J. Math. Kyoto Univ. (JMKYAZ)
44-2 (2004), 451–456

An application of unstable K-theory

By

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Abstract

In [1] and [2], we introduced and investigated "unstable K-theory" $U_n(X) = [X, U(n)]$ and showed the relation with the Adams e-invariant. In this paper, we offer a theorem relating to $U_n(X)$ and show an application in connection with stable splittings of G_2.

1. Introduction

In this paper, we work in the pointed category, i.e., assume all spaces are base-pointed and all homotopy sets are base-point preserving homotopy sets.

Let $U(n)$ be the unitary group and X be a pointed finite CW-complex. Then the homotopy set $U_n(X) = [X, U(n)]$ forms a group by the point-wise multiplication. We call this group as the "unstable K-theory" of X, for the reason that $U_n(X)$ is isomorphic to $\widetilde{K}^1(X)$ for sufficiently large n. When $\mathrm{dim}X \le 2n$, this group $U_n(X)$ fits in the next exact sequence (1.1) where $\Theta(X)$ maps $\alpha \in \widetilde{K}^0(X)$ to $s_n(\alpha) = n!ch_n\alpha$ (see [1] for detail).

$$\widetilde{K}^0(X) \xrightarrow{\Theta(X)} \mathrm{H}^{2n}(X; \mathbb{Z}) \xrightarrow{\Phi(X)} U_n(X) \xrightarrow{\Pi(X)} \widetilde{K}^1(X) \to 0. \tag{1.1}$$

This group $U_n(X)$ is, even if $\mathrm{dim}X \le 2n$, not commutative in general. But we use the notations of abelian groups for this group, i.e., write its unit 0 and denote its operation by $+$.

Now we consider a suspended map $\Sigma f : \Sigma Y \to \Sigma X$ where finite CW-complexes ΣX and ΣY satisfy:

$$\mathrm{dim}\Sigma Y = 2n-1,\ \mathrm{dim}\Sigma X < 2n-1. \tag{1.2}$$

$$\widetilde{K}^1(Y) = 0,\ \widetilde{K}^0(X) = 0. \tag{1.3}$$

Also let k be an integer and we denote the k-fold map $\Sigma X \to \Sigma X$ by k. Then we have a commutative diagram:

$$\begin{array}{ccccccc} \Sigma Y & \xrightarrow{\Sigma f} & \Sigma X & \longrightarrow & C & \xrightarrow{\rho} & \Sigma^2 Y \\ \downarrow{=} & & \downarrow{k} & & \downarrow{\pi} & & \downarrow{=} \\ \Sigma Y & \xrightarrow[k\circ\Sigma f]{} & \Sigma X & \longrightarrow & C' & \xrightarrow[\rho']{} & \Sigma^2 Y \end{array} \tag{1.4}$$

Received February 13, 2004

where C and C' are mapping cones of Σf and $k \circ \Sigma f$ respectively and two rows are usual cofibrations. Then our main result is the following.

Theorem 1.1. *The induced map $\pi^* : U_n(C') \to U_n(C)$ is surjective and for any $\alpha \in \mathrm{Ker}\pi^*$, $k\alpha = 0$.*

From this theorem, we deduce Corollary 2.1 which estimates the order of a suspended map by means of unstable K-theory.

On the other hand, as an application, we consider the exceptional Lie group G_2. G_2 has the cell decomposition like

$$G_2 \simeq S^3 \cup e^5 \cup e^6 \cup e^8 \cup e^9 \cup e^{11} \cup e^{14}. \tag{1.5}$$

We denote the i-skeleton of G_2 by ${G_2}^{(i)}$. In [3], S. Oka showed the next result.

Theorem 1.2. *$G_2^{(6)}$ is not stable retract of $G_2^{(11)}$.*

In this paper we offer another simple proof of this result using the unstable K-theory $U_6(\Sigma G_2^{(11)})$.

