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On the existence and uniqueness of solutions to stochastic equations in infinite dimension with integral-Lipschitz coefficients

By

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Abstract

In this paper, we study the existence and uniqueness of solutions to stochastic equations in infinite dimension with an integral-Lipschitz condition for the coefficients.

1. Introduction

In this paper, we study the solvability of the following stochastic equation on a separable Hilbert space H:

$$dX(t) = (AX(t) + b(t, X(t)))dt + \sigma(t, X(t))dB(t), X(0) = x,$$

or, more precisely,

$$X(t) = e^{At}x + \int_0^t e^{A(t-s)}b(s, X(s))ds + \int_0^t e^{A(t-s)}\sigma(s, X(s))dB(s), \tag{1}$$

where $B(\cdot)$ is a cylindrical Wiener process valued in a separable Hilbert space K defined on a filtered probability space $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$, A is the infinitesimal generator of a C_0-semigroup e^{At} on H, and $x \in H$.

It is well known that under a Lipschitz condition on the coefficients b, σ (see, e.g. [2], [3]), eq. (1) has a unique mild solution. Furthermore, Da Prato and Zabczyk have studied this equation under the hypothesis that A is almost m-dissipative, and b is continuous and monotone. But they have supposed that σ is Lipschitz.

On the other hand, Yamada and Watanabe in [11] and [8] proved the uniqueness of solutions of stochastic differential equations (in finite dimension) with some non-Lipschitz coefficients; and Yamada in [10] proved the existence of solutions of stochastic differential equations (in finite dimension) by the successive approximation.

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The aim of this paper is to establish the existence and uniqueness of the solution to eq. (1) under the following so-called integral-Lipschitz condition:

$$|b(t,x_1)-b(t,x_2)|^2+|\sigma(t,x_1)-\sigma(t,x_2)|^2\le\rho(|x_1-x_2|^2), \tag{2}$$

where $\rho:(0,+\infty)\to(0,+\infty)$ is a continuous, increasing, concave function satisfying

$$\rho(0+)=0,\qquad\int_0^1\frac{dr}{\rho(r)}=+\infty.$$

A typical example of (2) is:

$$|b(t,x_1)-b(t,x_2)|+|\sigma(t,x_1)-\sigma(t,x_2)|\le|x_1-x_2|\left(\ln\frac{1}{|x_1-x_2|}\right)^{\frac{1}{2}}.$$

In particular, we do not need the Lipschitz condition for σ. This result can be considered as an infinite-dimensional counterpart of the one found by Yamada in [10]. Nevertheless, one will see that our proof is much simpler than the one of [10].

The main result of our paper concerns the existence and uniqueness of the solution to eq. (1) under a "weaker" condition for b, that is:

$$|b(t,x_1)-b(t,x_2)|\le\rho(|x_1-x_2|).$$

In this case, the uniqueness result for finite dimensional stochastic differential equations could be found in Watanabe and Yamada [8]. But our proof of existence by the successive approximation is new, even in the finite dimensional case.

A typical example for this condition is:

$$|b(t,x_1)-b(t,x_2)|\le|x_1-x_2|\ln\frac{1}{|x_1-x_2|}.$$

Nevertheless, we suppose that A is m-dissipative and as a by-product, we obtain a uniqueness result for the one-dimensional stochastic differential equation which is slightly better than the well-known Yamada-Watanabe one in [11] and whose proof is simpler.

Finally, we apply our method to the study of backward stochastic differential equations and we obtain some existence and uniqueness result which is slightly stronger than the result of Mao [5].

The paper is organized as follows: in Section 2, we give some preliminaries. In Section 3, we prove the existence and uniqueness result to eq. (1) under the integral-Lipschitz condition on the coefficients. In Section 4, we prove the existence and uniqueness of solution to the dissipative stochastic equations under some "weaker" condition for b, and the corresponding result for the one-dimensional stochastic differential equation. And in the last section, we turn to the study of backward stochastic differential equations.

2. Preliminaries

2.1. Stochastic equations in infinite dimensions

Let $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ be a filtered probability space. Let H, K be two separable Hilbert spaces. We denote their norms by $|\cdot|$ and their scalar products by $\langle \cdot, \cdot \rangle$. A stochastic linear function on K is a linear mapping from K to $L^0(\Omega, \mathcal{F}, P)$ (see [2] and [3]).

Definition 2.1. We say that $B(t), t \in \mathbf{R}_+$ is a cylindrical Wiener process in a Hilbert space K, if:

(1) $\forall t \in \mathbf{R}_+$, $B(t)$ is a stochastic linear function on K.

(2) $\forall n \in \mathbf{N}^*, \forall h_1, h_2, \ldots, h_n \in K$, $\{(B(t)h_1, B(t)h_2, \ldots, B(t)h_n), t \in \mathbf{R}_+\}$ is a Wiener process (not necessarily standard) with values in $\mathbf{R}^n$.

(3) $\forall h_1, h_2 \in K, \forall t \in \mathbf{R}_+$,

$$E(B(t)h_1)(B(t)h_2) = t\langle h_1, h_2 \rangle.$$

We assume that $\mathcal{F}_t$ is the natural filtration of B.

For any Hilbert space H_1, we denote by $L^2_{\mathcal{F}}([0,T]; H_1)$ the set of all the $\mathcal{F}_t$-progressively measurable processes $x(\cdot)$, with values in H_1, such that

$$|x(\cdot)| = \left(E\left[\int_0^T |x(t)|^2 dt \right] \right)^{1/2} < +\infty.$$

Obviously $L^2_{\mathcal{F}}([0,T]; H_1)$ is a Hilbert space. We can define a stochastic integral for $x(\cdot) \in L^2_{\mathcal{F}}([0,T]; H_1)$:

$$\int_0^t x(s) dB_1(s), \forall t \in [0,T],$$

where $\{B_1(s), s \in [0,T]\}$ is a real Wiener process on $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$.

