{\nu}}(\Gamma_M,H))$ depending on these factors,

$$U_{\bar{\nu},\bar{\theta}}(g):f(\bar{P})\in\mathrm{L}^2_{\bar{\nu}}(\Gamma_M,H)\mapsto\sqrt{\frac{d\bar{\nu}_{\bar{g}}}{d\bar{\nu}}}(\bar{P})\bar{\theta}(\bar{P},g)f(\bar{g}^{-1}(\bar{P}))\in\mathrm{L}^2_{\bar{\nu}}(\Gamma_M,H). \tag{3.28}$$

As before there correspondes a representation on the set $\hat{\mathrm{L}}_{\bar{\nu}}^{2}(\hat{M}^{\infty}, H)$ of all square summable H-valued symmetric functions on $\hat{M}^{\infty}$ defined by

$$U_{\hat{\nu},\hat{\theta}}(g): f(\hat{P}) \in \hat{\mathrm{L}}_{\hat{\nu}}^{2}(\hat{M}^{\infty}, H) \mapsto \sqrt{\frac{d\bar{\nu}_{\bar{g}}}{d\bar{\nu}}}(\bar{P})\hat{\theta}(\hat{P}, g) f(\hat{g}^{-1}(\hat{P})) \in \hat{\mathrm{L}}_{\hat{\nu}}^{2}(\hat{M}^{\infty}, H). \tag{3.29}$$

Theorem 3.6 (Irreducibility). *The natural representation given by* (3.28), *where we assume that* $\hat{\theta}$ *is canonical and strongly Borelian, is irreducible if and only if so is* $\bar{\theta}$.

Proof. The necessity is obvious. The sufficiency is proved in a similar way with that of Theorem 2.15. Take any open set G of M. Put

$$H_G(U_{\bar{\nu},\bar{\theta}}) := \{f \in \mathrm{L}_{\bar{\nu}}^{2}(\Gamma_M, H) \mid U_{\bar{\nu},\bar{\theta}}(g)(f) = f \quad \text{for} \quad {}^{\forall} g \in \mathrm{Diff}_0^{*}(G)\},$$

and

$$\Delta_G := \{\bar{P} \in \Gamma_M \mid \bar{P} \cap G = \emptyset\}.$$

It is no problem to see that $f \in H_G(U_{\bar{\nu},\bar{\theta}})$, whenever $f(\bar{P}) = 0$ for $\bar{\nu}$-a.e.$\bar{P} \in \Delta_G^c$.

Conversely suppose that $f \in H_G(U_{\bar{\nu},\bar{\theta}})$. We wish to prove that $f(\bar{P}) = 0$ for $\bar{\nu}$-a.e.$\bar{P} \in \Delta_G^c$. Take a relatively compact open subset Y of G being diffeomorphic to a connected open subset of $\boldsymbol{R}^d$, such one as in (c) of the proof of Theorem 2.12, and put

$$B_Y^n := \{\gamma \subset Y \mid {}^{\#}|\gamma| = n\}.$$

We have a natural decomposition

$$\Gamma_M = \cup_{n=0}^{\infty} B_Y^n \times \Gamma_{M \setminus Y},$$

of which the first and the second projection are denoted by π_n' and π_n'', respectively. Put

$$\nu_n := \bar{\nu} \mid B_Y^n \times \Gamma_{M \setminus Y} / \beta_n, \ \nu_n' := \nu_n \circ (\pi_n')^{-1} \quad \text{and} \quad \nu_n'' := \nu_n \circ (\pi_n'')^{-1},$$

where $\beta_n := \nu(B_Y^n \times \Gamma_{M \setminus Y})$. It follows from lemma 1 in p.13 in [25] that

$$\bar{\nu} = \sum_{n=0}^{\infty} \nu_n \simeq \bar{n} := \beta_1(\mu \times \nu_1'') + \sum_{n \neq 1} \beta_n \nu_n' \times \nu_n''.$$

Using the natural map T defined by

$$T: \varphi(\bar{P}) \in \mathrm{L}_{\bar{\nu}}^{2}(\Gamma_M, H) \mapsto \sqrt{\frac{d\bar{\nu}}{d\bar{n}}}(\bar{P}) \varphi(\bar{P}) \in \mathrm{L}_{\bar{n}}^{2}(\Gamma_M, H), \tag{3.30}$$

we have

$$h := Tf \in H_G(U_{\bar{n},\bar{\theta}}), \tag{3.31}$$

and for any $g \in \mathrm{Diff}_0^*(Y)$

$$\sqrt{\frac{d\mu_g}{d\mu}(P_k)}C(\hat{P})^{-1}\left(\frac{d\mu_g}{d\mu}(P_k)\right)^{\sqrt{-1}H_k^{[P]}} C(P_1, \cdots, P_{k-1}, g^{-1}(P_k), P_{k+1}, \cdots)\cdot \tag{3.32}$$
$$h(P_1, \cdots, P_{k-1}, g^{-1}(P_k), P_{k+1} \cdots) = h(\hat{P})$$

for $\hat{n}$-a.e.$\hat{P} \in Y_k$, where as usual we identify h with the corresponding symmetric function on $\hat{M}^\infty$, and

$$Y_k := \{\hat{P} \in \hat{M}^\infty \,||\bar{P} \cap Y| = 1 \quad \textit{and} \quad P_k \in Y\}.$$

