J. Math. Kyoto Univ. (JMKYAZ)
40-1 (2000), 155–163

Homogeneous generalized functions which are rotation invariant

By

Ji-Young NA and Soon-Yeong CHUNG

Abstract

We characterize generalized functions including distributions and ultradistributions which are rotation invariant and homogeneous as follows:

If u is a generalized function in $\boldsymbol{R}^n$ with $n \geq 2$ which is rotation invariant and homogeneous of real degree k then it can be written as

$$u = \begin{cases} a|x|^k + b\Delta^{\frac{-n-k}{2}}\delta, & \text{if } -n-k \text{ is an even nonnegative integer,} \\ a|x|^k, & \text{otherwise.} \end{cases}$$

In addition, we find a structure theorem of rotation invariant ultradistributions with support at the origin.

1. Introduction

A theory of invariance under a transformation group is one of the most important subjects in harmonic analysis and its applications to physics (see [4], [8], [9]). It is well known that a distribution u in $\boldsymbol{R}^n$ which is rotation invariant and homogeneous of degree k, comes out to be $u = |x|^k$ in $\boldsymbol{R}^n \backslash \{0\}$ (see [3, Section 23]).

In this paper we give an expression in the whole of $\boldsymbol{R}^n$ of the generalized functions, including distributions and ultradistributions, which are rotation invariant and homogeneous. To be precise, we show that if u is an ultradistribution in $\boldsymbol{R}^n$ with $n \geq 2$, homogeneous of real degree k and rotation invariant, then u can be written as

$$u = \begin{cases} a|x|^k + b\Delta^{\frac{-n-k}{2}}\delta, & \text{if } -n-k \text{ is an even nonnegative integer,} \\ a|x|^k, & \text{otherwise,} \end{cases}$$

where Δ is the Laplace operator $\Delta = \Sigma_{j=1}^{n} \dfrac{\partial}{\partial x_j^2}$ and δ is the Dirac measure in $\boldsymbol{R}^n$.

Besides, proving this theorem we find a structure theorem of rotation invariant ultradistributions supported at the origin.

1991 *Mathematics Subject Classification.* Primary 46F05; Secondary 43A99.

Partially supported by GARC-KOSEF and Korea Research Foundation
Communicated by Prof. K .Ueno, June 28, 1999

2. Ultradistributions and main results

Throughout this paper, referring the Euclidean space $\boldsymbol{R}^n$ we assume $n \geq 2$, since all that we are going to consider in this paper is trivial if $n=1$.

It is seen in [3] that if a distribution T in $\boldsymbol{R}^n$ is invariant under rotation and homogeneous of real degree k, then T has the form $T=c|x|^k$ away from the origin. In this section we find an expression of T which holds in the whole of $\boldsymbol{R}^n$. Actually we do this work for ultradistributions including distributions.

First, we introduce an ultradistribution. Let M_p, $p=0,1,2,\cdots$, be a sequence of positive numbers and let Ω be an open subset of $\boldsymbol{R}^n$. An infinitely differentiable function ϕ on Ω is called an ultradifferentiable function of class (M_p) (of class $\{M_p\}$, respectively) if for any compact set K of Ω and for each $h>0$ (for some $h>0$, respectively)

$$|\phi|_{M_p,K,h} = \sup_{\substack{x \in K \\ \alpha \in \boldsymbol{N}_0^n}} \frac{|\partial^\alpha \phi(x)|}{h^{|\alpha|} M_{|\alpha|}}$$

is finite.

We impose the following conditions on M_p:

(M.1) $M_p^2 \leq M_{p-1} M_{p+1}$, $p=1,2,\cdots$.

(M.2) There are positive constants A and H such that

$$M_p \leq AH^p \min_{0 \leq q \leq p} M_q M_{p-q}, \quad p=0,1,\cdots.$$

(M.3) There is a constant $A>0$ such that

$$\sum_{q=p+1}^{\infty} \frac{M_{q-1}}{M_q} \leq Ap \frac{M_p}{M_{p+1}}, \quad p=1,2,\cdots.$$

For example, the sequence $M_p = p!^s (s>1)$ satisfies all conditions above.

We denote by $\mathscr{E}_{(M_p)}(\Omega)$ ($\mathscr{E}_{\{M_p\}}(\Omega)$, respectively) the space of all ultradifferentiable functions of class (M_p) (of class $\{M_p\}$, respectively) on Ω.