Next, we consider the space $\mathcal{L}^2(K; H)$ which is the set of Hilbert-Schmidt operators from K into H, i.e.,

$$\mathcal{L}^2(K; H) = \left\{ \psi \in \mathcal{L}(K; H) \,\middle|\, \sum_{n=1}^{\infty} |\psi e_n|^2 < +\infty \right\},$$

where $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis of K. $\mathcal{L}^2(K; H)$ is a Hilbert space and its norm is still denoted by $|\cdot|$.

For any $\psi(\cdot) \in L^2_{\mathcal{F}}([0,T]; \mathcal{L}^2(K; H))$ we can define the stochastic integral $\int_0^T \psi(t) dB(t) : L^2_{\mathcal{F}}([0,T]; \mathcal{L}^2(K; H)) \to L^2(\Omega, \mathcal{F}_T, P; H)$ as follows:

$$\int_0^T \psi(t) dB(t) = \sum_{n=1}^{\infty} \int_0^T \psi(t) e_n d(B(t) e_n).$$

The right hand of the above equality is well defined since

$$E\left[\left|\sum_{n=1}^{\infty}\int_0^T \psi(t)e_n d(B(t)e_n)\right|^2\right] = E\left[\sum_{n=1}^{\infty}\int_0^T |\psi(t)e_n|^2 dt\right]$$
$$= E\left[\int_0^T |\psi(t)|^2 dt\right] < +\infty.$$

Let $b(\cdot,x)$, $\sigma(\cdot,x)$ be stochastic processes depending on $x \in H$, such that:

$$\forall x \in H, b(\cdot,x) \in L^2_{\mathcal{F}}([0,T];H), \sigma(\cdot,x) \in L^2_{\mathcal{F}}([0,T];\mathcal{L}^2(K;H)), \tag{3}$$

$$\forall x \in H, |b(t,x)|^2 + |\sigma(t,x)|^2 \le \beta_1^2(t) + \beta_2^2(t)|x|^2, \tag{4}$$

where $\beta_1 \in L^2_{\mathcal{F}}([0,T];\mathbf{R}_+), \beta_2 \in L^2([0,T];\mathbf{R}_+)$.

For a given C_0-semigroup e^{At} with the infinitesimal generator A, we have the following well-known result (see, e.g. [2], [3]):

Proposition 2.1. *We suppose* (3), (4), *and the following Lipschitz condition*: $\forall x_1, x_2 \in H$,

$$|b(t,x_1) - b(t,x_2)|^2 + |\sigma(t,x_1) - \sigma(t,x_2)|^2 \le c^2|x_1 - x_2|^2, \tag{5}$$

where $c > 0$ *is a constant*; *then there exists a unique process* $X(\cdot) \in C_{\mathcal{F}}([0,T]; L^2(\Omega,\mathcal{F},P;H))$ *satisfying the following stochastic equation*: *for* $t \in [0,T]$,

$$X(t) = e^{At}x + \int_0^t e^{A(t-s)}b(s,X(s))ds + \int_0^t e^{A(t-s)}\sigma(s,X(s))dB(s). \tag{6}$$

Such a solution is called a mild solution of (6).

The aim of this paper is to study the solvability of eq. (6) under some weaker condition than the Lipschitz condition (5).

2.2. Dissipative infinitesimal generator

This subsection is adapted from [3].

The infinitesimal generator $A : D(A) \subset H \to H$ is said to be dissipative if and only if that for any $x \in D(A)$,

$$\langle Ax, x\rangle \le 0.$$

A is called m-dissipative if the range of $\lambda I - A$ is the whole space H for some $\lambda > 0$ (and then for any $\lambda > 0$). Finally A is called almost m-dissipative if $A - \alpha I$ is m-dissipative for some $\alpha \in \mathbf{R}$.

The Yosida approximation $A_\alpha, \alpha > 0$, of an m-dissipative infinitesimal generator A is defined by

$$A_\alpha = AJ_\alpha = \frac{1}{\alpha}(J_\alpha - I), \tag{7}$$

where

$$J_\alpha = (I - \alpha A)^{-1}. \tag{8}$$

We list some useful properties of J_α and A_α.

Lemma 2.1. *Let $A : D(A) \to H$ be an m-dissipative infinitesimal generator, and let J_α and A_α be defined by* (7) *and* (8) *respectively.*

(i) *For any $\alpha > 0$, we have*

$$|J_\alpha| \leq 1.$$

(ii) *For any $\alpha > 0$, A_α is dissipative, and bounded*:

$$|A_\alpha| \leq \frac{2}{\alpha},$$

and

$$|A_\alpha x| \leq |Ax|, \quad \forall x \in D(A).$$

(iii) *We have*

$$\lim_{\alpha \to 0} J_\alpha x = x, \quad \forall x \in H.$$

2.3. Technical lemmas

The following lemma is the starting point of this paper. The first part of this lemma can be found in [1].

Lemma 2.2. *Let $\rho : (0, +\infty) \to (0, +\infty)$ be a continuous, increasing function satisfying*

$$\rho(0+) = 0, \quad \int_0^1 \frac{dr}{\rho(r)} = +\infty,$$

and let u be a measurable, non-negative function defined on $(0, +\infty)$ satisfying

$$u(t) \leq a + \int_0^t \beta(s)\rho(u(s))ds, \quad t \in (0, +\infty), \tag{9}$$

where $a \in \mathbf{R}_+$, and $\beta \in L^1_{loc}([0, \infty); \mathbf{R}_+)$. We have:

(1) *If $a = 0$, then $u(t) = 0$, for $t \in [0, +\infty)$,* a.e.