Thus considering local translations as before, we find that

$$C(\hat{P})^{-1}C(P_1, \cdots, P_{k-1}, Q_k, P_{k+1}, \cdots)h(P_1, \cdots, P_{k-1}, Q_k, P_{k+1}, \cdots) = h(\hat{P}) \tag{3.33}$$

for $\mu \times \hat{n}$-a.e.$(Q_k, \hat{P}) \in U \times Y_k(U)$, where U is a sufficiently small neighbourhood of any point in Y and $Y_k(U) := Y_k \cap \{\hat{P} \in \hat{M}^\infty \,| P_k \in U\}$. It follows from Fubini's theorem that there exists an $A_k \in U$ so that

$$C(\hat{P})h(\hat{P}) = h(\hat{P}, A_k) \tag{3.34}$$

for $\hat{n}$-a.e.$\hat{P} \in Y_k(U)$, where

$$h(\hat{P}, A_k) := C(P_1, \cdots, P_{k-1}, A_k, P_{k+1}, \cdots)h(P_1, \cdots, P_{k-1}, A_k, P_{k+1}, \cdots).$$

Thus substituting (3.34) to (3.32), we get for all $g \in \mathrm{Diff}_0^*(U)$

$$\sqrt{\frac{d\mu_g}{d\mu}(P_k)}\left(\frac{d\mu_g}{d\mu}(P_k)\right)^{\sqrt{-1}H_k^{[P]}} h(\hat{P}, A_k) = h(\hat{P}, A_k),$$

for $\hat{n}$-a.e.$\hat{P} \in Y_k(U)$. It follows that $h(\hat{P}) = 0$ and hence $f(\hat{P}) = 0$ for $\hat{n}$-a.e.$\hat{P} \in Y_k(U)$ via the same pocedure as before. Therefore we have shown that $f(\bar{P}) = 0$ for $\bar{n}$-a.e.$\bar{P} \in \{\bar{Q} \in \Gamma_M |\ |\bar{Q} \cap Y| = 1\}$, whenever $f \in H_G(U_{\bar{v},\bar{\theta}})$ and hence $f(\bar{P}) = 0 \bmod \bar{v}$ on Δ_G^c. Now we have gotten a result corresponding to (2.61), and the same arguments work on with those parts after (2.61). This completes the proof.

Theorem 3.7 (Equivalence). *Let $\bar{v}_i$ $(i = 1, 2)$ be $\mathrm{Diff}_0^*(M)$-quasi-invariant probability measures on $(\Gamma_M, \mathfrak{B})$ and $\bar{\theta}_i$ be canonical, strongly Borelian 1-cocycles on $\Gamma_M \times \mathrm{Diff}_0^*(M)$. Then $(U_{\bar{v}_1,\bar{\theta}_1}, \mathrm{L}^2_{\bar{v}_1}(\Gamma_M, H_1))$ and $(U_{\bar{v}_2,\bar{\theta}_2}, \mathrm{L}^2_{\bar{v}_2}(\Gamma_M, H_2))$ are equivalent if and only if*

$$\bar{v}_1 \simeq \bar{v}_2, \tag{3.35}$$

and

(3.36) $\bar{\theta}_1$ and $\bar{\theta}_2$ are cohomologous.

Proof. The sufficiency is obvious. In order to check the necessity take any Bore set Ω of Γ_M and put

$$P^i_\Omega: f(\bar{P}) \in \mathrm{L}^2_{\bar{v}_i}(\Gamma_M, H_i) \mapsto \chi_\Omega(\bar{P}) f(\bar{P}) \in \mathrm{L}^2_{\bar{v}_i}(\Gamma_M, H_i),$$

as the projections. Then for an intertwing unitary operator A between the natural representations,

$$P^2_\Omega A = A P^1_\Omega \tag{3.37}$$

is a direct consequence of the preceding arguments. Therefore $v_1(\Omega)=0$ implies $v_2(\Omega)=0$ and vice versa. This proves (3.35).

Next put

$$T: f(\bar{P}) \in \mathrm{L}^2_{\bar{v}_1}(\Gamma_M, H_2) \mapsto f(\bar{P}) \sqrt{\frac{d\bar{v}_1}{d\bar{v}_2}}(\bar{P}) \in \mathrm{L}^2_{\bar{v}_2}(\Gamma_M, H_2).$$

Then we have

$$P^1_\Omega C = C P^2_\Omega \tag{3.38}$$

for a map defined by $C := A^{-1} T$. It follows that there exists a $U(H_2, H_1)$-valued measurable map $V(\bar{P})$ defined on Γ_M such that for any $f \in \mathrm{L}^2_{\bar{v}_1}(\Gamma_M, H_2)$

$$C(f)(\bar{P}) = V(\bar{P})(f(\bar{P})) \tag{3.39}$$

for $\bar{v}_1$-a.e.$\bar{P}$. By the definition of A and T,

$$U_{\bar{v}_1, \bar{\theta}_1} \circ C = C \circ U_{\bar{v}_1, \bar{\theta}_2},$$

in other words

$$V(\bar{P}) \bar{\theta}_2(\bar{P}, g) = \bar{\theta}_1(\bar{P}, g) V(\bar{g}^{-1} \bar{P})$$

holds for $\bar{v}_1$-a.e.$\bar{P}$. This proves the assertion.