In the following, we always use $\mathbb{Z}$ as the coefficient ring of cohomology and we omit to write them. Also we do not distinguish maps and their homotopy classes.

2. Main result

We consider the suspended map $\Sigma f : \Sigma Y \to \Sigma X$ under the assumption (1.2), (1.3) and prove Theorem 1.1.

First we observe that from (1.3)

$$\widetilde{K}^1(C) = \widetilde{K}^1(C') = 0.$$

Also from (1.2),

$$\rho^* : \mathrm{H}^{2n}\left(\Sigma^2 Y\right) \xrightarrow{\cong} \mathrm{H}^{2n}\left(C\right),\ \rho'^* : \mathrm{H}^{2n}\left(\Sigma^2 Y\right) \xrightarrow{\cong} \mathrm{H}^{2n}\left(C'\right)$$

are isomorphisms. Therefore $\pi^* = H^{2n}(\pi) : \mathrm{H}^{2n}(C') \to \mathrm{H}^{2n}(C)$ is an isomorphism as well.

Applying the short exact sequences obtained from (1.1) to C and C', we have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widetilde{K}^0(C)/\mathrm{Ker}\Theta(C) & \xrightarrow{\Theta(C)} & \mathrm{H}^{2n}(C) & \longrightarrow & U_n(C) & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \overline{K^0(\pi)}} & & \uparrow{\scriptstyle H^{2n}(\pi)} & & \uparrow{\scriptstyle \pi^*} & & \\
0 & \longrightarrow & \widetilde{K}^0(C')/\mathrm{Ker}\Theta(C') & \xrightarrow{\Theta(C')} & \mathrm{H}^{2n}(C') & \longrightarrow & U_n(C') & \longrightarrow & 0,
\end{array}$$

where $\overline{K^0(\pi)}$ is the induced map obtained from $K^0(\pi) : \widetilde{K}^0(C') \to \widetilde{K}^0(C)$. This diagram implies that π^* is surjective. Moreover, since $H^{2n}(\pi)$ is isomorphic, we see $\mathrm{Coker}\overline{K^0(\pi)} \cong \mathrm{Ker}\pi^*$ by the snake lemma.

Now, applying K-theory to the diagram (1.4), we have

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widetilde{K}^0(\Sigma^2 Y) & \longrightarrow & \widetilde{K}^0(C) & \longrightarrow & \widetilde{K}^0(\Sigma X) & \longrightarrow & 0 \\ & & {\scriptstyle =}\big\uparrow & & {\scriptstyle K^0(\pi)}\big\uparrow & & {\scriptstyle K^0(k)=k}\big\uparrow & & \\ 0 & \longrightarrow & \widetilde{K}^0(\Sigma^2 Y) & \longrightarrow & \widetilde{K}^0(C') & \longrightarrow & \widetilde{K}^0(\Sigma X) & \longrightarrow & 0 \end{array}$$

and, by the snake lemma again, we see $\mathrm{Coker}K^0(\pi) \cong \mathrm{Coker}k$. This map k is just the map multiplying k. Therefore, for any element $\alpha \in \mathrm{Coker}K^0(\pi)$, $k\alpha = 0$. Since $\mathrm{Coker}\overline{K^0(\pi)}$ is a factor group of $\mathrm{Coker}K^0(\pi)$, the same is true for the element of $\mathrm{Coker}\overline{K^0(\pi)} \cong \mathrm{Ker}\pi^*$. □

Now we offer a corollary. As above, we consider a suspended map $\Sigma f : \Sigma Y \to \Sigma X$ under the assumptions (1.2) and (1.3), and the cofibration sequence $\Sigma Y \xrightarrow{\Sigma f} \Sigma X \to C_{\Sigma f} \xrightarrow{\rho} \Sigma^2 Y$. Apply the exact sequence (1.1) to ΣX and we see $U_n(\Sigma X) = 0$ and $\rho^* : U_n(\Sigma^2 Y) \to U_n(C_{\Sigma f})$ is surjective.