(2) *If $a > 0$, we define $\nu(t) = \int_{t_0}^t (ds/\rho(s)), t \in \mathbf{R}_+$, where $t_0 \in (0, +\infty)$, then,*

$$u(t) \leq \nu^{-1}\left(\nu(a) + \int_0^t \beta(s)ds\right). \tag{10}$$

Proof. We define the function on $\mathbf{R}_+$:

$$R(t) = a + \int_0^t \beta(s)\rho(u(s))ds,$$

then as ρ is non-decreasing,

$$\frac{dR}{dt}(t) = \beta(t)\rho(u(t)) \leq \beta(t)\rho(R(t)).$$

Hence,

$$\frac{d}{dt}\nu(R(t)) = \frac{1}{\rho(R(t))}\beta(t)\rho(u(t)) \leq \beta(t),$$

from which we deduce:

$$\nu(R(t)) \leq \nu(a) + \int_0^t \beta(s)ds. \tag{11}$$

On the other hand, from the definition, ν is obviously a strictly increasing function, and $\nu(0+) = -\infty$. Then ν admits a strictly increasing inverse function, and $\nu^{-1}(-\infty) = 0$.

(1) If $a = 0$, then,

$$\nu(R(t)) = -\infty,$$

and

$$R(t) = 0,$$

from which we deduce the desired result.

(2) If $a > 0$, then from (11),

$$R(t) \leq \nu^{-1}\left(\nu(a) + \int_0^t \beta(s)ds\right),$$

and we obtain (10). □

Examples.

(i) The standard Lipschitz case: $\rho(t) = t, t > 0$, then from (10), we get:

$$u(t) \leq ae^{\int_0^t \beta(s)ds}.$$

(ii) The so-called Log-Lipschitz case: $\rho(t) = t\ln(1/t)$, for $t \in (0, t_0]$, $t_0 > 0$ is small enough, then from (10), we obtain:

$$u(t) \leq a^{e^{-\int_0^t \beta(s)ds}},$$

for $a > 0$ and $t > 0$ which are small enough.

(iii) If $\rho(t) = t\ln(1/t)\ln\ln(1/t)$, for $t \in (0, t_0]$, $t_0 > 0$ is small enough, then from (10), we obtain:

$$u(t) \leq \exp\left(-\left\{\ln\frac{1}{a}\right\}^{e^{-\int_0^t \beta(s)ds}}\right),$$

for $a > 0$ and $t > 0$ which are small enough.

The second lemma is trivial, we include it for completeness.

Lemma 2.3. *We put, for a given* $\epsilon > 0$,

$$F_\epsilon(x) = (|x|^2 + \epsilon)^{\frac{1}{2}}, \quad x \in H,$$

then, $F_\epsilon \in C^2(H; \mathbf{R})$, *and*

$$F'_\epsilon(x) = \frac{x}{(|x|^2 + \epsilon)^{\frac{1}{2}}}, \qquad F''_\epsilon(x) = \frac{1}{(|x|^2 + \epsilon)^{\frac{1}{2}}} \left(I - \frac{x \otimes x}{|x|^2 + \epsilon} \right).$$

In particular,

$$|F'_\epsilon(x)| \le 1, \qquad |F''_\epsilon(x)| \le \frac{2}{(|x|^2 + \epsilon)^{\frac{1}{2}}}.$$

3. Existence and uniqueness of the solution to stochastic equations in infinite dimension

Let $(\Omega, \mathcal{F}, P)$ be a probability space carrying a cylindrical Wiener process $B = \{B_t, t \ge 0\}$ with values in K (see Section 2 for a brief definition), and let $\{\mathcal{F}_t\}$ be the σ-field generated by B (that is, $\mathcal{F}_t = \sigma(B_s, 0 \le s \le t)$). We make the standard P-augmentation to each $\mathcal{F}_t$ such that $\mathcal{F}_t$ contains all the P-null sets of $\mathcal{F}$. Then $\{\mathcal{F}_t\}$ is right continuous and $\{\mathcal{F}_t\}$ satisfies the usual hypothesis. Let $T > 0$ be an arbitrarily fixed number.

Let $b(\cdot, x)$, $\sigma(\cdot, x)$ be stochastic processes depending on $x \in H$ satisfying (3) and (4). The aim of this section is to establish the solvability of (6) under the following so-called integral-Lipschitz condition: for any $x_1, x_2 \in H$,

$$|b(t, x_1) - b(t, x_2)|^2 + |\sigma(t, x_1) - \sigma(t, x_2)|^2 \le \beta^2(t) \rho(|x_1 - x_2|^2), \tag{12}$$

where $\beta \in L^2([0, T]; \mathbf{R}_+)$, and $\rho : (0, +\infty) \to (0, +\infty)$ is a continuous, non-decreasing, concave function satisfying:

$$\rho(0+) = 0, \qquad \int_0^1 \frac{dr}{\rho(r)} = +\infty. \tag{13}$$

Theorem 3.1. *We suppose* (3) *and the integral-Lipschitz condition* (12). *Then there exists a unique process* $X(\cdot) \in C_{\mathcal{F}}([0, T]; L^2(\Omega, \mathcal{F}, P; H)) (= C_{\mathcal{F}})$ *satisfying the stochastic equation* (6). *Furthermore, if we define*

$$\nu(t) = \int_{t_0}^t \frac{ds}{\rho(s)}, \quad t \in \mathbf{R}_+,$$

where $t_0 \in (0, +\infty)$,

$$M = \max_{t \in [0, T]} |e^{At}|,$$

and we denote by $X(\,\cdot\,; x)$ *be a mild solution of* (6) *starting from* x *at time* $t = 0$, *then for any* $t \ge 0$,

$$E[|X(t; x_1) - X(t; x_2)|^2] \le \nu^{-1} \left(\nu(3M^2 |x_1 - x_2|^2) + 3M^2(1 + T) \int_0^t \beta^2(s) ds \right).$$

Remark 3.1. (3) and the integral-Lipschitz condition (12) imply (4), see [10].

Proof. We begin with the proof of uniqueness.