Acknowledgement. I wish to express my thanks to Professor T.Hirai at Kyoto University for introducing me this subject and for kind advice. I also thank to an opportunity of discussing with Professor Ismagilov at Baumann Moscow Technical University who gave me many valuable suggestions and informations.

DEPARTMENT OF MATHEMATICS
FUKUI UNIVERSITY

References

[1] C. Chevalley, Theory of Lie groups, Princeton University Press, 1946.

[2] G. A. Goldin, J. Grodrick, R. T. Powers and D. H. Sharp, Non relativisitic current algebra in the N/V limit, J. Math. Phys., **15** (1974), 217–228.

[3] S. Helgason, Differential geometry, Lie groups, and symmetric spaces, Academic Press, 1978.

[4] T. Hirai, Construction of irreducible unitary representations of the infinite symmetric group $\mathfrak{S}_\infty$, J. Math. Kyoto Univ., **31** (1991), 495–541.

[5] T. Hirai, Irreducible unitary representations of the group of diffeomorphisms of a non-compact manifold, ibid., **33** (1993), 827–864.

[6] T. Hirai and H. Shimomura, Relations between unitary representations of diffeomorphism groups and those of the infinite symmetric group or of related permutation groups, ibid., **37** (1997), 261–316.

[7] R. S. Ismagilov, Unitary representations of the group of diffeomorphisms of a circle, Funct. Anal. Appl., **5** (1971), 45–53 (=Funct. Anal., **5** (1971), 209–216 (English Translation)).

[8] R. S. Ismagilov, On unitary representations of the group of diffeomorphisms of a compact manifold, Math. USSR Izvestija, **6** (1972), 181–209.

[9] R. S. Ismagilov, Unitary representations of the group of diffeomorphisms of the space $\boldsymbol{R}^n$, $n \geq 2$, Funct. Anal. Appl., **9** (1975), 71–72 (=Funct.Anal., **9** (1975), 154–155 (English Translation)).

[10] R. S. Ismagilov, On the group of volume-preserving diffeomorphisms, Math. USSR Izvestija, **17** (1981), 95–127.

[11] R. S. Ismagilov, Imbedding of a group of measure-preserving diffeomorphisms into a semidirect product and its unitary representations, Math.Sbornik, **41** (1982), 67–81.

[12] R. S. Ismagilov, Representations of infinite-dimensional groups, Trans.Math.Monographs Amer. Math.Soc., **152** (1996).

[13] A. A. Kirillov, Unitary representations of the group of diffeomorphisms and of some of its subgroups, Sel. Math. Sov., **1** (1981), 351–372.

[14] J. A. Leslie, On a differentiable structure for the group of diffeomorphisms, Topology, **6** (1967), 263–271.

[15] Yu. A. Neretin, The complementary series of representations of the group of diffeomorphisms of the circle, Russ.Math.Surv., **37** (1982), 229–230.

[16] H. Omori, Theory of infinite dimensional Lie groups, Kinokuniya shoten, 1978 (in Japanese).

[17] H. Omori, Y. Maeda, A. Yoshioka and O. Kobayashi, On regular Fréchet Lie groups IV, Tokyo J. Math., **5** (1982), 365–398.

[18] E. Shavgulidze, Mesures quasi-invariantes sur les groupes de difféomorphismes des variétés riemaniennes, C. R. Acad. Sci., **321** (1995) 229–232.

[19] K. Shiga, Manifold theory II, Kisosūgaku, Iwanamikouza, Iwanami, 1976 (in Japanese).

[20] H. Shimomura, Poisson measures on the configuration space and unitary representations of the group of diffeomorphisms, J. Math. Kyoto Univ., **34** (1994), 599–614.

[21] H. Shimomura, Ergodic decomposition of probability measures on the configuration space, ibid., **35** (1995), 611–630.

[22] H. Shimomura, 1-cocycles on the group of diffeomorphisms, ibid., **38** (1998), 695–725.

[23] H. Shimomura and T. Hirai, On group topologies on the group of diffeomorphisms, RIMS kōkyūroku, **1017** (1997), 104–115.

[24] N. Tatsuuma, H. Shimomura and T. Hirai, On group topologies and unitary representations of inductive limits of topological groups and the case of the group of diffeomorphisms, J. Math. Kyoto Univ., **38** (1998), 551–578.

[25] A. M. Vershik, I. M. Gel'fand and M. I. Graev, Representations of the group of diffeomorphism, Usp.Mat.Nauk., **30** (1975), 3–50 (=Russ. Math.Surv., **30** (1975), 3–50).