

# Entropies, convexity, and functional inequalities

On  $\Phi$ -entropies and  $\Phi$ -Sobolev inequalities

By

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## Abstract

Our aim is to provide a short and self contained synthesis which generalise and unify various related and unrelated works involving what we call  $\Phi$ -Sobolev functional inequalities. Such inequalities related to  $\Phi$ -entropies can be seen in particular as an inclusive interpolation between Poincaré and Gross logarithmic Sobolev inequalities. In addition to the known material, extensions are provided and improvements are given for some aspects. Stability by tensor products, convolution, and bounded perturbations are addressed. We show that under simple convexity assumptions on  $\Phi$ , such inequalities hold in a lot of situations, including hyper-contractive diffusions, uniformly strictly log-concave measures, Wiener measure (paths space of Brownian Motion on Riemannian Manifolds) and generic Poisson space (includes paths space of some pure jumps Lévy processes and related infinitely divisible laws). Proofs are simple and relies essentially on convexity. We end up by a short parallel inspired by the analogy with Boltzmann-Shannon entropy appearing in Kinetic Gases and Information Theories.

## 1. Introduction

Let  $\Phi : \mathcal{I} \rightarrow \mathbb{R}$  be a smooth *convex* function defined on a closed interval  $\mathcal{I}$  of  $\mathbb{R}$  not necessarily bounded. Let  $\mu$  be a positive measure on a Borel space  $(\Omega, \mathcal{F})$ . The  $\Phi$ -entropy functional  $\mathbf{Ent}_\mu^\Phi$  is defined on the set of  $\mu$ -integrable functions  $f : (\Omega, \mathcal{F}) \rightarrow (\mathcal{I}, \mathbb{B}(\mathcal{I}))$  by the following formula:

$$\mathbf{Ent}_\mu^\Phi(f) := \int_\Omega \Phi(f) d\mu - \Phi\left(\int_\Omega f d\mu\right).$$

Obviously, such a formula makes sense only when  $\int_\Omega f d\mu \in \mathcal{I}$ , which is always the case when  $\mu$  is a probability measure. Unless otherwise stated, the  $\Phi$ -entropy in the sequel will be always considered for probability measures. One

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has then that

$$(1.1) \quad \mathbf{Ent}_\mu^\Phi(f) = \mathbf{E}_\mu(\Phi(f)) - \Phi(\mathbf{E}_\mu f).$$

In addition, depending on  $\mathcal{I}$ , the integrability condition on  $f$  can be relaxed and Jensen inequality implies that the  $\Phi$ -entropy functional takes its values in  $\mathbb{R}_+ \cup \{+\infty\}$ . Moreover, it is convex with respect to its functional argument *at fixed mean*. As we will see, the global convexity requires more assumptions on  $\Phi$ . If  $f$  is  $\mu$ -a.s. constant,  $\mathbf{Ent}_\mu^\Phi(f)$  vanishes, and the converse is true when  $\Phi$  is strictly convex. Sometimes, we will drop the  $\mu$  subscript in  $\mathbf{Ent}_\mu^\Phi$ . For any random variable  $X : (\Omega', \mathcal{F}') \rightarrow (\Omega, \mathcal{F})$ , we will denote

$$\mathbf{Ent}^\Phi(f(X)) := \mathbf{Ent}_{\mathcal{L}(X)}^\Phi(f).$$

The classical variance and entropy can be recovered since we have

$$(1.2) \quad \mathbf{Ent}_\mu^{x \mapsto x^2} = \mathbf{Var}_\mu \quad \text{and} \quad \mathbf{Ent}_\mu^{x \mapsto x \log x} = \mathbf{Ent}_\mu.$$

Notice that the  $\Phi$ -entropy functional  $f \mapsto \mathbf{Ent}_\mu^\Phi(f)$  is neither homogeneous nor translation invariant in general. Nevertheless, since it is non-negative and vanishes when its argument  $f$  is a constant function, it can be a good candidate as a left hand side of a Sobolev like functional inequality where the right hand side is a Dirichlet form.

Actually, the term “ $\Phi$ -entropy” is quite arbitrarily chosen, since we can speak about “ $\Phi$ -variance” too, but this term is perhaps more adapted to the quantity

$$(1.3) \quad \mathbf{Var}_\mu^\Phi(f) := \mathbf{E}_\mu(\Phi(f - \mathbf{E}_\mu f)),$$

which is translation invariant and gives the classical variance  $\mathbf{Var}_\mu$  when  $\Phi(x) = x^2$ , but the entropy  $\mathbf{Ent}_\mu$  cannot be recovered. One can remember the famous quotation from John Von Neumann about *entropy*, which can be found for example in [ABC<sup>+</sup>00, Chap. 10]. Notice that similar  $\Phi$ -entropies appears with that name in a slightly different forms in a lot of papers related to Information Theory and Convex Analysis fields, see for example [BR82b], [BR82a], [BR82c], [TV93] and [BTT86] and references therein. The  $\Phi$ -entropy is related to the so called  $(h, \Phi)$ -entropies, see for example [MMPS97] and references therein. The  $\Phi$ -entropy is also known as  $J$ -divergence ( $J$  stands for Jensen). See also [Csi63, Csi72] for the similar notion of  $\phi$ -divergence.

Let  $(X_t)_{t \geq 0}$  be a Markov process on a Polish space  $\Omega$  equipped with its Borel  $\sigma$ -field, say for example  $(\mathbb{R}^d, \mathbb{B}(\mathbb{R}^d))$ . We define the classical associated Markov semi-group  $(\mathbf{P}_t)_{t \geq 0}$  acting on  $\mathcal{C}_b(\Omega, \mathbb{R})$  by:

$$(1.4) \quad \mathbf{P}_t(f)(x) := \mathbf{E}_\mu(f(X_t) \mid X_0 = x).$$

Let us assume that there exists an invariant measure  $\mu$ , i.e. a positive Borel measure  $\mu$  on  $\Omega$  stable by  $(\mathbf{P}_t)_{t \geq 0}$ . When  $\mu$  is a *probability*, we get that  $\mathcal{L}(X_0) = \mu$  implies  $\mathcal{L}(X_t) = \mu$  for all  $t \in \mathbb{R}_+$ . We denote by  $\mathbf{L} := \partial_{t=0^+} \mathbf{P}_t$  the

infinitesimal generator of  $(\mathbf{P}_t)_{t \geq 0}$  with domain  $\mathcal{D}(\mathbf{L}) \in L^2(\mu)$ , and by  $\mathbf{\Gamma}$  the associated “carré du champ” operator defined by:

$$(1.5) \quad \mathbf{\Gamma}(g, h) := \frac{1}{2}(\mathbf{L}(gh) - g\mathbf{L}h - h\mathbf{L}g).$$

By convention,  $\mathbf{\Gamma}g := \mathbf{\Gamma}(g, g)$ . For Markov processes,  $\mathbf{\Gamma}g$  is always non-negative. The invariance of a probability measure  $\mu$  is equivalent to

$$\forall f \in \mathcal{D}(\mathbf{L}), \quad \mathbf{E}_\mu(\mathbf{L}f) = 0.$$

We say that a positive measure  $\mu$  is symmetric if and only if  $\mathbf{L}$  is symmetric in  $L^2(\mu)$ , i.e.  $\langle f, \mathbf{L}g \rangle_{L^2(\mu)} = \langle g, \mathbf{L}f \rangle_{L^2(\mu)}$  for all  $f$  and  $g$  in  $\mathcal{D}(\mathbf{L})$ . Measure  $\mu$  is invariant if it is symmetric but the converse is false in general. Symmetric measures lead to an integration by parts formula

$$(1.6) \quad -\langle g, \mathbf{L}h \rangle_{L^2(\mu)} = \langle \mathbf{\Gamma}(g, h) \rangle_{L^2(\mu)} = \langle \mathbf{\Gamma}(h, g) \rangle_{L^2(\mu)}.$$

The reader may find an introduction to the analysis of Markov semi-groups in [Bak94] and [Bak02] for example, where the delicate problem of the existence of an algebra of functions  $\mathcal{A}$  stable by semi-group and generator is addressed. By invariance of  $\mu$  and Jensen inequality for  $\Phi$ :

$$\begin{aligned} \mathbf{Ent}_\mu^\Phi(\mathbf{P}_t f) &= \mathbf{E}_\mu(\Phi(\mathbf{P}_t f)) - \Phi(\mathbf{E}_\mu f) \\ &\leq \mathbf{E}_\mu(\mathbf{P}_t \Phi(f)) - \Phi(\mathbf{E}_\mu f) \\ &= \mathbf{Ent}_\mu^\Phi(f). \end{aligned}$$

On the other hand, if the semi-group is  $L^2$ -ergodic,  $\mathbf{Ent}_\mu^\Phi(\mathbf{P}_t f)$  converges to 0 when  $t$  tends to  $+\infty$ , and we get that:

$$0 = \mathbf{Ent}_\mu^\Phi(\mathbf{P}_\infty f) \leq \mathbf{Ent}_\mu^\Phi(\mathbf{P}_t f) \leq \mathbf{Ent}_\mu^\Phi(\mathbf{P}_0 f) = \mathbf{Ent}_\mu^\Phi(f).$$

Actually, one can show that any  $\Phi$ -entropy related to the invariant measure of a Markov process is non-increasing along the associated Markovian semi-group:

**Proposition 1.1** (DeBruijn like property for Markov semi-groups).

Let  $(X_t)_{t \geq 0}$  be a Markov process on a Polish space  $\Omega$  equipped with its Borel  $\sigma$ -field. Let  $(\mathbf{P}_t)_{t \geq 0}$  be the associated Markov semi-group with infinitesimal generator  $\mathbf{L}$  and “carré du champ”  $\mathbf{\Gamma}$ . Assume that  $\mu$  is an invariant probability measure. Then, for any suitable function  $f : \Omega \rightarrow \mathcal{I}$  and any  $t > 0$ :

$$(1.7) \quad \partial_t \mathbf{Ent}_\mu^\Phi(\mathbf{P}_t f) = \mathbf{E}_\mu(\Phi'(\mathbf{P}_t f) \mathbf{L} \mathbf{P}_t f) \leq 0.$$

When  $\mu$  is symmetric, one has the following formulation

$$(1.8) \quad \partial_t \mathbf{Ent}_\mu^\Phi(\mathbf{P}_t f) = -\mathbf{E}_\mu(\mathbf{\Gamma}(\Phi'(\mathbf{P}_t f), \mathbf{P}_t f)).$$

Moreover, when  $(X_t)_{t \geq 0}$  is a diffusion process, one has:

$$(1.9) \quad \partial_t \mathbf{Ent}_\mu^\Phi(\mathbf{P}_t f) = -\mathbf{E}_\mu(\Phi''(\mathbf{P}_t f) \mathbf{\Gamma} \mathbf{P}_t f).$$

*Proof.* Equality in (1.7) follows immediately from definition of  $\mathbf{Ent}_\mu^\Phi$  and  $\mathbf{L}$ . The symmetric case (1.8) comes by integration by parts (1.6). Let us show that the right hand side of (1.7) is  $\leq 0$ . Jensen inequality for convex function  $\Phi$  and probability measure  $\mathbf{P}_t$  yields  $\Phi(\mathbf{P}_t(g)) \leq \mathbf{P}_t(\Phi(g))$  for any  $t > 0$ , and hence  $\Phi'(g) \mathbf{L}g \leq \mathbf{L}\Phi(g)$ . Thus, by invariance of  $\mu$ , we get that  $\mathbf{E}_\mu(\Phi'(g) \mathbf{L}g) \leq 0$ , which gives the result when  $g = \mathbf{P}_t f$ . Finally, the diffusion case (1.9) comes from the fact that we have then the so called “chain rule formula”

$$(1.10) \quad \Gamma(\alpha(g), h) = \alpha'(g) \Gamma(g, h).$$

Recall that the operator  $\mathbf{L}$  is a diffusion operator if and only if for any  $f_1, \dots, f_k$  in  $\mathcal{D}(\mathbf{L})$  and any smooth function  $\alpha : \mathbb{R}^k \rightarrow \mathbb{R}$  such that  $\alpha(f_1, \dots, f_k) \in \mathcal{D}(\mathbf{L})$ :

$$(1.11) \quad \mathbf{L}(\alpha(f_1, \dots, f_k)) = \sum_{i=1}^k (\partial_i \alpha)(f_1, \dots, f_k) \mathbf{L}f_i + \sum_{1 \leq i, j \leq k} (\partial_{ij}^2 \alpha)(f_1, \dots, f_k) \Gamma(f_i, f_j).$$

We have to mention that the diffusion property makes really sense only in continuous space settings and implies roughly that  $\mathbf{L}$  is a second order linear partial differential operator without constant part, cf. [Bak02, Bak94]. Finally, one can observe that the convexity of  $\Phi$  is needed only in order to give the sign in (1.7): the  $\Phi$ -entropy  $\mathbf{Ent}_\mu^\Phi$  is non-increasing along the Markov semi-group when  $\Phi$  is convex.