As $X(\,\cdot\,;x)$ is a mild solution of (6), obviously we have

$$\begin{aligned} X(t;x_1) - X(t;x_2) = e^{At}(x_1 - x_2) \\ &+ \int_0^t e^{A(t-s)}(b(s,X(s;x_1)) - b(s,X(s;x_2)))ds \\ &+ \int_0^t e^{A(t-s)}(\sigma(s,X(s;x_1)) - \sigma(s,X(s;x_2)))dB_s. \end{aligned}$$

and

$$\begin{aligned} &|X(t;x_1) - X(t;x_2)|^2 \\ &\quad \le 3\Bigg\{|e^{At}(x_1-x_2)|^2 \\ &\quad + \left(\int_0^t e^{A(t-s)}[b(s,X(s;x_1)) - b(s,X(s;x_2))]ds\right)^2 \\ &\quad + \left(\int_0^t e^{A(t-s)}[\sigma(s,X(s;x_1)) - \sigma(s,X(s;x_2))]dB_s\right)^2\Bigg\}. \end{aligned}$$

Now let us put:

$$u(t) = \sup_{0\le r\le t} E[|X(r;x_1) - X(r;x_2)|^2],$$

then,

$$u(t) \le 3M^2|x_1-x_2|^2 + 3M^2(1+T)\int_0^t \beta^2(s)E\{\rho(|X(s;x_1) - X(s;x_2)|^2)\}ds.$$

As ρ is concave and increasing, we deduce:

$$u(t) \le 3M^2|x_1-x_2|^2 + 3M^2(1+T)\int_0^t \beta^2(s)\rho(u(s))ds.$$

From (10), we obtain:

$$u(t) \le \nu^{-1}\left(\nu(3M^2|x_1-x_2|^2) + 3M^2(1+T)\int_0^t \beta^2(s)ds\right).$$

In particular, if $x_1 = x_2$, we obtain the uniqueness of the solution to (6).

Now we turn to the proof of existence to (6). We define the Picard sequence of processes $\{X^n(\cdot), n \ge 0\} \subset C_{\mathcal{F}}$ as follows:

$$X^0(t) = x, \quad t \in [0,T],$$

and

$$
\begin{aligned}
X^{n+1}(t) &= e^{At}x + \int_0^t e^{A(t-s)} b(s, X^n(s))ds \\
&\quad + \int_0^t e^{A(t-s)} \sigma(s, X^n(s))dB_s, \qquad \text{for} \quad t \in [0,T].
\end{aligned} \tag{14}
$$

Because of the assumptions (3) and (4), the sequence $\{X^n(\cdot), n \geq 0\} \subset C_{\mathcal{F}}$ is well defined.

Furthermore, we establish the a priori estimates for $\{E[|X^n(t)|^2], n \geq 1\}$.

From (14), we deduce that

$$
E[|X^{n+1}(t)|^2] \leq 3M^2|x|^2 + 3M^2(1+T)E\left\{\int_0^t (\beta_1^2(s) + \beta_2^2(s)|X^n(s)|^2)ds\right\}.
$$

Hence,

$$
\begin{aligned}
E[|X^{n+1}(t)|^2] &\leq 3M^2|x|^2 + 3M^2(1+T)E\left[\int_0^T \beta_1^2(s)ds\right] \\
&\quad + 3M^2(1+T)\int_0^t \beta_2^2(s)E[|X^n(s)|^2]ds.
\end{aligned}
$$

Set

$$
p(t) = \left\{3M^2|x|^2 + 3M^2(1+T)E\left[\int_0^T \beta_1^2(s)ds\right]\right\} \exp\left\{3M^2(1+T)\int_0^t \beta_2^2(s)ds\right\}, \tag{15}
$$

then p is the solution of

$$
p(t) = 3M^2|x|^2 + 3M^2(1+T)E\left[\int_0^T \beta_1^2(s)ds\right] + 3M^2(1+T)\int_0^t \beta_2^2(s)p(s)ds. \tag{16}
$$

By recurrence, we prove easily that for any $n \geq 0$,

$$
E[|X^n(t)|^2] \leq p(t).
$$

Set

$$
u_{k+1,n}(t) = \sup_{0 \leq r \leq t} E[|X^{k+1+n}(r) - X^{k+1}(r)|^2].
$$

From the definition of the sequence $\{X^n(\cdot), n \geq 0\}$,

$$
\begin{aligned}
X^{k+1+n}(t) - X^{k+1}(t) &= \int_0^t e^{A(t-s)}(b(s, X^{k+n}(s)) - b(s, X^k(s)))ds \\
&\quad + \int_0^t e^{A(t-s)}(\sigma(s, X^{k+n}(s)) - \sigma(s, X^k(s)))dB_s.
\end{aligned}
$$

Hence,

$$u_{k+1,n}(t) \leq 2M^2(1+T)\int_0^t \beta^2(s)\rho(u_{k,n}(s))ds.$$

Set

$$v_k(t) = \sup_n u_{k,n}(t), \quad 0 \leq t \leq T,$$

then,

$$0 \leq v_{k+1}(t) \leq 2M^2(1+T)\int_0^t \beta^2(s)\rho(v_k(s))ds.$$

Finally, we define:

$$w(t) = \lim \sup_{k\to+\infty} v_k(t), \quad t \geq 0.$$

As ρ is continuous and

$$v_k(t) \leq 2p(t),$$

$$0 \leq w(t) \leq 2M^2(1+T)\int_0^t \beta^2(s)\rho(w(s))ds, \quad 0 \leq t \leq T.$$

Hence,

$$w(t) = 0, \quad t \in [0,T].$$

That is, $\{X^n(\cdot), n \geq 0\}$ is a Cauchy sequence in $C_{\mathcal{F}}$. We denote the limit in $C_{\mathcal{F}}$ of this sequence by $X(\cdot)$. Then there exists a subsequence $X^{n_l}(\cdot)$ and a process $X'(\cdot)$ such that

$$\lim_{l\to+\infty} X^{n_l}(t,\omega) = X(t,\omega), \qquad \text{for} \quad (t,\omega)\ dt\otimes dP-\text{a.e.},$$

and

$$|X^{n_l}(t,\omega)| \leq |X'(t,\omega)|, \qquad \text{for} \quad (t,\omega)\ dt\otimes dP-\text{a.e.}$$

Let $l \to +\infty$ in (14), we prove easily that X is a mild solution of (6). The proof of the existence of the solution to (6) is now complete. □

Remark 3.2. As stated in the introduction, the existence by successive approximation was already done by Yamada [10] in the finite dimensional case. As one can see, our method is simpler as we have borrowed the method introduced by Chemin and Lerner [1]. More importantly, in infinite dimensional case, there is no existence result when the coefficients are only continuous. Hence, our result concerning the existence of solution is new.