The topologies of such spaces are defined as follows:

A sequence $\phi_j \to 0$ in $\mathscr{E}_{(M_p)}(\Omega)$ ($\mathscr{E}_{\{M_p\}}(\Omega)$, respectively) if for any compact set K of Ω and for every $h>0$ (for some $h>0$, respectively) we have

$$\sup_{\substack{x \in K \\ \alpha \in \boldsymbol{N}_0^n}} \frac{|\partial^\alpha \phi_j(x)|}{h^{|\alpha|} M_{|\alpha|}} \to 0 \text{ as } j \to \infty.$$

In addition, we denote by $\mathscr{D}_{(M_p)}(\Omega)$ ($\mathscr{D}_{\{M_p\}}(\Omega)$, respectively) the space of all ultradifferentiable functions of class (M_p) (of class $\{M_p\}$, respectively) on Ω with compact support.

As usual, we denote by $\mathscr{E}'_{(M_p)}(\Omega)$ ($\mathscr{E}'_{\{M_p\}}(\Omega)$, respectively) the strong dual space of $\mathscr{E}_{(M_p)}(\Omega)$ (of $\mathscr{E}_{\{M_p\}}(\Omega)$, respectively) and we call its elements ultradistributions of Beurling type (of Roumieu type, respectively) with compact support in Ω. The spaces $\mathscr{D}'_{(M_p)}(\Omega)$ and $\mathscr{D}'_{\{M_p\}}(\Omega)$ are also defined similarly as in the distributions $\mathscr{D}'(\Omega)$. For more details on the ultradistributions $\mathscr{E}'_{(M_p)}(\Omega)$, $\mathscr{E}'_{\{M_p\}}(\Omega)$, $\mathscr{D}'_{(M_p)}(\Omega)$, and $\mathscr{D}'_{\{M_p\}}(\Omega)$ we refer the reader to [2], [5] and [6].

In what follows, $*$ denotes (M_p) or $\{M_p\}$ throughout this paper.

Now we introduce the homogeneity and the spherical average for the generalized functions.

Let $l_\varepsilon = \varepsilon I$ where I is the $n \times n$ identity matrix and $\varepsilon > 0$. For an ultradistribution u an ultradistribution $u \circ l_\varepsilon$ is defined by

$$<u \circ l_\varepsilon, \phi> = \frac{1}{\varepsilon^n} <u, \phi\left[\frac{x}{\varepsilon}\right]>, \quad \phi \in \mathscr{D}_*.$$

From now on when we refer to a degree k of homogeneity we assume that k should be a real number.

Definition 2.1. *An ultradistribution u in $\boldsymbol{R}^n$ is homogeneous of degree k if for all* $\varepsilon > 0$

$$u \circ l_\varepsilon = \varepsilon^k u.$$

Then using the same method as in [3] we can easily show the following:

Lemma 2.2. *An ultradistribution u in Ω is homogeneous of degree k if and only if it satisfies the Euler equation*

$$ku = \sum_{j=1}^{n} x_j \frac{\partial u}{\partial x_j}, \quad x \in \Omega.$$

Definition 2.3. *For a continuous function ϕ in Ω, the spherical average of ϕ is defined to be a function*

$$\phi_S(r) = \frac{1}{|S^{n-1}|} \int_{S^{n-1}} \phi(r\omega)\, d\omega,$$

where $|S^{n-1}|$ denotes the surface area of the $(n-1)$-dimensional unit sphere.

Definition 2.4. (i) *For an ultradistribution $u \in \mathscr{D}'_*$ ($\mathscr{E}'_*$, respectively), the spherical average of u is an ultradistribution u_S defined by the relation*

$$<u_S, \phi> = <u, \phi_S>$$

for all $\phi \in \mathscr{D}_*$ ($\mathscr{E}_*$, respectively).

(ii) *An ultradistribution $u \in \mathscr{D}'_*$ is said to be rotation invariant if $u = u_S$.*

In the above we note that since ϕ_S is also ultradifferentiable, $<u, \phi_S>$ is well-defined. Hence, we can readily see that u_S defines an ultradistribution. Moreover, it is known in [1] that if f is a rotation invariant continuous function, then $f=f_S$ and if u is a rotation invariant distribution, then $u=u_S$.