Finally, notice that the term  $-\Phi(\mathbf{E}_\mu f)$  in the definition of  $\mathbf{Ent}_\mu^\Phi(f)$  plays no role in the non-increasing property along the semi-group since by invariance of  $\mu$ , one has  $-\Phi(\mathbf{E}_\mu \mathbf{P}_t(f)) = -\Phi(\mathbf{E}_\mu f)$ . Therefore, one can investigate the non-increasing property along the semi-group for generic  $(h, \Phi)$ -entropies defined by  $h(\mathbf{E}_\mu(\Phi(f)))$ .  $\square$

Property (1.7) tells us that any  $\Phi$ -entropy related to the invariant measure on a Markov process is non-increasing along the Markovian semi-group. Actually, an exponential decrease a  $\Phi$ -entropy along the semi-group is equivalent to a functional inequality for  $\mu$ :

**Corollary 1.1** (Exponential decrease of  $\mathbf{Ent}^\Phi$  along a semi-group).

*There is an equivalence between:*

$$(1.12) \quad \exists c \in \mathbb{R}_+^*, \quad \forall f : \Omega \rightarrow \mathcal{I}, \quad \mathbf{Ent}_\mu^\Phi(f) \leq -c \mathbf{E}_\mu(\Phi'(f) \mathbf{L}f),$$

and

$$(1.13) \quad \exists c \in \mathbb{R}_+^*, \quad \forall t \geq 0, \quad \forall f : \Omega \rightarrow \mathcal{I}, \quad \mathbf{Ent}_\mu^\Phi(\mathbf{P}_t f) \leq e^{-t/c} \mathbf{Ent}_\mu^\Phi(f).$$

*Proof.* Obvious from the DeBruijn like property stated in Proposition 1.1 by taking the derivative in  $t$ .  $\square$

Provided that  $\mathcal{I} = \mathbb{R}$ , one can adapt Corollary 1.1 to the “ $\Phi$ -variance” given by (1.3). For Markov processes, the functional  $\mathbf{J}_\mu^\Phi$  defined by:

$$(1.14) \quad \mathbf{J}_\mu^\Phi(f) := - \int_\Omega \Phi'(f) \mathbf{L}f d\mu(x)$$

can be seen as a generalisation of Fisher information, which corresponds to the cases  $\Phi(x) = x \log x$  or  $\Phi(x) = x^2$  (both are equivalent for diffusions due to the chain rule (1.10)). More generally, when  $\mathcal{I} = \mathbb{R}_+$ , one can define the Kullback-Leibler  $\Phi$ -entropy (or relative  $\Phi$ -entropy) and the  $\Phi$ -Fisher functionals for any couple of positive measures  $\mu$  and  $\nu$  by:

$$(1.15) \quad \mathbf{Ent}^\Phi(\nu | \mu) = \begin{cases} \int_\Omega \widehat{\Phi} \left( \frac{d\nu}{d\mu} \right) d\mu & \text{if } \nu \ll \mu \\ +\infty & \text{if not,} \end{cases}$$

where  $\widehat{\Phi}(u) := \Phi(u) - \Phi(1)u$ , and

$$(1.16) \quad \mathbf{J}^\Phi(\nu | \mu) = \begin{cases} - \int_\Omega \Phi' \left( \frac{d\nu}{d\mu} \right) \mathbf{L} \left( \frac{d\nu}{d\mu} \right) d\mu & \text{if } \nu \ll \mu \\ +\infty & \text{if not.} \end{cases}$$

Observe that  $\widehat{\Phi}$  inherits the convexity property from  $\Phi$  and that  $(\widehat{\Phi}(0), \widehat{\Phi}(1)) = (\Phi(0), 0)$  and that  $(\widehat{\Phi})'' = \Phi''$ : one can always “correct” a convex function by an affine additive part in such a way that the new function vanishes at a fixed point. If  $\nu$  is a probability measure with  $d\nu := f d\mu$ , one has

$$\mathbf{Ent}^\Phi(\nu | \mu) = \mathbf{Ent}_\mu^\Phi(f) \quad \text{and} \quad \mathbf{J}^\Phi(\nu | \mu) = \mathbf{J}_\mu^\Phi(f).$$

It would be interesting to study the role played by such functionals in Large Deviation Principles, since the “normal” entropy appears as a rate function, cf. [DS89]. The Kullback-Leibler relative entropy, which corresponds to  $\Phi(x) = x \log x$  appears for example in Sanov Theorem as a particular convex conjugate functional on probability measures spaces. One can hope a sort of  $\Phi$ -Sanov like Theorem involving the  $\Phi$ -relative entropy, and the extremal case of the variance is interesting.

**Definition 1.1** ( $\Phi$ -Sobolev inequalities). In accordance with Corollary 1.1, a probability measure  $\mu$  associated to a Markov process with generator  $\mathbf{L}$  satisfies to a  $\Phi$ -Sobolev inequality of constant  $c \in \mathbb{R}_+$  on the class  $\mathcal{A} \subset \{f : \Omega \rightarrow \mathcal{I}\}$  if and only if

$$(1.17) \quad \forall f \in \mathcal{A}, \mathbf{Ent}_\mu^\Phi(f) \leq c \mathbf{E}_\mu(\mathcal{E}^\Phi(f)),$$

where  $f \mapsto \mathcal{E}^\Phi(f)$  is a non-negative “energy” functional vanishing when  $f$  is constant. The precise choice of such functional will depend on the structure on the involved space  $\Omega$  and measure  $\mu$ . When  $\mathcal{E}^\Phi(f) = -\Phi'(f) \mathbf{L}f$ , the “traditional” logarithmic Sobolev and Poincaré inequalities correspond respectively to  $\Phi(x) = x \log x$  and to  $\Phi(x) = x^2$ . In abstract settings, a  $\Phi$ -Sobolev inequality for  $\mu$  takes the form when  $\mathcal{I} = \mathbb{R}_+$

$$(1.18) \quad \forall \nu \ll \mu, \mathbf{Ent}^\Phi(\nu | \mu) \leq c \mathbf{J}^\Phi(\nu | \mu).$$

Beware that (1.17) gives (1.18) but the converse seems to be true only when  $\Phi(x) = x \log x$  (nevertheless, it is always true for densities  $f$  with respect

to  $\mu$ ). Here again, one can prefer the name “ $\Phi$ -Poincaré” inequalities, but it seems to be an illogical choice since log-Sobolev inequalities are not called “log-Poincaré”! “Our”  $\Phi$ -Sobolev inequalities are close in form to the “Orlicz-Poincaré” inequalities mentioned in [GZ03, p. 125–126]. The interested reader may find an extensive study of similar but quite different inequalities involving  $\Phi(x) = |x|^p$  with  $p \geq 1$  in [BZ02] and [BCR04].

Notice that the two classical  $\Phi$ -entropies in (1.2) share the same special property:  $1/\Phi''$  is affine, i.e. both convex and concave in the same time. Hence, if one try to generalise these two cases, one can assume that  $1/\Phi''$  is convex. As we will see, it turns out that it is not a good choice in view of deriving coercive inequalities like Poincaré or log-Sobolev inequalities, for which the concavity of  $1/\Phi''$  is needed. Such a condition can be found for example in the unrelated works [LO00], [Hu00] and [BR82a]. Nevertheless, convexity may gives inverse forms of such inequalities. Here are some possible *additional* assumptions on  $\Phi$ :

- ( $\mathcal{H}1$ )  $(u, v) \mapsto \Phi''(u) v^2$  is non-negative and convex on  $\mathcal{I} \times \mathcal{I}$ ;
  - ( $\mathcal{H}2$ )  $(u, v) \mapsto \Phi(u + v) - \Phi(u) - \Phi'(u) v$  is non negative and convex on  $\mathcal{I}^{(2)}$ ;
  - ( $\mathcal{H}2'$ )  $\Phi''$  is convex, non-negative and non-increasing on  $\mathcal{I}$ ;
- where  $\mathcal{I}^{(2)} := \{(u, v) \in \mathbb{R}^2, (u, u + v) \in \mathcal{I} \times \mathcal{I}\}$ . For convenience, we denote by  $\Psi$  the real valued function defined on  $\mathcal{I}^{(2)}$  by

$$(1.19) \quad \Psi(u, v) := \Phi(u + v) - \Phi(u) - \Phi'(u) v.$$

Notice that ( $\mathcal{H}1$ ) is equivalent to the convexity of  $\Phi$  and  $\Phi''$  and  $-1/\Phi''$ , see Remark 11 page 355. On the other hand, ( $\mathcal{H}2'$ ) implies ( $\mathcal{H}2$ ), and ( $\mathcal{H}2'$ ) is equivalent to state that  $\Phi$ ,  $\Phi''$  and  $-\Phi'$  are convex. Basic examples for both ( $\mathcal{H}1$ ) and ( $\mathcal{H}2$ ) and ( $\mathcal{H}2'$ ) are given by

- $\Phi(x) = x \log x$  on  $\mathcal{I} = \mathbb{R}_+$ ;
- $\Phi(x) = x^p$  with  $1 < p < 2$  on  $\mathcal{I} = \mathbb{R}_+$ ;
- $\Phi(x) = x^2$  on  $\mathcal{I} = \mathbb{R}$ .

In such examples, the associated  $\Phi$ -entropy  $\mathbf{Ent}_\mu^\Phi$  is homogeneous. Hypothesis ( $\mathcal{H}1$ ) is suitable in continuous settings, whereas hypothesis ( $\mathcal{H}2$ ) and ( $\mathcal{H}2'$ ) are useful in discontinuous ones. The bivariate convexity of  $(u, v) \mapsto \Phi''(u) v^2$  under ( $\mathcal{H}1$ ) and of  $\Psi$  under ( $\mathcal{H}2$ ) is the key property to derive  $\Phi$ -Sobolev inequalities like in (1.17). Such functional inequalities may be investigated in many situations involving Markov processes:

Time – Space	Continuous space	Discontinuous space
Continuous time	Diffusions processes	Poisson space and Lévy processes
Discrete time	Discrete Markov chains and martingales	

Of course, Lévy processes are martingales, but it is not the case for all diffusions. The discrete time case is not really addressed in this work, but one can consider Random Walks or Bernoulli processes, and some answers can be found for example in [GP03] and references therein. For each case, one may be interested in inequalities on the whole paths space or more simply in the law at fixed

time (infinite for invariant measure by ergodicity). There is a lack of chain rule formula (1.10) for  $\Gamma$  operator in discrete space settings. However, i.i.d. increments can then help to recover a Brownian Motion like behaviour. In some cases, inequalities at fixed time can be tensorised to give multi-time inequalities which appears as inequalities for marginals of the paths space measure. A limiting procedure can be then used to recover inequalities on paths space (it is the so called cylindrical method). Roughly, we use here two types of proofs in order to extract  $\Phi$ -Sobolev inequalities from “Markovianity”. The first kind makes use of the semi-group and the second kind, more powerful, is based on a martingale representation and gives directly results on paths spaces. Finally, it is probably possible to establish  $\Phi$ -Sobolev inequalities under  $(\mathcal{H}1)$  on some loop spaces, as what is done in [GM98b] and [GM98a] for the logarithmic Sobolev inequality.