4. Existence and uniqueness of the solution to dissipative stochastic equations in infinite dimension

In this section, we give our main result about the existence and uniqueness result to (6) under some weaker condition than the condition (12).

4.1. Dissipative stochastic equations in infinite dimension

In this subsection, we assume always that A is m-dissipative. Then the Yosida approximations $A_\alpha, \alpha > 0$ are bounded, m-dissipative, and

$$\lim_{\alpha \to 0} e^{A_\alpha t} x = e^{At} x, \quad \forall x \in H.$$

Theorem 4.1. *We assume the following one-sided integral-Lipschitz condition for b and σ: for any $x_1, x_2 \in H$,*

(17)
$$2\langle b(t, x_1) - b(t, x_2), x_1 - x_2 \rangle + |\sigma(t, x_1) - \sigma(t, x_2)|^2 \leq \beta^2(t) \rho(|x_1 - x_2|^2).$$

Then there exists at most one mild solution in $C_{\mathcal{F}}$ to (6).

Proof. Let us suppose that there exist two mild solutions $X^1(\cdot)$, $X^2(\cdot) \in C_{\mathcal{F}}$ to (6). And set

(18)
$$X^1_\alpha(t) = e^{A_\alpha t} x + \int_0^t e^{A_\alpha (t-s)} b(s, X^1(s)) ds + \int_0^t e^{A_\alpha (t-s)} \sigma(s, X^1(s)) dB_s,$$

and

(19)
$$X^2_\alpha(t) = e^{A_\alpha t} x + \int_0^t e^{A_\alpha (t-s)} b(s, X^2(s)) ds + \int_0^t e^{A_\alpha (t-s)} \sigma(s, X^2(s)) dB_s.$$

As A_α is bounded, we can rewrite (18) and (19) in their differential forms, and we deduce:

$$\begin{aligned} dX^1_\alpha(t) &= (A_\alpha X^1_\alpha(t) + b(t, X^1(t)))dt + \sigma(t, X^1(t)) dB_t, \\ dX^2_\alpha(t) &= (A_\alpha X^2_\alpha(t) + b(t, X^2(t)))dt + \sigma(t, X^2(t)) dB_t. \end{aligned}$$

In particular, for any $\alpha > 0$,

$$E\left[\sup_{0 \leq t \leq T} (|X^1_\alpha(t)|^2 + |X^2_\alpha(t)|^2)\right] < +\infty.$$

Applying the Itô formula to $|X^1_\alpha(t) - X^2_\alpha(t)|^2$, we obtain:

$$\begin{aligned} d(|X^1_\alpha(t) - X^2_\alpha(t)|^2) = 2&\langle A_\alpha(X^1_\alpha(t) - X^2_\alpha(t)), X^1_\alpha(t) - X^2_\alpha(t) \rangle dt \\ &+ 2\langle X^1_\alpha(t) - X^2_\alpha(t), b(t, X^1(t)) - b(t, X^2(t)) \rangle dt \\ &+ 2\langle X^1_\alpha(t) - X^2_\alpha(t), \sigma(t, X^1(t)) - \sigma(t, X^2(t)) \rangle dB_t \\ &+ |\sigma(t, X^1(t)) - \sigma(t, X^2(t))|^2 dt. \end{aligned}$$

As

$$E\left[\left(\int_0^T |\langle X_\alpha^1(t) - X_\alpha^2(t), \sigma(t, X^1(t)) - \sigma(t, X^2(t))\rangle|^2 dt\right)^{\frac{1}{2}}\right]$$
$$\leq E\left[\sup_{0\leq t\leq T} |X_\alpha^1(t) - X_\alpha^2(t)| \left(\int_0^T |\sigma(t, X^1(t)) - \sigma(t, X^2(t))|^2 dt\right)^{\frac{1}{2}}\right]$$
$$\leq \frac{1}{2}E\left[\sup_{0\leq t\leq T}(|X_\alpha^1(t) - X_\alpha^2(t)|^2)\right] + \frac{1}{2}E\left[\int_0^T |\sigma(t, X^1(t)) - \sigma(t, X^2(t))|^2 dt\right]$$
$$< +\infty,$$

$\int_0^t \langle X_\alpha^1(s) - X_\alpha^2(s), \sigma(s, X^1(s)) - \sigma(s, X^2(s))\rangle dB(s), t \geq 0$ is a martingale, and as a consequence,

$$E\left[\int_0^t \langle X_\alpha^1(t) - X_\alpha^2(t), \sigma(t, X^1(t)) - \sigma(t, X^2(t))\rangle dB(s)\right] = 0,$$

from which we deduce:

$$\begin{aligned} &E[|X_\alpha^1(t) - X_\alpha^2(t)|^2] \\ &\quad \leq 2E\left[\int_0^t \langle X_\alpha^1(s) - X_\alpha^2(s), b(s, X^1(s)) - b(s, X^2(s))\rangle ds\right] \\ &\qquad + E\left[\int_0^t |\sigma(s, X^1(s)) - \sigma(s, X^2(s))|^2 ds\right]. \end{aligned}$$

When α goes to 0, we get, using (17) and the concavity of ρ,

$$\begin{aligned} E[|X^1(t) - X^2(t)|^2] &\leq E\left[\int_0^t \beta^2(s)\rho(|X^1(s) - X^2(s)|^2) ds\right] \\ &\leq \int_0^t \beta^2(s)\rho(E[|X^1(s) - X^2(s)|^2]) ds. \end{aligned}$$

Finally, Lemma 2.2 gives the uniqueness result. □

Remark 4.1. The condition (17) is weaker than the condition (12), because the concavity of ρ implies that

$$\rho(r) \geq r\rho(1), \quad r \in (0, 1].$$

As for existence, we need some stronger conditions.