For each defining sequence M_p we define the associated function of M_p on $(0, \infty)$ by

$$M(t)=\sup_p \log \frac{t^p M_0}{M_p}.$$

Then (M.1) implies

$$M_p = M_0 \sup_{t>0} \frac{t^p}{\exp M(t)}, \quad p=1,,2,3,\cdots \tag{2.1}$$

(see [6]).

Now we will characterize rotation invariant ultradistributions with support at the origin.

Theorem 2.5. *Let $u \in \mathscr{E}'_*(\boldsymbol{R}^n)$ have its support at the origin and $*=\{M_p\}$ ($*=(M_p)$, respectively). If u is rotation invariant, then there exists an entire function F in $\boldsymbol{C}$ such that on $\boldsymbol{R}^n$*

$$u=F(\Delta)\delta$$

and for every $L>0$ there exists $C>0$ (there exist $L>0$ and $C>0$, respectively) such that

$$|F(z^2)| \le C \exp M(L|z|), \ z \in \boldsymbol{C},$$

where Δ is the Laplace operator $\Delta=\Sigma_{j=1}^n \frac{\partial^2}{\partial x_j^2}$ and δ is the Dirac measure in $\boldsymbol{R}^n$.

Proof. We prove only the case where $*=\{M_p\}$. In view of the Paley-Wiener type theorem in [7], the Fourier-Laplace transform $\hat{u}(\zeta)$ of u is an entire function in $\boldsymbol{C}^n$ and satisfies that for every $L>0$ there exists $C>0$ such that

$$|\hat{u}(\zeta)| \le C \exp M(L|\zeta|), \ \zeta \in \boldsymbol{C}^n. \tag{2.2}$$

Moreover, the function $\hat{u}(\xi)$, $\xi \in \boldsymbol{R}^n$, is also rotation invariant, since u is rotation invariant. For each $\xi \in \boldsymbol{R}^n$ we choose a rotation matrix A so that $\xi = Ae_n$, where e_n is a unit vector in the direction of the n-th coordinate for $\boldsymbol{R}^n$. Then it follows that

$$\hat{u}(\xi)=\hat{u}(|\xi|Ae_n)=\hat{u}(|\xi|e_n)=\hat{u}(0,\cdots,0,\pm|\xi|).$$

Expanding $\hat{u}(\zeta)$ into

$$\hat{u}(\zeta)=\sum_{\alpha}\frac{\partial^{\alpha}\hat{u}(0)}{\alpha!}\zeta^{\alpha},\ \zeta\in \boldsymbol{C}^{n},$$

we have for every $\xi\in \boldsymbol{R}^{n}$

$$\begin{aligned}\hat{u}(\xi)&=\hat{u}(0,\cdots,0,\pm|\xi|)\\&=\sum_{k=0}^{\infty}\frac{\partial_{n}^{k}\hat{u}(0)}{k!}(\pm|\xi|)^{k}\\&=\sum_{k=0}^{\infty}\frac{\partial_{n}^{2k}\hat{u}(0)}{(2k)!}(|\xi|)^{2k},\end{aligned}$$

where $\partial_{j}=\dfrac{\partial}{\partial x_{j}}$, $j=1,2,\cdots,n$.

By the identity theorem of entire functions the above equality still holds for complex vectors $\zeta\in \boldsymbol{C}^{n}$. In other words,

$$\hat{u}(\zeta)=\sum_{k=0}^{\infty}\frac{\partial_{n}^{2k}\hat{u}(0)}{(2k)!}(\zeta_{1}^{2}+\cdots+\zeta_{n}^{2})^{k}$$

for $\zeta=(\zeta_{1},\cdots,\zeta_{n})\in \boldsymbol{C}^{n}$.

Now we define $F(z)$ on $\boldsymbol{C}$ by

$$F(z)=(2\pi)^{-n}\sum_{k=0}^{\infty}\frac{\partial_{n}^{2k}\hat{u}(0)}{(2k)!}(-z)^{k}. \tag{2.3}$$

Then $F(z)$ is an entire function in $\boldsymbol{C}$ and

$$F(-(\zeta_{1}^{2}+\cdots+\zeta_{n}^{2}))=(2\pi)^{-n}\hat{u}(\zeta),\ \zeta\in \boldsymbol{C}^{n}.$$

Moreover, it follows from the Fourier inversion formula that

$$\begin{aligned}u&=(2\pi)^{-n}\hat{\hat{u}}\\&=\sum_{k=0}^{\infty}\frac{\partial_{n}^{2k}\hat{u}(0)}{(2k)!}(-1)^{k}(\partial_{1}^{2}+\cdots+\partial_{n}^{2})^{k}\delta\\&=F(\Delta)\delta.\end{aligned}$$

On the other hand, using (2.2) and (2.3) we have

$$\begin{aligned}|F(z^{2})|&=|\hat{u}(0,\cdots,0,iz)|\\&\leq C\exp M(L|z|),\ z\in \boldsymbol{C}.\end{aligned}$$

This completes the proof.