The DeBruijn like property stated in proposition 1.1 can be seen as a sort of special case of Gâteaux directional derivative of  $\Phi$ -entropy like functionals, see for example [BR82a], where the  $\Phi$ -entropy is called  $J$ -Divergence. Actually, one can state the following Proposition:

**Proposition 1.2.** *Assume that  $\mu$  is a probability measure and that  $\Phi$  fulfils  $(\mathcal{H}1)$ . Let  $L^{1,\Phi}(\mu)$  be the set of measurable functions  $f : (\Omega, \mathcal{F}) \rightarrow \mathcal{I}$  such that  $f \in L^1(\mu)$  and  $\Phi(f) \in L^1(\mu)$ . Then  $L^{1,\Phi}(\mu)$  is convex, the  $\Phi$ -entropy functional  $f \in L^{1,\Phi}(\mu) \mapsto \mathbf{Ent}_\mu^\Phi(f)$  is convex, and for any  $f \in L^{1,\Phi}(\mu)$ , one has the following duality formula:*

$$(1.20) \quad \mathbf{Ent}_\mu^\Phi(f) = \sup_{h \in L^{1,\Phi}(\mu)} \left\{ \mathbf{E}_\mu((\Phi'(h) - \Phi'(\mathbf{E}_\mu h))(f - h)) + \mathbf{Ent}_\mu^\Phi(h) \right\}.$$

*Proof.* For any couple  $f, g$  in  $L^{1,\Phi}(\mu)$  and any  $t \in [0, 1]$ , one has

$$h_t := tf + (1 - t)g \in L^1(\mu),$$

and  $\Phi(h_t) \in L^1(\mu)$  by convexity of  $\Phi$ . Thus  $h_t \in L^{1,\Phi}(\mu)$  and  $L^{1,\Phi}(\mu)$  is convex. Consider now the real function  $\alpha : [0, 1] \rightarrow \mathbb{R}_+$  defined by  $\alpha(t) := \mathbf{Ent}_\mu^\Phi(h_t)$  and let us show that it is convex. Assume first that  $f$  and  $g$  are bounded. An easy computation gives

$$\alpha'(t) = \mathbf{E}_\mu(\Phi'(h_t)(f - g)) - \Phi'(\mathbf{E}_\mu(h_t))(\mathbf{E}_\mu(f - g)),$$

and

$$\alpha''(t) = \mathbf{E}_\mu(\Phi''(h_t)(f - g)^2) - \Phi''(\mathbf{E}_\mu(h_t))(\mathbf{E}_\mu(f - g))^2,$$

which is non-negative by virtue of Jensen inequality for the bivariate function  $(u, v) \mapsto \Phi''(u)v^2$ . Alternatively, one can use the concavity of  $1/\Phi''$  and Cauchy-Schwarz inequality. Therefore,  $\alpha$  is continuous on  $[0, 1]$  and convex on  $(0, 1)$ . In other words, for any  $s, t, \lambda \in [0, 1]$ ,

$$\alpha(\lambda s + (1 - \lambda)t) \leq \lambda\alpha(s) + (1 - \lambda)\alpha(t).$$

At this stage, the dominated convergence Theorem allows us to drop the boundedness assumption on  $f$  and  $g$ . In particular,  $\alpha(\lambda) \leq \lambda\alpha(1) + (1-\lambda)\alpha(0)$  can be written

$$\mathbf{Ent}_\mu^\Phi(\lambda f + (1-\lambda)g) \leq \lambda \mathbf{Ent}_\mu^\Phi(f) + (1-\lambda) \mathbf{Ent}_\mu^\Phi(g),$$

and the convexity of  $f \in L^{1,\Phi}(\mu) \mapsto \mathbf{Ent}_\mu^\Phi(f)$  is established. Notice that the convexity of  $\Phi$  implies that the expression of  $\alpha'$  is valid for any  $f$  and  $g$  in  $L^{1,\Phi}(\mu)$ . Since every convex function is the envelope of its tangents, cf. [RW98, HUL01], one gets the following variational formula:

$$\mathbf{Ent}_\mu^\Phi(f) = \alpha(1) = \sup_{t \in [0,1]} \{\alpha(t) + \alpha'(t)(1-t)\},$$

which can be rewritten as (1.20) since for any fixed  $f$  in  $L^{1,\Phi}(\mu)$ ,

$$L^{1,\Phi}(\mu) = \{tf + (1-t)g, \text{ where } g \in L^{1,\Phi}(\mu) \text{ and } t \in [0,1]\}.$$

Notice that the value of the sup in (1.20) is achieved for  $h = f$ .  $\square$

The convexity of  $\Phi$ -entropy like functionals is well known, see for example [BR82c], [BR82a, Thm. 2] and [LO00, Lem. 4]. One can show that this convexity is in some sense equivalent to  $(\mathcal{H}1)$  via well chosen  $f$  and  $h$  functions in (1.20). Formally, the Fréchet derivatives of this functional are given by

$$\left(D\mathbf{Ent}_\mu^\Phi\right)(f)(h) = \mathbf{E}_\mu([\Phi'(f) - \Phi'(\mathbf{E}_\mu f)]h),$$

and

$$\left(D^2\mathbf{Ent}_\mu^\Phi\right)(f)(h, h) = \mathbf{E}_\mu(\Phi''(f)h^2) - \Phi''(\mathbf{E}_\mu f)(\mathbf{E}_\mu h)^2.$$

We recover the well known formulas for variance and entropy by considering the appropriate  $\Phi$ :

$$(1.21) \quad \mathbf{Var}_\mu(f) = \sup_h \{2 \mathbf{Cov}_\mu(f, h) - \mathbf{Var}_\mu(h)\},$$

and

$$(1.22) \quad \mathbf{Ent}_\mu(f) = \sup_h \{\mathbf{E}_\mu(f \log h) - \mathbf{E}_\mu(f) \log \mathbf{E}_\mu(h)\}.$$

The duality formula (1.20) for  $\Phi$ -entropies can be found in Convex Analysis and Information Theory literature, at least for discrete probability measures, see for example [BTT86]. It is quite straightforward via the convexity of  $\Phi$ -entropies under  $(\mathcal{H}1)$ . We have to mention that Pascal Massart & al gave recently an elementary proof of this variational formula, cf. [BBLM04, Mas03]. See [BR82a] and [MMPS97] for further developments and a Bayesian and Riemannian Geometry point of view.



**Remark 1** (Bi-convexity of relative  $\Phi$ -entropy). Assume that  $\mathcal{I} = \mathbb{R}_+$  and let  $(\Omega, \mathcal{F})$  be a measurable space equipped by a positive Borel measure  $\mu$ . Let  $\alpha$  and  $\beta$  be two probability measures on  $(\Omega, \mathcal{F})$ , absolutely continuous with respect to  $\mu$ , with densities  $p$  and  $q$  respectively. Then, the relative  $\Phi$ -entropy  $\mathbf{Ent}^\Phi(\beta | \alpha)$  defined in (1.15) is given by

$$\mathbf{Ent}_\mu^\Phi(q; p) := \int_\Omega \widehat{\Phi}\left(\frac{q}{p}\right) p \, d\mu.$$

An easy computation shows that the convexity of  $\Phi$  induces the bi-convexity of the bivariate function  $(u, v) \in \mathbb{R}_+^2 \mapsto \widehat{\Phi}(v/u) u$ . Thus  $(p, q) \mapsto \mathbf{Ent}_\mu^\Phi(q; p)$  is also bivariate convex. Notice that at fixed  $f$ , the  $\Phi$ -entropy itself  $\mu \mapsto \mathbf{Ent}_\mu^\Phi(f)$  is concave!

**Remark 2** (Convexity and nullity for constants). One can be surprised by the condition  $(\mathcal{H}1)$  requested for the convexity of the  $\Phi$ -entropy functional  $f \mapsto \mathbf{Ent}_\mu^\Phi(f)$ . Actually, it is due to the presence of the  $-\Phi(\mathbf{E}_\mu f)$  term in the definition of  $\mathbf{Ent}_\mu^\Phi$ . If one removes this term, the resulting functional  $f \mapsto \mathbf{E}_\mu(\Phi(f))$  is always convex since  $\Phi$  is convex and  $(\mathcal{H}1)$  is not needed. Nevertheless, in that case, the functional does not vanishes when its argument  $f$  is constant, and thus it becomes useless as the left hand side of a Sobolev like functional inequality.

## 2. Phi-Sobolev for diffusions and log-concave measures

Proposition 1.1 translates the fact that the derivative in time of the  $\Phi$ -entropy along a Markovian semi-group can be expressed in terms of the derivatives in space  $\Gamma$ , i.e. Fisher information. As we will see, for diffusions, a commutation formula between the semi-group and the derivatives in space yields finally local  $\Phi$ -Sobolev inequalities. We learnt the following Theorem for hypercontractive diffusions from [Hu00, Sect. 3], where it is stated in a slightly different manner, see also [Bak02].

**Theorem 2.1** ( $\Phi$ -Sobolev inequality for diffusions). *Let  $(\mathcal{M}, g)$  be a connected complete Riemannian manifold and let  $(X_t)_{t \geq 0}$  be a diffusion process on  $\mathcal{M}$  with symmetric invariant positive measure  $\mu$  absolutely continuous with respect to the Riemannian volume measure. If a  $\text{CD}(\rho, \infty)$  criterion is satisfied with  $\rho \geq 0$  (cf. (2.5), Section 2.1, page 335), then under  $(\mathcal{H}1)$ , one has:*

(1) *For any smooth function  $f : \mathcal{M} \rightarrow \mathcal{I}$  and any  $t \in \mathbb{R}_+$ :*

$$(2.1) \quad \mathbf{Ent}_{\mathbf{P}_t}^\Phi(f) \leq \frac{1 - e^{-2\rho t}}{2\rho} \mathbf{P}_t(\Phi''(f) \Gamma f),$$

where  $(\mathbf{P}_t)_{t \geq 0}$  and  $\Gamma$  are as in (1.4) and (1.5) and where the constant is  $t/2$  when  $\rho = 0$ .

(2) *If  $\rho > 0$  and  $\mu$  is a probability measure and  $(X_t)_{t \geq 0}$  is  $L^2$ -ergodic then, for any smooth function  $f : \mathcal{M} \rightarrow \mathcal{I}$ :*

$$(2.2) \quad \mathbf{Ent}_\mu^\Phi(f) \leq \frac{1}{2\rho} \mathbf{E}_\mu(\Phi''(f) \Gamma f).$$

*Proof.* The second part may be deduced from the first one via ergodicity by letting  $t$  tends to  $+\infty$ . Let us give a direct proof. Let  $\mathbf{L}$  be the infinitesimal generator of the diffusion semi-group. By the ergodic property and Fubini Theorem, it follows that

$$\begin{aligned} \mathbf{Ent}_\mu^\Phi(f) &:= \mathbf{E}_\mu(\Phi(f)) - \Phi(\mathbf{E}_\mu f) \\ &= \mathbf{E}_\mu(\Phi(\mathbf{P}_0 f)) - \mathbf{E}_\mu(\Phi(\mathbf{P}_\infty f)) \\ &= - \int_0^\infty dt \mathbf{E}_\mu(\partial_t \Phi(\mathbf{P}_t f)) \\ &= - \int_0^\infty dt \mathbf{E}_\mu(\Phi'(\mathbf{P}_t f) \mathbf{L} \mathbf{P}_t f) \\ &= + \int_0^\infty dt \mathbf{E}_\mu(\Phi''(\mathbf{P}_t f) \mathbf{\Gamma} \mathbf{P}_t f), \end{aligned}$$

where the last equality is obtained by integration by parts (1.6) which is a consequence of the symmetry of  $\mu$  for  $\mathbf{L}$ . Now, by the diffusion property, the  $\text{CD}(\rho, \infty)$  criterion is equivalent to the following commutation formula (cf. Section 2.1 page 335):

$$(2.3) \quad \sqrt{\mathbf{\Gamma} \mathbf{P}_t f} \leq e^{-\rho t} \mathbf{P}_t \sqrt{\mathbf{\Gamma} f}.$$

Therefore,

$$\Phi''(\mathbf{P}_t f) \mathbf{\Gamma} \mathbf{P}_t f \leq \exp(-2\rho t) \Phi''(\mathbf{P}_t f) \left( \mathbf{P}_t \sqrt{\mathbf{\Gamma} f} \right)^2,$$

and then by Jensen inequality with the bivariate function  $\Phi''(u) v^2$  which is convex under  $(\mathcal{H}1)$ :

$$\Phi''(\mathbf{P}_t(f)) \mathbf{P}_t \left( \sqrt{\mathbf{\Gamma} f} \right)^2 \leq \mathbf{P}_t(\Phi''(f) \mathbf{\Gamma} f).$$