Theorem 4.2. *We suppose* (3) *and the following condition*:

$|b(t,x)|^2 + |\sigma(t,x)|^2 \leq \beta_1(t) + \beta_2(t)|x|^2,\ \beta_1 \in L^p_{\mathcal{F}}([0,T]; H),\ \beta_2 \in L^p([0,T]; H),$

for some $p > 2$. *We assume also*

$$|b(t,x_1) - b(t,x_2)| \leq \beta(t)\rho_1(|x_1 - x_2|),$$
$$|\sigma(t,x_1) - \sigma(t,x_2)|^2 \leq \beta(t)\rho_2(|x_1 - x_2|^2),$$

where $\beta \in L^1([0,T];\mathbf{R}_+), \rho_1, \rho_2 : (0,+\infty) \to (0,+\infty)$ *are continuous, concave and increasing, and both of them satisfy* (13). *Furthermore, we assume that*

$$\rho_3(r) = \frac{\rho_2(r^2)}{r}, \quad r \in (0,+\infty)$$

is also continuous, concave and increasing, and

$$\rho_3(0+) = 0, \qquad \int_0^1 \frac{dr}{\rho_1(r) + \rho_3(r)} = +\infty.$$

Then there exists a unique solution to the equation (6).

Before the proof of this theorem, let us give one simple example to explain why the condition for b here is "weaker" than that of Theorem 3.1.

Example. If

$$\rho_1(r) = r \ln \frac{1}{r},$$
$$\rho_2(r) = r \ln \frac{1}{r},$$

then the conditions for Theorem 4.2 are satisfied but not for Theorem 3.1.

Proof. We define a sequence of processes $\{X^n(\cdot), n \geq 0\} \subset C_{\mathcal{F}}$ as follows:

$$X^0(t) = x, \quad t \in [0,T],$$

and

$$\begin{aligned} X^{n+1}(t) = e^{At}x &+ \int_0^t e^{A(t-s)} b(s, X^n(s))ds \\ &+ \int_0^t e^{A(t-s)} \sigma(s, X^{n+1}(s))dB_s, \qquad \text{for} \quad t \in [0,T]. \end{aligned} \tag{20}$$

Because of the assumptions of this theorem and Theorem 3.1, the sequence $\{X^n(\cdot), n \geq 0\} \subset C_{\mathcal{F}}$ is well defined.

It is important to note that (14) in Theorem 3.1 is a usual Picard approximation, while given X^n, (20) is a stochastic equation and Theorem 3.1 is applied to find X^{n+1} in a unique way.

Furthermore,

$$\begin{aligned} &E[|X^{n+1}(t)|^2] \\ &\quad \leq 3M^2|x|^2 + 3M^2(1+T)E\left[\int_0^T \beta_1^2(s)ds\right] \\ &\qquad + 3M^2T \int_0^t \beta_2^2(s)E[|X^n(s)|^2]ds + 3M^2 \int_0^t \beta_2^2(s)E[|X^{n+1}(s)|^2]ds. \end{aligned}$$

Obviously,

$$E[|X^0(t)|^2] \le p(t).$$

Let us suppose that

$$E[|X^n(t)|^2] \le p(t),$$

where p is defined by (15) which is the solution of (16).

Then

$$\begin{aligned} E[|X^{n+1}(t)|^2] \\ &\le 3M^2|x|^2 + 3M^2(1+T)E\left[\int_0^T \beta_1^2(s)ds\right] \\ &\quad + 3M^2 T\int_0^t \beta_2^2(s)p(s)ds + 3M^2\int_0^t \beta_2^2(s)E[|X^{n+1}(s)|^2]ds, \end{aligned}$$

from which we deduce that

$$E[|X^{n+1}(t)|^2] \le p(t).$$

The above recurrence proves that:

$$E[|X^n(t)|^2] \le p(t).$$

Set

$$u_{k+1,n}(t) = \sup_{0\le r\le t} E[|X^{k+1+n}(r) - X^{k+1}(r)|].$$

From the definition of the sequence $\{X^n(\cdot), n \ge 0\}$,

$$\begin{aligned} X^{k+1+n}(t) - X^{k+1}(t) \\ &= \int_0^t e^{A(t-s)}(b(s, X^{k+n}(s)) - b(s, X^k(s)))ds \\ &\quad + \int_0^t e^{A(t-s)}(\sigma(s, X^{k+n+1}(s)) - \sigma(s, X^{k+1}(s)))dB_s. \end{aligned}$$

We define

$$\begin{aligned} X_\alpha^{n+1}(t) &= e^{A_\alpha t}x + \int_0^t e^{A_\alpha(t-s)}b(s, X^n(s))ds \\ &\quad + \int_0^t e^{A_\alpha(t-s)}\sigma(s, X^{n+1}(s))dB_s, \qquad \text{for} \quad t\in[0,T]. \end{aligned}$$

Note that as $|x|$ is not C^2, one has to approximate $|x|$ by $F_\epsilon \in C^2$ which is studied in Lemma 2.3. Applying the Itô formula to $F_\epsilon(X_\alpha^{k+1+n} - X_\alpha^{k+1}(t))$, and taking the expectation, we get

$$\begin{aligned} E[F_\epsilon(X_\alpha^{k+1+n} - X_\alpha^{k+1}(t))] \\ &\le E\left[\int_0^t |b(s, X^{k+n}(s)) - b(s, X^k(s))|ds\right] \\ &\quad + E\left[\int_0^t \frac{|\sigma(s, X^{k+1+n}(s)) - \sigma(s, X^{k+1}(s))|^2}{(|X^{k+1+n}(s) - X^{k+1}(s)|^2+\epsilon)^{\frac{1}{2}}}ds\right]. \end{aligned}$$