Now we are in a position to state the main theorem of this paper.

Theorem 2.6. *Let u be an element of* $\mathscr{D}'_*(\boldsymbol{R}^n)$. *If u is rotation invariant and homogeneous of degree k, then there exist real numbers a and b such that*

$$u=\begin{cases} a|x|^k+b\Delta^{\frac{-n-k}{2}}\delta, & \textit{if } -n-k \textit{ is an even nonnegative integer,} \\ a|x|^k, & \textit{otherwise,} \end{cases}$$

where Δ *is the Laplace operator* $\Delta=\Sigma_{j=1}^n \frac{\partial^2}{\partial x_j^2}$ and δ *is the Dirac measure in* $\boldsymbol{R}^n$.

Proof. Here we prove only the case $*=\{M_p\}$. In fact, the case $*=(M_p)$ can be done similarly with only a slight modification. If $\psi(x)$ is an ultradifferentiable function in $\boldsymbol{R}$ supported by the half-axis $0<r<\infty$, then the function

$$\mathscr{L}\psi(x)=\phi(x)=\psi(|x|)$$

is an ultradifferentiable function in $\boldsymbol{R}^n$, vanishing in a neighborhood of the origin. The mapping $\mathscr{L}$ determined by this equation is a continuous linear transformation of the ultradifferentiable functions on the half-axis into the ultradifferentiable functions on $\boldsymbol{R}^n$ so that the linear functional

$$<S, \psi> = <u, \mathscr{L}\psi>$$

is an ultradistribution on $r>0$.

Since $\mathscr{L}(\psi\circ l_\varepsilon)=(\mathscr{L}\psi)\circ l_\varepsilon$ and u is homogeneous of degree k, we can easily see that S is homogeneous of degree $n+k-1$. Using the Euler equation

$$\sum_{j=1}^n x_j\frac{\partial S}{\partial x_j}=(n+k-1)S \text{ in } \boldsymbol{R}^n$$

and the chain rule

$$\sum_{j=1}^n x_j\frac{\partial}{\partial x_j}=r\frac{d}{dr}, \quad r=|x|\neq 0,$$

we have a differential equation

$$r\frac{dS}{dr}=(n+k-1)S, \; r>0.$$

Solving this differential equation in $\boldsymbol{R}^n\setminus\{0\}$ we have

$$S=Cr^{n+k-1}, \; r>0$$

for some real constant C.

It follows that the ultradistribution T defined by

$$<T,\phi> = <u,\phi> - \frac{C}{|S^{n-1}|}\int |x|^k\phi(x)dx, \ \ \phi\in\mathscr{D}_*$$

is an ultradistribution defined on $\boldsymbol{R}^n\setminus\{0\}$, which vanishes on all ultradifferentiable functions that are functions only of radius $|x|$. Moreover, it is easy to see that T is rotation invariant.

Let $\phi(x)$ be an arbitrary ultradifferentiable function vanishing for $|x|<\varepsilon$. Then the spherical average ϕ_S of ϕ is an ultradifferentiable function only of radius $|x|$ and it follows that

$$<T,\phi> = <T_S,\phi> = <T,\phi_S> = 0,$$

which implies $T=0$ outside the origin. Therefore, if we put a constant $a=\frac{C}{|S^{n-1}|}$ then $u-a|x|^k$ is a rotation invariant ultradistribution which is homogeneous of degree k and has its support at the origin. Thus by Theorem 2.5 there is an entire function F in $\boldsymbol{C}$ such that

$$u-a|x|^k=F(\Delta)\delta \tag{2.4}$$

and for every $L>0$

$$|F(z^2)|\le C\exp M(L|z|), \ \ z\in\boldsymbol{C}$$

for some constant $C>0$. Since for every $m\in\boldsymbol{N}_0$ and every $l>0$ there is a constant $A>0$ such that