Corollary 2.1. *If Σf has its order k in $[\Sigma Y, \Sigma X]$, the order of any element of* $\mathrm{Ker}(\rho^* : U_n(\Sigma^2 Y) \to U_n(C_{\Sigma f}))$ *is a factor of k.*

Proof. Apply Theorem 1.1 to Σf and its order k. Then $C' = C_{k\Sigma f} \simeq \Sigma X \vee \Sigma^2 Y$ and ${\rho'}^* : U_n(\Sigma^2 Y) \to U_n(C')$ is an isomorphism. Since $\pi^* {\rho'}^* = \rho^* : U_n(\Sigma^2 Y) \to U_n(C_{\Sigma f})$, the statement follows. □

3. Application

In [3], some spaces were considered, whose K-groups are isomorphic to those of spheres, but whose homology groups are not isomorphic to those of spheres. Namely, for given element α in the stable homotopy group of spheres $\pi^S_{k-1}(S^0)$ where k is even and the order of α is q, the following spaces are introduced:

$$X^n_\alpha = S^n \cup_\alpha e^{n+k} \cup_q e^{n+k+1}, \tag{3.1}$$

$$Y^n_\alpha = S^{n-k-1} \cup_q e^{n-k} \cup_\alpha e^n. \tag{3.2}$$

And it was showed that

$$\widetilde{K}^*(X^n_\alpha) \cong \widetilde{K}^*(S^n), \qquad \widetilde{K}^*(Y^n_\alpha) \cong \widetilde{K}^*(S^n) \tag{3.3}$$

as additive groups. Also these spaces are related to the exceptional Lie group G_2. Take the cellular decomposition (1.5) of G_2. Then

$${G_2}^{(6)} \simeq X^3_\eta, \qquad {G_2}^{(11)}/{G_2}^{(6)} \simeq Y^{11}_\eta \tag{3.4}$$

where η is the generator of $\pi^S_1(S^0) \cong \mathbb{Z}/2\mathbb{Z}$ (see [3]).

Now we shall prove Theorem1.2 using “unstable K-theory”.

Since $\mathrm{H}^3(G_2) \cong \mathrm{H}^{11}(G_2) \cong \mathbb{Z}$, we take their generators x_3, x_{11} respectively. The K-theory of G_2 and its Chern character are known as follows (see [5]).

Theorem 3.1.

$$K^*(G_2) \cong \bigwedge(\alpha, \beta), \quad \alpha, \beta \in \widetilde{K}^1(G_2),$$

$$ch(\alpha) = 2x_3 + \frac{2}{5!}x_{11}, \quad ch(\beta) = 10x_3 - \frac{50}{5!}x_{11}. \tag{3.5}$$

Thus $ch(\alpha\beta) = x_{11}x_3$ is a generator of $\mathrm{H}^{14}(G_2)$. Consider the cofibration $G_2^{(11)} \to G_2 \to S^{14}$ and the naturality of the Chern character implies that $\widetilde{K}^0(S^{14}) \xrightarrow{\cong} \widetilde{K}^0(G_2)$. Then we obtain the exact sequence

$$0 \to \widetilde{K}^1(G_2) \to \widetilde{K}^1(G_2^{(11)}) \to \widetilde{K}^0(S^{14}) \xrightarrow{\cong} \widetilde{K}^0(G_2) \to \widetilde{K}^0(G_2^{(11)}) \to 0,$$

i.e., $\widetilde{K}^0(\Sigma G_2^{(11)}) \cong \widetilde{K}^0(\Sigma G_2)$ and $\widetilde{K}^1(\Sigma G_2^{(11)}) = 0$.