Sending $\alpha \to 0$, and then letting $\epsilon \to 0$, we deduce that

$$u_{k+1,n}(t) \leq \int_0^t \beta(s)\rho_1(u_{k,n}(s))ds + \int_0^t \beta(s)\rho_3(u_{k+1,n}(s))ds.$$

Set

$$v_k(t) = \sup_n u_{k,n}(t), \quad 0 \leq t \leq T,$$

then,

$$0 \leq v_{k+1}(t) \leq \int_0^t \beta(s)\rho_1(v_k(s))ds + \int_0^t \beta(s)\rho_3(v_{k+1}(s))ds.$$

Finally, we define:

$$w(t) = \lim \sup_{k \to +\infty} v_k(t), \quad t \geq 0,$$

then

$$0 \leq w(t) \leq \int_0^t \beta(s)(\rho_1 + \rho_3)(w(s))ds, \quad 0 \leq t \leq T.$$

Hence,

$$w(t) = 0, \quad t \in [0, T].$$

Hence, $\{X^n(\cdot), n \geq 0\}$ is a Cauchy sequence in $C_{\mathcal{F}}([0,T];L^1(\Omega,\mathcal{F},P;H))$. We denote the limit in $C_{\mathcal{F}}([0,T];L^1(\Omega,\mathcal{F},P;H))$ of this sequence by $X(\cdot)$.

Then there exists a subsequence $X^{n_l}(\cdot)$, such that:

$$X^{n_l}(t,\omega) \to X(t,\omega), \quad (t,\omega)\ dt \otimes dP - \text{a.e.}$$

Taking into consideration that $\beta_1 \in L^p_{\mathcal{F}}([0,T];H), \beta_2 \in L^p([0,T];H)$, we can prove in a similar way that

$$\sup_{n \geq 0} \sup_{0 \leq t \leq T} E[|X^n(t)|^p] < +\infty, \qquad \text{where} \quad p > 2.$$

Hence, $X^n(t)$ is uniformly integrable in $L^2(\Omega, \mathcal{F}_t, P; H)$, and thus

$$\lim_{l \to +\infty} E[|X^{n_l}(t,\omega) - X(t,\omega)|^2] = 0.$$

Let $n \to +\infty$ in (20), we prove easily that X is a mild solution of (6). The proof of the existence of the solution to (6) is now complete. □

Remark 4.2. In the finite dimensional case, similar uniqueness result is given in [8]. Nevertheless, our uniqueness result is slightly stronger that of [8], even in the finite dimensional case. As for the existence part, our existence result is completely new, even in the finite dimensional case by the successive approximation.

4.2. One-dimensional stochastic differential equation

In this subsection, we consider the following one-dimensional stochastic differential equation:

$$X(t) = x + \int_0^t b(s, X(s))ds + \int_0^t \sigma(s, X(s))dB_s, \tag{21}$$

where $b : \mathbf{R}_+ \times \mathbf{R} \to \mathbf{R}$, $\sigma : \mathbf{R}_+ \times \mathbf{R} \to \mathbf{R}$ are two measurable functions. We suppose that:

$$(b(t, x_1) - b(t, x_2))(x_1 - x_2) \le \beta(t)\rho(|x_1 - x_2|^2), \tag{22}$$

$$|\sigma(t, x_1) - \sigma(t, x_2)|^2 \le \beta(t)\tilde{\rho}(|x_1 - x_2|), \tag{23}$$

where $\beta \in L^1_{loc}(\mathbf{R}_+; \mathbf{R}_+)$; $\rho : (0, +\infty) \to (0, +\infty)$ is a continuous, increasing, concave function satisfying (13); and $\tilde{\rho}$ is a Borel locally bounded function from $(0, +\infty)$ into itself such that there exists a $r_0 > 0$,

$$\int_0^{r_0} \frac{dr}{\tilde{\rho}(r)} = +\infty.$$

Theorem 4.3. *Let us assume* (22) *and* (23). *Then the pathwise uniqueness holds for the one-dimensional stochastic differential equation* (21).

Proof. Let X^1, X^2 be two solutions of (21), then

$$X^1(t) - X^2(t) = \int_0^t (b(s, X^1(s)) - b(s, X^2(s)))ds + \int_0^t (\sigma(s, X^1(s)) - \sigma(s, X^2(s)))dB_s.$$

As for any fixed $\epsilon > 0$,

$$\begin{aligned}
&\int_0^t 1_{0<X^1(s)-X^2(s)\le\epsilon}\tilde{\rho}(X^1(s) - X^2(s))^{-1}d\langle X^1 - X^2\rangle_s \\
&= \int_0^t 1_{0<X^1(s)-X^2(s)\le\epsilon}\tilde{\rho}(X^1(s) - X^2(s))^{-1}(\sigma(s, X^1(s)) - \sigma(s, X^2(s)))^2 ds \\
&\le \int_0^t \beta(s)ds < +\infty,
\end{aligned}$$

we obtain from Lemma 3.3 ([7, p. 370]),

$$L^0(X^1 - X^2) = 0.$$

Applying Tanaka's formula to $|X^1 - X^2|$, we obtain:

$$\begin{aligned}
&|X^1(t) - X^2(t)| \\
&\quad = \int_0^t \operatorname{sgn}(X^1(s) - X^2(s))d(X^1(s) - X^2(s)) \\
&\quad = \int_0^t \operatorname{sgn}(X^1(s) - X^2(s))(b(s, X^1(s)) - b(s, X^2(s)))ds \\
&\qquad + \int_0^t \operatorname{sgn}(X^1(s) - X^2(s))(\sigma(s, X^1(s)) - \sigma(s, X^2(s)))dB_s.
\end{aligned}$$

Set

$$\tau_N = \inf\{t \geq 0 : |X^1(t) - X^2(t)| \geq N\},$$

then τ_N is a stopping time and

$$\begin{aligned}
&|X^1(t\wedge\tau_N) - X^2(t\wedge\tau_N)| \\
&\quad = \int_0^{t\wedge\tau_N} \operatorname{sgn}(X^1(s) - X^2(s))(b(s, X^1(s)) - b(s, X^2(s)))ds \\
&\qquad + \int_0^{t\wedge\tau_N} \operatorname{sgn}(X^1(s) - X^2(s))(\sigma(s, X^1(s)) - \sigma(s, X^2(s)))dB_s.
\end{aligned}$$