Alternatively, one can use the concavity of  $1/\Phi''$  and the Cauchy-Schwarz inequality. The desired results follows immediately from the invariance of  $\mu$ :  $\mathbf{E}_\mu(\mathbf{P}_t f) = \mathbf{E}_\mu(f)$ . The proof of the first part (2.1) is quite similar. One just has to replace  $\mu$  by  $\mathbf{P}_s$ , and  $\mathbf{P}_t$  by  $\mathbf{P}_{t-s}$ . The integration by parts and the chain rule leading to the expression in  $\Phi''$  must be replaced by the following, relying on the diffusion property of  $\mathbf{L}$ :

$$\begin{aligned} \partial_s \mathbf{P}_s(\Phi(\mathbf{P}_{t-s}(f))) &:= \mathbf{P}_s(\mathbf{L}\Phi(\mathbf{P}_{t-s}(f)) - \Phi'(\mathbf{P}_{t-s}(f)) \mathbf{L} \mathbf{P}_{t-s}(f)) \\ &= \mathbf{P}_s(\Phi''(\mathbf{P}_{t-s}(f)) \mathbf{\Gamma} \mathbf{P}_{t-s}(f)), \end{aligned}$$

where  $0 < s \leq t$ . Notice that the diffusion property is used directly here, which was not the case for the non local inequality, for which it was used via the strong commutation property given by the  $\text{CD}(\rho, \infty)$  criterion. As we will see, there is a lack of the diffusion property in discontinuous space settings, and one has to replace  $(u, v) \mapsto \Phi''(u) v^2$  by  $\Psi$  which is convex under  $(\mathcal{H}2)$ .  $\square$

See [Hu00] for Gross like hypercontractivity, F.K.G. inequalities and related aspects of  $\Phi$ -entropies for diffusion semi-groups. We have to mention that the diffusion property leads to a more general result which contains F.K.G. and  $\Phi$ -Sobolev inequalities as sub-cases, as explained in [Hu00, Thm. 4.2]. Namely, it states that if  $\Phi_1, \Phi_2 : \mathcal{I} \subset \mathbb{R} \rightarrow \mathbb{R}$  are two smooth functions such that a particular  $5 \times 5$  matrix involving the derivatives of  $\Phi_1$  and  $\Phi_2$  is positive definite, then for any smooth  $f$  and  $g$ :

$$\begin{aligned} \mathbf{Ent}_\mu^{\Phi_1, \Phi_2}(f, g) \leq & \mathbf{E}_\mu(\Phi_1''(f) \Phi_2(g) \Gamma f) \\ & + \mathbf{E}_\mu(\Phi_1(f) \Phi_2''(g) \Gamma g) \\ & + 2 \mathbf{E}_\mu(\Phi_1'(f) \Phi_2'(g) \Gamma(f, g)), \end{aligned}$$

where the quantity

$$\mathbf{Ent}_\mu^{\Phi_1, \Phi_2}(f, g) := \mathbf{E}_\mu(\Phi_1(f) \Phi_2(g)) - \Phi_1(\mathbf{E}_\mu(f)) \Phi_2(\mathbf{E}_\mu(g))$$

is a sort of  $(\Phi_1, \Phi_2)$ -covariance. One has in particular

$$\mathbf{Ent}_\mu^{x \mapsto x, x \mapsto x} = \mathbf{Cov}_\mu \quad \text{and} \quad \mathbf{Ent}_\mu^{x \mapsto 1, x \mapsto \Phi(x)} = \mathbf{Ent}_\mu^\Phi.$$

**Remark 3** (Non-negativity of functions). It is possible in some cases to reduce the analysis to non-negative functions. Namely, since  $\mathbf{P}_t(|f|) \geq |\mathbf{P}_t(f)|$ , one can write:

$$\begin{aligned} \mathbf{E}_\mu(f \mathbf{L}f) &= \lim_{t \rightarrow 0^+} \frac{1}{2t} \mathbf{E}_\mu(f \mathbf{P}_t(f) - f^2) \\ &\leq \lim_{t \rightarrow 0^+} \frac{1}{2t} \mathbf{E}_\mu(|f| \mathbf{P}_t(|f|) - |f|^2) \\ &= \mathbf{E}_\mu(|f| \mathbf{L}|f|), \end{aligned}$$

which gives  $\mathbf{E}_\mu(\Gamma|f|) \leq \mathbf{E}_\mu(\Gamma f)$ . More generally, one can show by the same way that  $\mathbf{E}_\mu(\Phi''(|f|) \Gamma|f|) \leq \mathbf{E}_\mu(\Phi''(|f|) \Gamma f)$ . Moreover, one can assume that  $f \geq \varepsilon > 0$  by Fatou Lemma.

### 2.1. Geometry and log-concavity

As we have seen in Theorem 2.1, a natural framework for  $\Phi$ -Sobolev inequalities related to diffusions is Riemannian manifolds. Let  $\mathbf{L}$  be a Markov generator with symmetric positive measure  $d\mu = \exp(U) dv_g$  on a complete connected Riemannian manifold  $(\mathcal{M}, g)$  equipped with its volume measure  $v_g$ . Function  $U : \mathcal{M} \rightarrow \mathbb{R}$  is taken smooth. A basic example is given by the Laplace-Beltrami operator  $\mathbf{L} = \Delta_g + \nabla U$  with vector field  $\nabla U$ . Back to the generic case of a Markov generator  $\mathbf{L}$  on  $(\mathcal{M}, g)$ , let us define the iterated functional quadratic forms  $\Gamma f := \Gamma(f, f)$  as is (1.5) and  $\mathbf{I}_2 f := \mathbf{I}_2(f, f)$  by

$$(2.4) \quad \mathbf{I}_2(f, g) := \frac{1}{2} (\Gamma(f, g) - \Gamma(f, \mathbf{L}g) - \Gamma(g, \mathbf{L}f)).$$

Then, for any  $(\rho, n) \in \mathbb{R} \times \mathbb{N}$ , the  $\text{CD}(\rho, n)$  (or Bakry-Emery  $\mathbf{I}_2$ ) criterion can be expressed as:

$$(2.5) \quad \mathbf{I}_2 \geq \rho \Gamma + \frac{1}{n} (\mathbf{L})^2,$$

where  $(\mathbf{L})^2 f := (\mathbf{L}f)^2$ . For diffusions generators  $\mathbf{L}$ ,  $\text{CD}(\rho, \infty)$  is equivalent to the strong commutation formula (2.3) between  $\sqrt{\Gamma}$  and  $\mathbf{P}_t$ , which is a direct consequence a Bochner formula<sup>\*1</sup>:

$$\mathbf{I}_2(f, f) = (\text{Ric}_g - \nabla^2 U)(\nabla f, \nabla f) + \|\text{Hess}f\|_2^2,$$

see for example [Bak97, Bak94], [Led00] and [ABC<sup>+</sup>00, Chap. 5]. Notice that with  $n = \infty$ , (2.5) appears as the infinitesimal form of (2.3), via the fact that:

$$\mathbf{P}_t f = f + t \mathbf{L}f + \frac{1}{2} t^2 \mathbf{L}\mathbf{L}f + o(t^2),$$

cf. [ABC<sup>+</sup>00, Sect. 5.4]. In absence of the diffusion property, the Markov semi-group verifies the following weaker commutation formula under  $\text{CD}(\rho, \infty)$ :

$$(2.6) \quad \Gamma \mathbf{P}_t f \leq e^{-2\rho t} \mathbf{P}_t \Gamma f.$$

This commutation formula leads to a  $\Phi$ -Sobolev inequality when  $(u, v) \mapsto \Phi''(u)v$  is convex on  $\mathcal{I} \times \mathbb{R}_+$ , which is the case if and only if  $\Phi''' = 0$ , i.e.  $\Phi(u) = au^2 + bu + c$  with  $(a, b, c) \in \mathbb{R}_+ \times \mathbb{R}^2$  (i.e. Poincaré inequality).

Let us consider the “simple” example where  $(\mathcal{M}, g)$  is the standard flat Euclidean space  $(\mathbb{R}^d, \mathbf{I}_d)$  and where  $\mathbf{L} = \Delta - \nabla W \cdot \nabla$ . The symmetric invariant measure  $\mu$  is then given by  $d\mu(x) = \exp(-W(x)) dx$  and can be obviously normalised as a probability measure if and only if  $\exp(-W) \in L^1(\mathbb{R}^d, \mathbb{R}, dx)$ . One has  $\Gamma f = |\nabla f|^2$  and  $\text{Ric}_g \equiv 0$  and  $\mathbf{I}_2 f = \|\nabla^2 f\|_2^2 + \nabla f \cdot \nabla^2 W \nabla f$ . Now, if there exists a real number  $\rho \geq 0$  such that

$$\forall x \in \mathbb{R}^d, \nabla^2 W(x) \geq \rho \mathbf{I}_d$$

as quadratic forms on  $\mathbb{R}^d$ , then  $\mathbf{L}$  satisfies  $\text{CD}(\rho, \infty)$ . Moreover, if  $\rho > 0$ , measure  $\mu$  is uniformly strictly log-concave and can be normalised as a probability measure. Obviously, measure  $\mu$  can be finite without being uniformly strictly log-concave.

**Corollary 2.1** ( $\Phi$ -Sobolev for unif. strictly log-concave densities). *Let  $\mu$  be a probability measure on  $\mathbb{R}^d$  absolutely continuous with respect to Lebesgue measure  $dx$  with smooth density. Assume that  $\mu$  is uniformly strictly log-concave, i.e. that there exists  $\rho > 0$  such that  $d\mu(x) = \exp(-H(x)) dx$  where  $H : \mathbb{R}^d \rightarrow \mathbb{R}$  is smooth and  $\langle \text{Hess}(H)(x)y, y \rangle \geq \rho \|y\|_2^2$  for any  $x$  and  $y$  in  $\mathbb{R}^d$ . Then under  $(\mathcal{H}1)$  and for any smooth function  $f : \mathbb{R}^d \rightarrow \mathcal{I}$ :*

$$(2.7) \quad \text{Ent}_\mu^\Phi(f) \leq \frac{1}{2\rho} \mathbf{E}_\mu \left( \Phi''(f) |\nabla f|^2 \right).$$

*In particular, when  $\mu = \mathcal{N}(m, \Sigma)$  is a Gaussian measure of mean vector  $m \in \mathbb{R}^d$  and positive definite covariance matrix  $\Sigma \in \text{Sym}_d^{+*}(\mathbb{R})$ , one has*

$$\rho^{-1} = \max(\sigma(\Sigma))$$

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<sup>\*1</sup>Actually Bochner-Lichnerowicz-Weitzenboch.

where  $\sigma(\Sigma)$  is the spectrum of  $\Sigma$ . Moreover, the following Brascamp-Lieb type inequality holds:

$$(2.8) \quad \mathbf{Ent}_{\mathcal{N}(m,\Sigma)}^\Phi(f) \leq \frac{1}{2} \mathbf{E}_{\mathcal{N}(m,\Sigma)}(\Phi''(f) \langle \Sigma \nabla f; \nabla f \rangle),$$

and inequality (2.8) remains valid when  $\Sigma$  is singular.