Theorem 3.2.

$$U_6(\Sigma G_2^{(11)}) \cong \mathbb{Z}/12\mathbb{Z}.$$

Proof. Apply above results to the exact sequence (1.1) and we obtain

$$U_6\left(\Sigma G_2^{(11)}\right) \cong \mathrm{Coker}\left(s_6 : \widetilde{K}^0\left(\Sigma G_2^{(11)}\right) \to \mathrm{H}^{12}\left(\Sigma G_2^{(11)}\right)\right).$$

From the naturality of s_6 and (3.5), the statement follows. □

Proposition 3.1.

$$U_6(Y_\eta^{12}) \cong \mathbb{Z}/360\mathbb{Z}.$$

Proof. By (1.1) and (3.3), we have

$$U_6(Y_\eta^{12}) \cong \mathrm{Coker}(s_6 : \widetilde{K}^0(Y_\eta^{12}) \to \mathrm{H}^{12}\left(Y_\eta^{12}\right)).$$

Let $\pi : Y_\eta^{12} \to S^{12}$ be the map which smashes the 11-skeleton. The generators $\gamma' \in \widetilde{K}^0(Y_\eta^{12})$ and $\gamma \in \widetilde{K}^0(S^{12})$ are related as $\pi^*(\gamma) = 2\gamma'$ ([3, Proposition 1.4]). Therefore, from the naturality of s_6,

$$s_6(\gamma') = \frac{6!}{2}c' = 360c',$$

where c' is the generator of $\mathrm{H}^{12}\left(Y_\eta^{12}\right)$.

$$\begin{array}{ccc} \widetilde{K}^0(S^{12}) & \xrightarrow{\pi^*} & \widetilde{K}^0(Y_\eta^{12}) \\ \downarrow{\scriptstyle s_6} & & \downarrow{\scriptstyle s_6} \\ \mathrm{H}^{12}\left(S^{12}\right) & \xrightarrow{\cong} & \mathrm{H}^{12}\left(Y_\eta^{12}\right) \end{array}$$

□

From (3.4), we have the cofibration sequence:

$$X_\eta^3 \hookrightarrow G_2^{(11)} \to Y_\eta^{11} \xrightarrow{h} X_\eta^4 \hookrightarrow \Sigma G_2^{(11)}.$$

Now we set that the order of $f = \Sigma^{2i} h$ in $[\Sigma^{2i} Y_\eta^{11}, \Sigma^{2i} X_\eta^4]$ is k and prove $k \neq 1$.

First we observe that the mapping cone C_f is homotopy equivalent to $\Sigma^{2i+1} G_2^{(11)}$. Also,

$$\dim \Sigma^{2i} Y_\eta^{11} = 2(6+i) - 1, \quad \dim \Sigma^{2i} X_\eta^4 < 2(6+i) - 1,$$
$$\widetilde{K}^1(\Sigma^{2i-1} Y_\eta^{11}) = 0 \quad \text{and} \quad \widetilde{K}^0(\Sigma^{2i-1} X_\eta^4) = 0.$$

Therefore we can apply Corollary 2.1, i.e., there exists a surjective homomorphism

$$U_{6+i}(\Sigma^{2i} Y_\eta^{12}) \to U_{6+i}(\Sigma^{2i+1} G_2^{(11)})$$

and any element α of its kernel satisfies $k\alpha = 0$.

On the other hand, from Theorem 3.2 and Proposition 3.1, we have

$$U_{6+i}(\Sigma^{2i} Y_\eta^{12}) \cong \mathbb{Z}/\left(360 \times \frac{(6+i)!}{6!}\right)\mathbb{Z},$$
$$U_{6+i}(\Sigma^{2i+1} G_2^{(11)}) \cong \mathbb{Z}/\left(12 \times \frac{(6+i)!}{6!}\right)\mathbb{Z}$$

(we used Corollary 4.1 of [1]). Therefore k must be a multiple of 30. Thus h is not stably null-homotopic and X_η^3 is not stable retract of $G_2^{(11)}$. $\square$

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