$$\begin{aligned}|F^{(m)}(0)| &= \left|\frac{m!}{(2\pi i)}\int_{|\zeta|=l^2}\frac{F(\zeta)}{\zeta^{m+1}}d\zeta\right| \\ &\le A\frac{m!}{l^{2m}}\exp M(Ll),\end{aligned}$$

we have from (2.1) for every $m\in\boldsymbol{N}_0$

$$\begin{aligned}|F^{(m)}(0)| &\le Am!\inf_{l>0}\frac{\exp M(Ll)}{l^{2m}} \\ &\le Am!\,L^{2m}\left[\sup_{t>0}\frac{t^{2m}}{\exp M(t)}\right]^{-1} \\ &= AM_0m!L^{2m}/M_{2m}.\end{aligned}$$

Expand $F(z)$ into the Taylor series

$$F(z)=\sum_{m=0}^{\infty}(-1)^m a_{2m}z^m, \ \ z\in\boldsymbol{C}.$$

Then for any $L>0$ there exists constant $C>0$ such that

(2.5) $$|a_{2m}| \le CL^{2m}/M_{2m},\ m \in N_0.$$

Since $T-a|x|^k$ is homogeneous of degree k, we have for every $\varepsilon>0$

(2.6) $$<(T-a|x|^k)\circ l_\varepsilon, \phi> = \varepsilon^k < T-a|x|^k, \phi>$$

for all $\phi \in \mathscr{D}_*$.

In view of the relation (2.4), the equality (2.6) can be rewritten as

(2.7) $$\sum_{m=0}^{\infty}(-1)^m a_{2m}\varepsilon^{-2m-n}\Delta^m\phi(0) = \varepsilon^k \sum_{m=0}^{\infty}(-1)^m a_{2m}\Delta^m\phi(0).$$

Since $\phi \in \mathscr{D}_*$, there exist $h>0$ and $C'>0$ such that

(2.8) $$|\Delta^m\phi(0)| \le C'h^{2m}M_{2m},\ m \in N_0.$$

We define a function f on $(0, \infty)$ by

$$f(t) = \sum_{m=0}^{\infty}(-1)^m a_{2m}\Delta^m\phi(0)\frac{1}{t^{2m+n}}.$$

Then making use of (2.5) and (2.8) we can show that f is well defined and real analytic on $(0, \infty)$. Comparing the coefficients of t^k both sides of (2.7) we have $a_{2m}=0$ if $k \ne -2m-n$. Therefore, (2.4) make it possible to write

$$u = \begin{cases} a|x|^k + b\Delta^{\frac{-n-k}{2}}\delta, & \text{if } -n-k \text{ is an even nonnegative integer,} \\ a|x|^k, & \text{otherwise,} \end{cases}$$

for some constants a and b. This completes the proof.

DEPARTMENT OF MATHEMATICS,
KOREA ADVANCED INSTITUTE OF SCIENCE AND TECHNOLOGY,
DAEJON 305–701, KOREA

DEPARTMENT OF MATHEMATICS,
SOGANG UNIVERSITY,
SEOUL 121–742, KOREA
e-mail: sychung@ccs.sogang.ac.kr

References

[1] J. L. Challifour, Generalized functions and Fourier analysis, W. A. Benjamin, Inc. Massachusettes, 1972.

[2] S.-Y. Chung and D. Kim, Representations of quasianalytic ultradistributions, Ark. Mat., **31**(1993), 51–60.

[3] W. F. Donoghue, Distributions and Fourier transforms, Academic Press, New York, 1969.

[4] C. S. Hers, Invariant distributions, Proc. Symposia in Pure Math., **35**(1979), 555–568.

[5] L. Hörmander, The analysis of linear partial differential operators I, Springer-Verlag, New York, 1983.

[6] H. Komatsu, Ultradistributions I, J. Fac. Sci. Univ. Tokyo, Sect. **IA** **20**(1973), 25–105.
[7] H. Komatsu, Ultradistributions II, J. Fac. Sci. Univ. Tokyo, Sect. **IA** **24**(1977), 607–628.
[8] S. Rallis and G. Schiffmann, Distributions invariantes par le groupe orthogonal, Lec. Notes in Math. 497, 1970, 494–642.
[9] A. Tengstrand, Distributions invariant under an orthogonal group of arbitrary signature, Math. Scand., **8**(1960), 201–218.