Notice that (2.7) is obtained by the comparison  $\text{Hess}(H) \geq \rho \mathbf{I}_d$  to the standard quadratic form  $\langle \cdot; \cdot \rangle = \|\cdot\|_2^2$  corresponding to the diagonal covariance matrix  $\mathbf{I}_d$ . Such a comparison is essentially unidimensional and imposes a geometric information loss. Recall that the Brascamp-Lieb inequality for the strictly log-concave probability measure  $\mu$  on  $\mathbb{R}^d$  reads

$$\forall f : \mathbb{R}^d \rightarrow \mathbb{R} \text{ smooth, } \mathbf{Var}_\mu(f) \leq \mathbf{E}_\mu(\langle (\text{Hess}(H))^{-1} \nabla f; \nabla f \rangle),$$

where  $d\mu(x) = \exp(-H(x)) dx$ . When  $\mu = \mathcal{N}(m, \Sigma)$  with non singular  $\Sigma$ , one has  $\text{Hess}(H) = \Sigma^{-1}$ . Unfortunately, to our knowledge, and except for Gaussian measures, Brascamp-Lieb forms of  $\Phi$ -Sobolev inequalities are available only for Poincaré inequalities, i.e. when  $\Phi(u) = u^2$ , cf. [BL00] for a counter example in the logarithmic Sobolev case. However, if  $\mu$  satisfies to a  $\Phi$ -Sobolev inequality on  $\mathbb{R}^d$  with constant  $c$  and right hand side  $\mathbf{E}_\mu(\Phi''(f) |\nabla f|^2)$ , one can ask about some “optimal” deterministic symmetric  $d \times d$  matrices  $S$  such that

$$\mathbf{Ent}_\mu^\Phi(f) \leq c \mathbf{E}_\mu(\Phi''(f) \langle S \nabla f; \nabla f \rangle).$$

Back to the case where  $\mathbf{L} = \mathbf{\Delta} - \nabla W \cdot \nabla$  on  $\mathbb{R}^d$ , an interesting “degenerated” case is given by  $W(x) = \|x\|_r^r$  with  $r \in \{0\} \cup (1, 2) \cup (2, +\infty)$  since  $\nabla^2 W(x)$  is then singular at  $x = 0$ . One has then only  $\text{CD}(0, \infty)$ , inducing a local version only:

$$(2.9) \quad \mathbf{Ent}_{\mathbf{P}_t}^\Phi(f) \leq t \mathbf{P}_t(\Phi''(f) |\nabla f|^2),$$

which is the Brownian Motion or (heat semi-group) behaviour corresponding to the case  $r = 0$  and  $d\mu(x) = dx$ . One can expect a better inequality in term of  $t$ -dependence of the obtained constant, but the semi-group method seems to fail since it relies on a commutation formula (2.3) which is poor when  $r \notin \{0, 2\}$ . A perturbative approach (cf. Proposition 3.2 page 345) may be used for the inequalities related to  $\mu$  when  $r > 2$ , which corresponds to  $t = +\infty$ , but the constant is not sharp in general (cf. [ABC<sup>+</sup>00, Chap. 6]).

When  $\nabla^2 W$  is constant,  $(X_t)_{t \geq 0}$  is a Brownian Motion or an Ornstein-Uhlenbeck process, and the commutation formula (2.3) is exact, i.e. an equality. One can try to investigate the commutation between  $f \mapsto \Phi''(f) \mathbf{\Gamma} f$  and  $\mathbf{P}_t$ , with a cost related to a sort of convex conjugate  $\Phi^*$  of  $\Phi$ . The idea is to relate  $\Phi$  with a criterion involving  $\mathbf{\Gamma}$  and  $\mathbf{I}_2$  (and thus  $W$  when  $\mathbf{L} = \mathbf{\Delta} - \nabla W \cdot \nabla$ ).

**2.2. Interpolation between Poincaré and logarithmic Sobolev**

For simplicity and notational convenience, we restrict ourselves here to  $\mathbb{R}^d$ . Assume that  $\Phi : \mathcal{I} \rightarrow \mathbb{R}$  is convex and smooth and that the probability measure  $\mu$  on  $\mathbb{R}^d$  satisfies to a  $\Phi$ -Sobolev inequality of the form:

$$\exists c > 0, \quad \forall f \in \mathcal{C}_b^\infty(\mathbb{R}^d, \mathcal{I}), \quad \mathbf{Ent}_\mu^\Phi(f) \leq c \mathbf{E}_\mu \left( \Phi''(f) |\nabla f|^2 \right).$$

Now, for any  $g \in \mathcal{C}_b^\infty(\mathbb{R}^d, \mathcal{I})$ , let us consider  $a \in \overset{\circ}{\mathcal{I}}$  such that  $\Phi''(a) > 0$  and  $f_\varepsilon := a + \varepsilon g$  with  $\varepsilon > 0$ . Then,  $f_\varepsilon \in \mathcal{C}_b^\infty(\mathbb{R}^d, \mathcal{I})$  for  $\varepsilon$  sufficiently small and by Taylor formula for  $\Phi$  and the  $\Phi$ -Sobolev inequality above for  $f_\varepsilon$ , one gets when  $\varepsilon$  goes to  $0^+$ :

$$\mathbf{Var}_\mu(g) \leq 2c \mathbf{E}_\mu \left( |\nabla g|^2 \right).$$

Therefore, Poincaré inequality can be seen in a sense as the weakest  $\Phi$ -Sobolev inequality since it is implied by any  $\Phi$ -Sobolev inequality, at least on  $\mathcal{C}_b^\infty(\mathbb{R}^d, \mathcal{I})$ . Moreover, let us assume that  $\mathbb{R}_+ \subset \mathcal{I}$  and let us take now  $f_\varepsilon = a + \varepsilon g^{p/2}$  where  $g$  is positive and  $p \in (1, 2]$ , then we get by the same way:

$$\mathbf{E}_\mu(g^p) - \mathbf{E}_\mu \left( g^{p/2} \right)^2 \leq 2cp^2 \mathbf{E}_\mu \left( |\nabla g|^2 g^{p-2} \right).$$

Now, since  $p \in (1, 2]$ , Jensen inequality yields  $\mathbf{E}_\mu \left( g^{p/2} \right)^2 \leq \mathbf{E}_\mu(g)^p$ , and therefore:

$$\mathbf{E}_\mu(g^p) - \mathbf{E}_\mu(g)^p \leq 2c \frac{p}{(p-1)} \mathbf{E}_\mu \left( p(p-1) g^{p-2} |\nabla g|^2 \right),$$

which is a  $\Phi$ -Sobolev inequality with  $\Phi(x) = x^p$  on  $\mathcal{I} = \mathbb{R}_+$ . However, the constant does not give the sharp one obtained in Corollary 2.1 when  $\mu$  is log-concave, and there is no way to recover a logarithmic Sobolev inequality by letting  $p$  tends to  $1^+$  like in (2.13) since  $2cp/(p-1)^2$  blows up near  $p = 1$ . This fact is not surprising since one can start from a Poincaré inequality in the latter, which is known to be strictly weaker than logarithmic Sobolev inequality (the simplest counter-example is given by the exponential probability law on the real line, see also [ABC<sup>+</sup>00, Chap. 6]). Recall that the  $\Phi$ -Sobolev inequality obtained in Corollary 2.1 for log-concave probability measures include the optimal Poincaré and logarithmic Sobolev inequalities by considering the appropriate  $\Phi$ , namely:

$$(2.10) \quad \forall f \in \mathbf{W}^{1,\infty}(\mathbb{R}^d, \mathbb{R}_+^*), \quad \mathbf{Ent}_\mu(f) \leq \frac{1}{2\rho} \mathbf{E}_\mu \left( \frac{|\nabla f|^2}{f} \right),$$

and

$$(2.11) \quad \forall f \in \mathbf{W}^{1,\infty}(\mathbb{R}^d, \mathbb{R}), \quad \mathbf{Var}_\mu(f) \leq \frac{1}{\rho} \mathbf{E}_\mu \left( |\nabla f|^2 \right),$$

and more generally, for any  $p \in (1, 2]$ :

$$(2.12) \quad \forall f \in \mathbf{W}^{1,\infty}(\mathbb{R}^d, \mathbb{R}_+^*), \quad \mathbf{Ent}_\mu^{x \mapsto x^p}(f) \leq \frac{1}{2\rho} \mathbf{E}_\mu \left( p(p-1) f^{p-2} |\nabla f|^2 \right).$$

Recall that  $\mathbf{Ent}_\mu = \mathbf{Ent}_\mu^{x \mapsto x \log x}$  and that  $\mathbf{Var}_\mu := \mathbf{Ent}_\mu^{x \mapsto x^2}$ . The  $\mathbf{W}^{1,\infty}$ -regularity is not optimal, but we are not interested here in such aspects. Notice that  $f^{p-2} |\nabla f|^2 = 4p^{-2} |\nabla f^{p/2}|^2$ . The particular constant in  $p$  appearing in (2.12) allows to derive the logarithmic Sobolev inequality by using the fact that for a positive  $f$ :

$$(2.13) \quad \lim_{p \rightarrow 1^+} \frac{\mathbf{E}_\mu(f^p) - \mathbf{E}_\mu(f)^p}{p - 1} = \partial_{p=1} \{ \mathbf{E}_\mu(f^p) - \mathbf{E}_\mu(f)^p \} = \mathbf{Ent}_\mu(f).$$

Therefore, the logarithmic Sobolev inequality appears as a sort of limiting case of a family of  $\Phi$ -Sobolev inequalities associated to  $\Phi(x) = x^p$  with  $p$  close to  $1^+$ , and the constant is sharp when  $\mu$  is log-concave. In another direction, by an appropriate change of function, one can reformulate the  $p$ -inequality (2.12) as follows:

$$\forall f \in \mathbf{W}^{1,\infty}(\mathbb{R}^d, \mathbb{R}_+^*), \mathbf{E}_\mu(f^2) - \mathbf{E}_\mu(f^{2/p})^p \leq \frac{2(p-1)}{p\rho} \mathbf{E}_\mu(|\nabla f|^2),$$

which gives by denoting  $q = 2/p$ , i.e.  $q \in [1, 2)$ :

$$\forall f \in \mathbf{W}^{1,\infty}(\mathbb{R}^d, \mathbb{R}_+^*), \mathbf{E}_\mu(f^2) - \mathbf{E}_\mu(f^q)^{2/q} \leq \frac{(2-q)}{\rho} \mathbf{E}_\mu(|\nabla f|^2).$$

By removing the positivity condition on  $f$  as explained in Remark 3, we get finally that for any  $q \in [1, 2)$ :

$$(2.14) \quad \forall f \in \mathbf{W}^{1,\infty}(\mathbb{R}^d, \mathbb{R}), \mathbf{E}_\mu(f^2) - \mathbf{E}_\mu(|f|^q)^{2/q} \leq \frac{(2-q)}{\rho} \mathbf{E}_\mu(|\nabla f|^2),$$

which is exactly the inequality studied in [Bec89] for the Gaussian measure and in [LO00], [Wan02], [Bar01], [BR03] and [BCR04] for further developments related to this particular case. Notice that in (2.14), the energy term  $\mathbf{E}_\mu(|\nabla f|^2)$  does not depend on  $q$ , and one can adopt the following more convenient formulation:

$$(2.15) \quad \forall f \in \mathbf{W}^{1,\infty}(\mathbb{R}^d, \mathbb{R}), \sup_{q \in [1,2)} \frac{\mathbf{E}_\mu(f^2) - \mathbf{E}_\mu(|f|^q)^{2/q}}{(2-q)} \leq \frac{1}{\rho} \mathbf{E}_\mu(|\nabla f|^2).$$

Generalisations of this type of statement where  $(2 - q)$  and  $1/\rho$  are replaced by a more general functions are addressed in [LO00] and [Wan02]. Actually, inequality (2.14) appears as a sort of infinite dimensional “dual” version of a Sobolev inequality. Namely, let  $(\mathcal{M}, g)$  be a smooth compact connected Riemannian manifold with dimension  $d \geq 3$  and Ricci curvature  $\rho > 0$ , and let  $\mu$  be the normalised Riemannian volume probability measure. The Sobolev inequality states that for any real valued smooth function  $f$  on  $M$ :

$$(2.16) \quad \mathbf{E}_\mu(|f|^q)^{2/q} - \mathbf{E}_\mu(f^2) \leq \frac{q+2}{2q} \frac{q-2}{\rho} \mathbf{E}_\mu(|\nabla f|^2),$$

where  $q := 2d/(d-2)$  and  $|\nabla f|^2$  denotes the length of the gradient of  $f$  on  $\mathcal{M}$ , see for example [Led00]. Such an inequality is stronger than logarithmic Sobolev inequality which is stronger than Poincaré inequality. When  $\mathcal{M}$  is the standard  $d$ -dimensional sphere  $\mathbb{S}^d(r) \subset \mathbb{R}^{d+1}$  of radius  $r = \sqrt{d}$ , one has

$$\rho = \frac{d-1}{r^2} = \frac{q+2}{2q}.$$

It is then well known that one can recover the optimal logarithmic Sobolev inequality for the standard Gaussian measure on  $\mathbb{R}^k$  when  $\mu = \mathcal{N}(0, I_d)$  by taking the projection  $\sigma_k : \mathbb{R}^{d+1} \rightarrow \mathbb{R}^k$  where  $k \leq d$  and by letting  $d$  tends to  $+\infty$  (i.e.  $q$  tends to 2). This fact is sometimes referred as the ‘‘Poincaré observation’’, see for example [Joh03] for a generalisation to the case where  $\mu$  is a Boltzmann-Gibbs product measure.