Put

$$u^N(t) = E[|X^1(t\wedge\tau_N) - X^2(t\wedge\tau_N)|],$$

then

$$\begin{aligned}
u^N(t) &\leq E\left[\int_0^{t\wedge\tau_N} \beta(s)\rho(|X^1(s) - X^2(s)|)ds\right] \\
&\leq \int_0^t \beta(s)\rho(u^N(s))ds.
\end{aligned}$$

Applying Lemma 2.2, we deduce that

$$X^1(t\wedge\tau_N) = X^2(t\wedge\tau_N), \qquad \text{for} \quad t \in [0,\infty).$$

Sending N to $+\infty$, we obtain the desired result. $\square$

Remark 4.3. The continuity of b is not needed in this theorem.

Remark 4.4. We can obtain a comparison result as the one in [9] under some similar conditions for b^1 or b^2.

5. Existence and uniqueness of the solution for backward stochastic differential equations

In this section, we consider the following backward stochastic differential equation (BSDE):

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s, \tag{24}$$

where $\xi \in L^2(\Omega, \mathcal{F}_T, P; \mathbf{R}^m)$,

$$f : [0,T] \times \Omega \times \mathbf{R}^m \times \mathbf{R}^{m\times d} \to \mathbf{R}^m$$

is $\mathcal{P}\otimes\beta(\mathbf{R}^m)\otimes\beta(\mathbf{R}^{m\times d})/\beta(\mathbf{R}^m)$ measurable ($\mathcal{P}$ denotes the predictable σ-field of $[0,T]\times\Omega$); for a certain constant $c > 0$ and a certain $\beta \in L^2_{\mathcal{F}}([0,T]; \mathbf{R}_+)$,

$$|f(t,y,z)| \leq \beta(t) + c(|y| + |z|), \tag{25}$$

$$|f(t,y_1,z_1) - f(t,y_2,z_2)|^2 \leq \rho(|y_1 - y_2|^2) + c^2|z_1 - z_2|^2, \tag{26}$$

where $\rho : (0,\infty) \to (0,\infty)$ is a continuous, concave, increasing function satisfying (13).

Theorem 5.1. *Under the assumptions* (25) *and* (26), (24) *admits a unique solution* $(Y, Z) \in L^2_{\mathcal{F}}([0,T]; \mathbf{R}^m) \times L^2_{\mathcal{F}}([0,T]; \mathbf{R}^{m\times d})$.

Proof. Let $(Y^1, Z^1), (Y^2, Z^2) \in L^2_{\mathcal{F}}([0,T]; \mathbf{R}^{m\times d})$ be two solutions of (24), then

$$Y^1_t - Y^2_t = \int_t^T (f(s, Y^1_s, Z^1_s) - f(s, Y^2_s, Z^2_s))ds - \int_t^T (Z^1_s - Z^2_s)dB_s.$$

Applying Itô's formula to $|Y^1_t - Y^2_t|^2$ and taking the expectation, we obtain:

$$\begin{aligned}
&E[|Y^1_t - Y^2_t|^2] + E\left[\int_t^T |Z^1_s - Z^2_s|^2 ds\right] \\
&\quad = 2E\left[\int_t^T (f(s, Y^1_s, Z^1_s) - f(s, Y^2_s, Z^2_s), Y^1_s - Y^2_s)ds\right] \\
&\quad \le 2E\left[\int_t^T (\rho^{\frac{1}{2}}(|Y^1_s - Y^2_s|^2) + c|Z^1_s - Z^2_s|)|Y^1_s - Y^2_s|ds\right] \\
&\quad \le E\left[\int_t^T (\rho(|Y^1_s - Y^2_s|^2)ds\right] + E\left[\int_t^T |Y^1_s - Y^2_s|^2 ds\right] \\
&\qquad + \frac{1}{2}E\left[\int_t^T |Z^1_s - Z^2_s|^2 ds\right] + 2c^2 E\left[\int_t^T |Y^1_s - Y^2_s|^2 ds\right].
\end{aligned}$$

Thus,

$$\begin{aligned}
&E[|Y^1_t - Y^2_t|^2] + \frac{1}{2}E\left[\int_t^T |Z^1_s - Z^2_s|^2 ds\right] \\
&\quad \le (2c^2+1)E\left[\int_t^T |Y^1_s - Y^2_s|^2 ds\right] + E\left[\int_t^T \rho(|Y^1_s - Y^2_s|^2)ds\right] \\
&\quad \le (2c^2+1)E\left[\int_t^T |Y^1_s - Y^2_s|^2 ds\right] + \int_t^T \rho(E[|Y^1_s - Y^2_s|^2])ds.
\end{aligned}$$

Now set:

$$u(t) = E[|Y^1_t - Y^2_t|^2],$$

then,

$$u(t) \le (2c^2+1)\int_t^T u(s)ds + \int_t^T \rho(u(s))ds.$$

As

$$\int_0^1 \frac{dr}{\rho(r) + (2c^2+1)r} = \infty,$$

we deduce from Lemma 2.2 that,

$$u(t) = 0,$$

And the uniqueness of the solution can be now easily proved.

As for the existence of solution, we proceed as Theorem 4.2: define a sequence of $(Y^n, Z^n), n \geq 0$, as follows:

$$Y^0 = 0, \qquad Z^0 = 0,$$

and

$$Y_t^{n+1} = \xi + \int_t^T f(s, Y_s^n, Z_s^{n+1})ds - \int_t^T Z_s^{n+1} dB_s.$$

Then the rest of the proof goes in a similar way as that in Theorem 3.1, and we omit it. $\square$

Remark 5.1. One can find similar result (but a little bit weaker than ours) in Mao [5]. Once again, the proof given here is much simpler.

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