**Remark 4** (From  $\Phi$ -Poincaré to  $\Phi$ -Sobolev). It is shown in [Wan02] that for any  $p \in [1, 2]$  and any  $f \in L^2(\Omega, \mu, \mathbb{R})$  where  $\mu$  is a probability measure on  $\Omega$  that:

$$\mathbf{E}_\mu(f^2) - \mathbf{E}_\mu(|f|^p)^{2/p} \leq \mathbf{Var}_\mu(f) + (1-p) \mathbf{Var}_\mu^{x \rightarrow x^p}(f)^{2/p}.$$

It appears as an extended version of a Rothaus-Deuschel-Stroock-Bakry inequality to the case  $p \in [1, 2]$ , cf. [ABC<sup>+</sup>00, Lemmas 4.3.7 and 4.3.8]. In presence of Poincaré inequality, it can be used perhaps to deduce a  $\Phi$ -Sobolev inequality for  $\Phi(x) = x^p$  from an hypothetic  $\Phi$ -Poincaré inequality involving the  $\Phi$ -variance (1.3).

**Remark 5** (Sets convexity). For any  $\alpha \in \{1, 2, 2'\}$ , let  $\mathbb{E}(\alpha, \mathcal{I})$  be the set of smooth convex functions from  $\mathcal{I}$  to  $\mathbb{R}$  such that  $(\mathcal{H}\alpha)$  holds. It is easy to see that these three sets are convex vector cones, i.e. stable by linear combinations with non-negative scalar coefficients. For example, for any  $\lambda := (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in (\mathbb{R}_+)^2 \times \mathbb{R}^2$  and any  $p \in (1, 2]$ , the function  $\Phi_{p, \lambda} : \mathbb{R}_+ \rightarrow \mathbb{R}$  defined by

$$\Phi_{p, \lambda}(x) := \lambda_1 x^p + \lambda_2 x \log x + \lambda_3 x + \lambda_4$$

is in  $\mathbb{E}(1, \mathbb{R}_+) \cap \mathbb{E}(2, \mathbb{R}_+) \cap \mathbb{E}(2', \mathbb{R}_+)$ . It is clear that functions  $\Phi_{2, \lambda}$  are extremal points of the convex cone  $\mathbb{E}(1, \mathbb{R}_+)$ . One can observe that for any probability measure  $\mu$  and any interval  $\mathcal{I}$  of  $\mathbb{R}$ , the functional valued functional  $\Phi \in \mathbb{E}(\alpha, \mathcal{I}) \mapsto \mathbf{Ent}_\mu^\Phi$  is the restriction of a linear functional over the cone  $\mathbb{E}(\alpha, \mathcal{I})$ , which is not sensitive to  $\lambda_3$  and  $\lambda_4$ . The same linearity property holds for the associated  $\Phi$ -Sobolev inequalities which are moreover invariant by any dilatation in  $\lambda$ . Thus, we understand that in view of  $\Phi$ -Sobolev inequalities,  $\lambda_3$  and  $\lambda_4$  plays no role and that  $\Phi_{2, e_1}$  and  $\Phi_{?, e_2}$  which corresponds respectively to classical variance and entropy are ‘‘extremal’’ cases under  $(\mathcal{H}1)$ . It could be nice to try to use a sort of Choquet integral representation by mean of extremal points, cf. [FLP01].



### 3. Perturbation, tensorisation, convolution, concentration

We explore here the stability of  $\Phi$ -Sobolev inequalities by tensorisation, convolution, and perturbation. We give also some concentration of measure consequences.

#### 3.1. Tensorisation

A sub-additivity property for  $\Phi$ -entropies under  $(\mathcal{H}1)$  can be found in [LO00, Cor. 3]. However, we provide in the sequel a short and simple proof relying on the convexity of the  $\Phi$ -entropy functional established in Proposition 1.2. Duality and sub-additivity formulas are well known for variance and entropy, cf. [Led97] and [ABC<sup>+</sup>00, Chap. 1]. The tensorisation property for  $\Phi$ -entropies can be seen as an extension of the sub-additivity of Kullback-Leibler relative entropy. Beware that it is different from the also well known sub-additivity of Shannon like entropies, cf. Section 6 page 356 for an explanation.

Recall that for two probability measures  $\nu$  and  $\mu$  on a probability space  $(\Omega, \mathcal{F})$  with  $\nu \ll \mu$ , the Kullback-Leibler relative entropy  $\mathbf{Ent}(\nu | \mu)$  is defined by:

$$\mathbf{Ent}(\nu | \mu) := \int_{\Omega} \frac{d\nu}{d\mu} \log \frac{d\nu}{d\mu} d\mu = \mathbf{E}_{\mu} \left( \frac{d\nu}{d\mu} \log \frac{d\nu}{d\mu} \right) = \mathbf{E}_{\nu} \left( \log \frac{d\nu}{d\mu} \right).$$

Let  $(\Omega_1 \times \dots \times \Omega_n, \mathcal{F}_1 \otimes \dots \otimes \mathcal{F}_n, \mu_1 \otimes \dots \otimes \mu_n)$  be a product probability space. Then, for any probability measure  $\nu \ll \mu_1 \otimes \dots \otimes \mu_n$ :

$$(3.1) \quad \mathbf{Ent}(\nu | \mu_1 \otimes \dots \otimes \mu_n) \leq \mathbf{E}_{\mu}(\mathbf{Ent}(\nu | \mu_1)) + \dots + \mathbf{E}_{\mu}(\mathbf{Ent}(\nu | \mu_n)),$$

where

$$\mathbf{Ent}(\nu | \mu_i) := \mathbf{Ent}_{\mu_i}(f) = \int_{\Omega_i} f \log f d\mu_i - \int_{\Omega_i} f d\mu_i \log \int_{\Omega_i} f d\mu_i,$$

and  $f := d\nu / \mu_1 \otimes \dots \otimes \mu_n$ . Moreover, equality holds if and only if  $\nu = \nu_1 \otimes \dots \otimes \nu_n$ . In other words, for any real valued integrable function  $f$  on the product space:

$$(3.2) \quad \mathbf{Ent}_{\mu_1 \otimes \dots \otimes \mu_n}(f) \leq \mathbf{E}_{\mu}(\mathbf{Ent}_{\mu_1}(f)) + \dots + \mathbf{E}_{\mu}(\mathbf{Ent}_{\mu_n}(f)).$$

This property of relative entropy remains true for  $\Phi$ -entropies when  $\Phi$  fulfils  $(\mathcal{H}1)$ :

**Proposition 3.1** (Tensorisation of  $\Phi$ -entropies). *Suppose that  $\Phi$  satisfies  $(\mathcal{H}1)$ . Let  $(\Omega_1 \times \dots \times \Omega_n, \mathcal{F}_1 \otimes \dots \otimes \mathcal{F}_n, \mu_1 \otimes \dots \otimes \mu_n)$  be a product probability space. Then, for any measurable function  $f : (\Omega, \mathcal{F}) \rightarrow (\mathcal{I}, \mathbb{B}(\mathcal{I}))$  one has:*

$$(3.3) \quad \mathbf{Ent}_{\mu_1 \otimes \dots \otimes \mu_n}^{\Phi}(f) \leq \mathbf{E}_{\mu} \left( \mathbf{Ent}_{\mu_1}^{\Phi}(f) \right) + \dots + \mathbf{E}_{\mu} \left( \mathbf{Ent}_{\mu_n}^{\Phi}(f) \right).$$

*Proof.* As for variance and entropy, (3.3) is a consequence of the convexity of the  $\Phi$ -entropy functional, and can be obtained by using the variational

(duality) formula for  $\Phi$ -entropies (1.20). Namely, for any  $h$  and  $i \in \{0, \dots, n\}$ , let  $h_i := \mathbf{E}_{\mu_1 \otimes \dots \otimes \mu_i}(h)$ , with  $h_0 = h$ . Thanks to the variational formula (1.20), one has:

$$\mathbf{Ent}_\mu^\Phi(f) \leq \mathbf{E}_\mu((\Phi'(h_0) - \Phi'(h_n))(f - h)) - \mathbf{Ent}_\mu^\Phi(h).$$

Since we can write

$$\Phi'(h_0) - \Phi'(h_n) = \sum_{i=1}^n (\Phi'(h_{i-1}) - \Phi'(h_i)),$$

the desired result follows by the variational formula (1.20) again and the fact that

$$\mathbf{Ent}_\mu^\Phi(h) \geq \mathbf{E}_\mu\left(\mathbf{Ent}_{\mu_i}^\Phi(h_{i-1})\right),$$

which is due to Jensen inequality for the convex function  $\Phi$ . □

The tensorisation property for  $\Phi$ -entropies can be used to tensorise  $\Phi$ -Sobolev inequalities as what is done for logarithmic Sobolev and Poincaré inequalities, which can then be viewed as particular cases. In essence,  $\Phi$ -Sobolev inequalities under  $(\mathcal{H}1)$  are then infinite dimensional since they hold on the product space with the maximum of the one dimensional constants.

### 3.2. Convolution

For any  $x \in \mathbb{R}^d$ , we denote by  $\tau_x$  the translation of vector  $x$  in  $\mathbb{R}^d$  acting on any function  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  by  $(\tau_x \cdot f)(y) := f(x + y)$ . One can state then the following immediate Corollary of Proposition 3.1.

**Corollary 3.1** (Stability of  $\Phi$ -Sobolev inequalities by convolution).

*Suppose that  $\Phi$  satisfies  $(\mathcal{H}1)$ . For any  $i \in \{1, \dots, n\}$ , let  $\mu_i$  be a probability measure on  $\mathbb{R}^d$  satisfying to a  $\Phi$ -Sobolev inequality of the form*

$$\exists c_i > 0, \quad \forall f : \mathbb{R}^d \rightarrow \mathcal{I} \text{ smooth}, \quad \mathbf{Ent}_{\mu_i}^\Phi(f) \leq c_i \mathbf{E}_{\mu_i}(\mathcal{E}^\Phi(f)),$$

where  $f \mapsto \mathcal{E}^\Phi(f)$  is a non-negative functional depending on  $\Phi$  and its derivatives such that  $\mathcal{E} \circ \tau_x = \tau_x \circ \mathcal{E}$  for any  $x \in \mathbb{R}^d$ . Then, one has for any smooth function  $f : \mathbb{R}^d \rightarrow \mathcal{I}$

$$(3.4) \quad \mathbf{Ent}_{\mu_1 * \dots * \mu_n}^\Phi(f) \leq (c_1 + \dots + c_n) \mathbf{E}_{\mu_1 * \dots * \mu_n}(\mathcal{E}^\Phi(f)).$$

The commutation property  $\mathcal{E} \circ \tau_x = \tau_x \circ \mathcal{E}$  is satisfied for example when one has  $\mathcal{E}^\Phi(f) = \Phi''(f) |\nabla f|^2$ , and one can then check the optimality of the obtained constant  $c_1 + \dots + c_n$  when  $\mu_i$  are Gaussian measures. The Poisson measures with suitable energy  $\mathcal{E}^\Phi$  give another example for which constant is optimal, cf. Section 5. We believe that Corollary 3.1 remains essentially the same if one replaces the Euclidean space  $\mathbb{R}^d$  by an infinite dimensional Banach space or by a topological Abelian group (and maybe any Lie group). It can be useful for processes with independent increments, in discrete or continuous time, and for infinitely divisible laws. However, we will use different methods in the sequel, since i.i.d. increments and/or infinite divisibility can be used by other ways to provide the same result.

**Remark 6** (Invariance by action of the translation group). Notice that hypothesis  $\mathcal{E} \circ \tau_x = \tau_x \circ \mathcal{E}$  for any  $x \in \mathbb{R}^d$  ensures that any associated  $\Phi$ -Sobolev inequality is “invariant by translations in base space”. Namely, if one has

$$\exists c > 0, \quad \forall f \in \mathcal{A}, \quad \mathbf{Ent}_\mu^\Phi(f) \leq c \mathbf{E}_\mu(\mathcal{E}^\Phi(f)),$$

then for any  $x \in \mathbb{R}^d$ ,

$$\forall f \in \tau_{-x} \cdot \mathcal{A}, \quad \mathbf{Ent}_{\tau_x \cdot \mu}^\Phi(f) \leq c \mathbf{E}_{\tau_x \cdot \mu}(\mathcal{E}^\Phi(f)),$$

where the action of  $\tau_x$  on measure  $\mu$  is given by  $\tau_x \cdot \mu := \mu * \delta_x$ . Thus, if the class of functions  $\mathcal{A}$  is invariant by translations, i.e.  $\tau_x \cdot \mathcal{A} = \mathcal{A}$  for any  $x \in \mathbb{R}^d$ , then any element of the orbit  $\{\mu * \delta_x, x \in \mathbb{R}^d\}$  satisfies to the  $\Phi$ -Sobolev inequality satisfied by  $\mu$ , with same constant  $c$  and class  $\mathcal{A}$ . Here again, we believe that things remain essentially the same if one replaces  $\mathbb{R}^d$  by an infinite dimensional Banach space or by a topological Abelian group. Notice that when  $\Phi(x)$  is of the form  $x \log x$  or  $x^p$  with  $p \in (1, 2]$ , the associated  $\Phi$ -Sobolev inequality on  $\mathbb{R}^d$  with  $\mathcal{E}(f) = \Phi''(f) |\nabla f|^2$  is homogeneous, and thus is additionally stable by dilatations.

Actually, translations are particular examples of Lipschitz functions and one can expect a sort of stability by Lipschitz transforms when the energy functional  $\mathcal{E}$  satisfies some stability. The following remark gives an answer.

**Remark 7** (Invariance by action of Lipschitz functions). Let  $\Theta : \mathbb{R}^d \rightarrow \mathbb{R}^{d'}$  be a measurable map, acting on a function  $f : \mathbb{R}^{d'} \rightarrow \mathbb{R}$  by  $(\Theta \cdot f)(y) := f(\Theta(y))$  for any  $y \in \mathbb{R}^d$ . Assume that the probability measure  $\mu$  on  $\mathbb{R}^d$  satisfies to the following  $\Phi$ -Sobolev inequality:

$$\exists c > 0, \quad \forall g \in \mathcal{A}, \quad \mathbf{Ent}_\mu^\Phi(g) \leq c \mathbf{E}_\mu(\mathcal{E}^\Phi(g)),$$

where  $\mathcal{E}^\Phi$  is a non-negative functional. Assume that there exists a constant  $\alpha > 0$  such that for any  $g \in \mathcal{A}$ ,  $(\mathcal{E}^\Phi \circ \Theta)(g) \leq \alpha (\Theta \circ \mathcal{E}^\Phi)(g)$ . Let us denote by  $\mathcal{A} \cdot \Theta$  the set of functions  $f : \mathbb{R}^{d'} \rightarrow \mathbb{R}$  such that  $\Theta \cdot f \in \mathcal{A}$ . Then, one has

$$\forall f \in \mathcal{A} \cdot \Theta, \quad \mathbf{Ent}_{\Theta \cdot \mu}^\Phi(f) \leq c \alpha \mathbf{E}_{\Theta \cdot \mu}(\mathcal{E}^\Phi(f)),$$

where  $\Theta \cdot \mu := \mu \circ \Theta^{-1}$  is the image measure of  $\mu$  by  $\Theta$ , i.e. the law of  $\Theta$  under  $\mu$ . An important example is given by  $\mathcal{E}^\Phi(g) = \Phi''(g) |\nabla g|^2$  and by any smooth map  $\Theta$  such that  $\|\text{Jac}(\Theta)\|_2 \leq \sqrt{\alpha}$ . In particular, when  $d = d'$ , such  $\Phi$ -Sobolev inequalities are stable up to constants by the action of the non-Abelian group of diffeomorphisms with “bounded Jacobian”. Actually, in many cases including  $\mathcal{E}^\Phi(g) = \Phi''(g) |\nabla g|^2$ , only a weak smoothness of  $\Theta$  is needed, and for example one can assume only that  $\Theta$  is Lipschitz when  $d' = 1$ .

As an immediate consequence of Corollary 3.1 and Remarks 6 and 7, one can deduce the following result.

**Corollary 3.2.** For any  $i \in \{1, \dots, n\}$ , let  $\mu_i$  be a probability measure on  $\mathbb{R}^{d_i}$  satisfying to a  $\Phi$ -Sobolev inequality of the form:

$$\exists, c_i > 0, \quad \forall f : \mathbb{R}^{d_i} \rightarrow \mathcal{I} \text{ smooth}, \quad \mathbf{Ent}_{\mu_i}^\Phi(f) \leq c_i \mathbf{E}_{\mu_i} \left( \Phi''(f) |\nabla f|^2 \right),$$

where  $\Phi$  satisfies  $(\mathcal{H}1)$ . Let  $d$  be in  $\mathbb{N}^*$ . For each  $i = 1, \dots, n$ , let  $\Theta_i : \mathbb{R}^{d_i} \rightarrow \mathbb{R}^d$  be a smooth (resp. Lipschitz when  $d = 1$ ) map with  $\|\text{Jac}(\Theta_i)\|_2 \leq \sqrt{\alpha_i}$  (resp.  $\|f\|_{\text{Lip}} \leq \sqrt{\alpha_i}$  when  $d' = 1$ ). Let  $\mu$  be the probability measure on  $\mathbb{R}^d$  defined by

$$(3.5) \quad \mu := (\Theta_1 \cdot \mu_1) * \dots * (\Theta_n \cdot \mu_n).$$

Then, for any smooth function  $f : \mathbb{R}^d \rightarrow \mathcal{I}$

$$(3.6) \quad \mathbf{Ent}_\mu^\Phi(f) \leq (c_1 \alpha_1 + \dots + c_n \alpha_n) \mathbf{E}_\mu \left( \Phi''(f) |\nabla f|^2 \right).$$

Probability measures like in (3.5) appear for example in Statistics as fixed points of some kernel estimators. A very special Gaussian case of (3.5) was studied in [BK02] when  $\Phi(x) = x \log x$  and  $(n, d_1, d) = (1, 2, 1)$ . In view of some applications to Statistics and Statistical Mechanics, the natural next step is to explore the stability of  $\Phi$ -Sobolev inequalities by general mixtures of the form

$$\mathbf{E}_\mu(f) := \int_T \mathbf{E}_{\mu_t}(f) \, d\nu(t);$$

and conditioning of the form  $\mu = \mathcal{L}((X_1, \dots, X_n) | F(X_1, \dots, X_n) = m)$  for example. But such aspects are far more delicate and intricate and will be hopefully the aim of forthcoming papers.

### 3.3. Perturbation

One can derive a perturbation property for the  $\Phi$ -Sobolev inequality via the following straightforward variational formula for  $\Phi$ -entropies:

$$(3.7) \quad \mathbf{Ent}_\mu^\Phi(f) = \inf_{a \in \overset{\circ}{\mathcal{I}} \subset \mathbb{R}} \mathbf{E}_\mu \underbrace{(\Phi(f) - \Phi(a) - \Phi'(a)(f - a))}_{\geq 0}.$$

This formula is nothing else but the consequence of Taylor formula for the convex function  $\Phi$ , and no more assumption on  $\Phi$  are required here. It can be found in [ABC<sup>+</sup>00, Lem. 3.4.2]. One can easily recover the well known formulas for variance and entropy by considering the appropriate  $\Phi$ :

$$\mathbf{Var}_\mu(f) = \inf_{a \in \mathbb{R}} \mathbf{E}_\mu((f - a)^2)$$

and

$$\mathbf{Ent}_\mu(f^2) = \inf_{a \in \mathbb{R}_+^*} \mathbf{E}_\mu(f^2 \log(f^2/a) - a + f^2).$$

Such a variational formula (3.7) allows a perturbation of  $\Phi$ -Sobolev inequalities via the method used by Holley and Stroock for Poincaré and logarithmic

Sobolev inequalities, cf. [HS87]. Namely, let  $\mu$  be a probability measure on  $(\Omega, \mathcal{F})$  and  $B : (\Omega, \mathcal{F}) \rightarrow (\mathbb{R}, \mathbb{B}(\mathbb{R}))$  be a bounded measurable function. If we define the probability measure  $\nu_B$  on  $(\Omega, \mathcal{F})$  by:

$$(3.8) \quad d\nu_B := (Z_{\mu,B})^{-1} \exp(B) d\mu,$$

where  $Z_{\mu,B} := \mathbf{E}_\mu(\exp(B))$ , then one can write:

$$\begin{aligned} \mathbf{Ent}_{\nu_B}^\Phi(f) &= \inf_{a \in \mathcal{I}} \int_{\Omega} (\Phi(f) - \Phi(a) - \Phi'(a)(f - a)) d\nu_B \\ &\leq \exp(-\inf(B) + \sup(B)) \inf_{a \in \mathcal{I}} \int_{\Omega} (\Phi(f) - \Phi(a) - \Phi'(a)(f - a)) d\mu \\ &= \exp(\text{osc}(B)) \mathbf{Ent}_\mu^\Phi(f). \end{aligned}$$

One can express the result as follows:

**Proposition 3.2** (Perturbation). *Let  $(\Omega, \mathcal{F}, \mu)$  be a probability space such that*

$$\exists c \in \mathbb{R}_+^*, \quad \forall f \in \mathcal{A}, \quad \mathbf{Ent}_\mu^\Phi(f) \leq c \mathbf{E}_\mu(\mathcal{E}(f)),$$

where  $\mathcal{A}$  is a class of real valued measurable functions on  $(\Omega, \mathcal{F})$  taking their values in  $\mathcal{I}$  and where  $\mathcal{E} : \mathcal{A} \rightarrow \mathbb{L}^1(\Omega, \mathcal{F}, \mu, \mathbb{R}_+)$  is a functional. Then, for any bounded measurable function  $B : (\Omega, \mathcal{F}) \rightarrow \mathbb{R}$ , one has:

$$\forall f \in \mathcal{A}, \quad \mathbf{Ent}_{\nu_B}^\Phi(f) \leq c e^{2 \text{osc}(B)} \mathbf{E}_{\nu_B}(\mathcal{E}(f)),$$

where  $\nu_B$  is like in (3.8).

### 3.4. Concentration of measure

It is well known that a logarithmic Sobolev inequality for  $\mu$  on  $\mathbb{R}^d$  of the form

$$\exists c > 0, \quad \forall f : \mathbb{R}^d \rightarrow \mathbb{R} \text{ smooth}, \quad \mathbf{Ent}_\mu^{x \mapsto x \log x}(f^2) \leq c \mathbf{E}_\mu(|\nabla f|^2)$$

gives, when applied to  $f = \exp(\lambda F)$  where  $F : \mathbb{R}^d \rightarrow \mathbb{R}$  is 1-Lipschitz, a Gaussian like exponential upper bound for the Laplace transform of  $\mathcal{L}_\mu(F)$ :  $\mathbf{E}_\mu(\exp(\lambda F)) \leq \exp(c t^2)$ . Such a bound can be then used via classical Chebyshev-Markov-Chernov-Cramér approach to give a concentration inequality for  $F$  around its mean:

$$\mu(|F - \mathbf{E}_\mu(F)| \geq t) \leq 2 \exp(-t^2/c).$$

The  $t^2$  comes from the fact that the Young-Fenchel-Legendre convex conjugate of  $x \mapsto p^{-1} x^p$  is  $x \mapsto q^{-1} x^q$  where  $q := p/(p - 1)$  is the Hölder conjugate of  $p$ , and thus  $q = 2$  when  $p = 2$ . This method is known as Herbst argument and gives precise and non-asymptotic bounds which strengthen large deviations results. The concentration bound obtained from logarithmic Sobolev in

discrete settings are only Poissonian, i.e.  $\exp(-c \min(t^2, t \log t))$ , due to the lack of chain rule. At the opposite side of the set of possible  $\Phi$  functions, the Poincaré inequality gives by the same method an exponential like concentration, i.e.  $\exp(-ct)$ . See for example [Led99, Led01] and [ABC<sup>+</sup>00, Chap. 7] for a general approach to concentration of measure via functional inequalities. It is tempting to study the concentration of measure consequences of generic  $\Phi$ -Sobolev inequalities, and one can expect intermediate exponential speeds between  $t^2$  and  $t$ . Let us recall what can be found in [LO00] and [Bar01]. Let  $\mu$  be a probability measure on  $\mathbb{R}^d$ , and  $a \in [0, 1]$  and  $r = 2/(2-a)$  (i.e.  $r \in [1, 2]$ ). Assume that there exists a constant  $C > 0$  such that for any  $q \in [1, 2]$  and any smooth function  $f : \mathbb{R}^d \rightarrow \mathbb{R}$ :

$$\mathbf{E}_\mu(f^2) - \mathbf{E}_\mu(|f|^q)^{2/q} \leq C(2 - q)^a \mathbf{E}_\mu(|\nabla f|^2).$$

We have seen already that such an inequality can be deduced by a change of function from a  $\Phi$ -Sobolev inequalities with  $\Phi(x) = x^{2/q}$ . Then, the following concentration of measure holds for any  $t > 0$  and  $\mu$ -integrable 1-Lipschitz function  $F : \mathbb{R}^d \rightarrow \mathbb{R}$ :

$$\mu(F - \mathbf{E}_\mu(F) > \sqrt{C}t) \leq \exp(-Kt^r),$$

where  $K > 0$  is a universal constant. See [Wan02, Bar01, LO00, BCR04] and references therein for further developments. One can find a quite recent account in [Lug04]. Some aspects of concentration of measure consequences in discrete space settings are addressed in [Wu00] and [Led99] and references therein.

Theorem 2.1 (page 333) gives  $\Phi$ -Sobolev inequalities for hypercontractive diffusions and uniformly strictly log-concave probability measures, i.e. what is under Bakry-Emery  $\mathbf{I}_2$  criterion (2.5). It is quite natural to ask for a general criterion to establish such inequalities beyond this scope. One can find some answers in [LO00] and [Wan02] for  $\Phi(x) = x^p$ . Namely, let  $a \in [0, 1]$  and  $r = 2/(2 - a)$  (i.e.  $r \in [1, 2]$ ), and consider the probability measure  $\nu_r$  on  $\mathbb{R}^d$  defined by:

$$d\nu_r(x) := Z_r^{-1} \exp(-\|x\|_r^r) dx.$$

Then there exists an universal constant  $C > 0$  such that for any  $q \in [1, 2]$  and any smooth function  $f : \mathbb{R}^d \rightarrow \mathbb{R}$ :

$$\mathbf{E}_{\nu_r}(f^2) - \mathbf{E}_{\nu_r}(|f|^q)^{2/q} \leq C(2 - q)^a \mathbf{E}_{\nu_r}(|\nabla f|^2),$$

which is exactly after the suitable change of functions the  $\Phi$ -Sobolev inequalities for  $\Phi(x) = x^{2/q}$ . One can try to investigate  $\Phi$ -Sobolev inequalities on the real line via Hardy type inequalities, like what is known for Poincaré and logarithmic Sobolev inequalities, cf. for example [ABC<sup>+</sup>00, Chap. 6]. As we will show in the sequel,  $\Phi$ -Sobolev inequalities can be established on paths space of some diffusions (under  $(\mathcal{H}1)$ ) and some Lévy processes under  $(\mathcal{H}2)$ , extending by this way what is already known in the literature for Poincaré and logarithmic Sobolev inequalities.

#### 4. Phi-Sobolev for Brownian Motion and Wiener space

Consider the standard Brownian Motion  $(B_t)_{t \geq 0}$  on  $\mathbb{R}^d$  starting from  $B_0 = 0$ . Since  $\mathcal{L}(B_t) = \mathcal{N}(0, tI_d)$  is uniformly strictly log-concave with constant  $1/t$ , it satisfies a  $\Phi$ -Sobolev inequality (2.2) of constant  $t/2$  under  $(\mathcal{H}1)$  for  $\Phi$ :

$$\forall t > 0, \forall f, \mathbf{Ent}^\Phi(f(B_t)) \leq \frac{t}{2} \mathbf{E}(\Phi''(f(B_t)) |\nabla f|^2(B_t)).$$

Let  $0 < t_1 < \dots < t_n$  be  $n$  successive times, and let  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  be a smooth function. Then, one can write:

$$\begin{aligned} \mathbf{Ent}^\Phi(F(B_{t_1}, B_{t_2}, \dots, B_{t_n})) &= \mathbf{Ent}^\Phi(F(Q_1, Q_1 + Q_2, \dots, Q_1 + \dots + Q_n)) \\ &= \mathbf{Ent}^\Phi(G(Q_1, \dots, Q_n)) \\ &= \mathbf{Ent}_{\mathcal{L}(Q_1, \dots, Q_n)}^\Phi(G), \end{aligned}$$

where  $Q_i := B_{t_i} - B_{t_{i-1}}$  and  $t_0 := 0$  and

$$G(x_1, \dots, x_n) := F(x_1, x_1 + x_2, \dots, x_1 + \dots + x_n).$$

But now, since Brownian Motion has i.i.d. increments, one gets

$$\mathcal{L}(Q_1, \dots, Q_n) = \mathcal{N}(0, (t_1 - t_0)I_d) \otimes \dots \otimes \mathcal{N}(0, (t_n - t_{n-1})I_d).$$

Therefore, the tensorisation property (3.3) yields the following result.

**Theorem 4.1** (Multi-times  $\Phi$ -Sobolev for Brownian Motion). *Let  $(B_t)_{t \geq 0}$  be a standard Brownian Motion on  $\mathbb{R}^d$ . Assume that  $\Phi$  fulfils  $(\mathcal{H}1)$ . Then for any sequence of times  $t_0 := 0 < t_1 < \dots < t_n$  and any smooth function  $F : \mathbb{R}^n \rightarrow \mathbb{R}$ :*

$$(4.1) \quad \mathbf{Ent}^\Phi(F(B_{t_1}, \dots, B_{t_n})) \leq \frac{1}{2} \mathbf{E}_{\mathcal{L}(B_{t_1}, \dots, B_{t_n})}(\Phi''(F) \mathcal{D}_{t_1, \dots, t_n}^2 F),$$

where

$$(4.2) \quad \mathcal{D}_{t_1, \dots, t_n}^2 F := \sum_{i=1}^n (t_i - t_{i-1}) \left( \sum_{j=i}^n \partial_j F \right)^2.$$

Moreover, the inequality remains true when  $t_0 := 0 \leq t_1 \leq \dots \leq t_n$ .

One can ask if (4.1) is a consequence of (2.8). If we define the  $n \times n$  square matrices  $T$  and  $Q$  by  $T := \text{diag}(\sqrt{t_1 - t_0}, \dots, \sqrt{t_n - t_{n-1}})$  and  $Q_{i,j} := \delta_{j \geq i}$ , then  $\Sigma := (TQ)^\top TQ$  is the covariance matrix of the centred Gaussian vector  $(B_{t_1}, \dots, B_{t_n})$ . In fact,  $\Sigma_{i,j} := t_i \wedge t_j$ , and the  $\langle \Sigma v; v \rangle$  quadratic form in the right hand side of (2.8) reads

$$(4.3) \quad \begin{aligned} \langle \Sigma v; v \rangle &= (TQv)^\top TQv \\ &= \sum_{i=1}^n (t_i - t_{i-1}) \left( \sum_{j=i}^n v_j \right)^2, \end{aligned}$$

which is exactly the quadratic form appearing in the right hand side (4.2) of (4.1). Therefore, for Gaussian measures, the  $\Phi$ -Sobolev inequality (4.1) obtained by applying the tensorisation formula (3.3) to the unidimensional case (2.7) (i.e. with  $d = 1$ ) is exactly the Brascamp-Lieb type  $\Phi$ -Sobolev inequality (2.8). The direct use of (2.7) for the multivariate Gaussian measure  $\mathcal{L}(B_{t_1}, \dots, B_{t_n})$  gives the quadratic form  $(\max \sigma(\Sigma)) \sum_{i=1}^n v_i^2$  where  $\sigma(\Sigma)$  is the spectrum of  $\Sigma = TQ$ . Thus, inequality (2.7) for Gaussian measures is a direct consequence of (4.1). Notice that  $\det(\Sigma) = (t_1 - t_0) \cdots (t_n - t_{n-1})$  and that  $\text{Tr}(\Sigma) = t_1 + \cdots + t_n$ .

As we will see, inequality (4.1) appears as a particular case of a more general one on paths space (i.e. for the Wiener measure). The same procedure may be used for any random walk (resp. Lévy process), provided that the law of the increment (resp. the infinitely divisible law at time  $t = 1$ ) satisfies to a  $\Phi$ -Sobolev inequality under  $(\mathcal{H}1)$  for  $\Phi$  (we need a tensorisation property). Infinitely divisible laws are particular cases of laws of sums of i.i.d. random variables. As we will see, one can use for such laws the semi-group approach directly via the associated Lévy process, to establish  $\Phi$ -Sobolev inequalities with linear constant in time when  $\Phi$  satisfies  $(\mathcal{H}2)$ , just like for Brownian Motion.

One can derive the  $\Phi$ -Sobolev inequality for the Wiener measure on  $\mathbb{R}^d$  by a cylindrical method, starting from the multi-times  $\Phi$ -Sobolev inequality (4.1) (established by tensorisation) by letting the number  $n$  of times considered tends to  $+\infty$ . However,  $\Phi$ -Sobolev inequalities involving a Malliavin derivative can be easily derived on paths space via a martingale representation approach and Itô formula as what was done for the log-Sobolev inequality in [CHL97]. Actually, one can state the following Theorem.

**Theorem 4.2** ( $\Phi$ -Sobolev for the Wiener measure). *Let  $\mathbf{W}_0(\mathbb{R}^d)$  be the paths space of continuous functions from  $[0, 1]$  to  $\mathbb{R}^d$  starting from 0 at  $t = 0$ , measured by the standard Wiener measure. Assume that  $\Phi$  fulfils  $(\mathcal{H}1)$ . Then for any random variable  $F \in L^2(\mathbf{W}_0(\mathbb{R}^d), \mathcal{I})$ :*

$$(4.4) \quad \mathbf{Ent}^\Phi(F) \leq \frac{1}{2} \mathbf{E} \left( \Phi''(F) |\mathcal{D}F|_{\mathbb{H}}^2 \right),$$

where  $\mathcal{D} : L^2(\mu) \rightarrow L^2(\mu, \mathbb{H})$  is the Malliavin gradient operator on  $\mathbf{W}_0(\mathbb{R}^d)$  and  $\mathbb{H}$  is the Cameron-Martin Hilbert space.

*Proof.* For every functional  $F : \mathbf{W}_0(\mathbb{R}^d) \rightarrow \mathcal{I}$ , consider the martingale  $M_t := \mathbf{E}(F | \mathcal{F}_t)$  where  $0 \leq t \leq 1$  and  $\mathcal{F}_t = \sigma(B_s, 0 \leq s \leq t)$  is the natural filtration. Then, one has

$$dM_t = \langle \mathbf{E}((DF)_t | \mathcal{F}_t), dB_t \rangle,$$

where  $(DF)_t$  denotes the directional derivative of  $F$ . Now, Itô formula yields:

$$\mathbf{E}(\Phi(M_t)) - \mathbf{E}(\Phi(M_0)) = \frac{1}{2} \mathbf{E} \left( \int_0^1 \Phi''(M_t) |\mathbf{E}((DF)_t | \mathcal{F}_t)|^2 dt \right),$$



which is nothing else but

$$\mathbf{Ent}^{\Phi}_{\mathcal{L}((B_t)_{0 \leq t \leq 1})}(F) = \frac{1}{2} \mathbf{E} \left( \int_0^1 \Phi''(\mathbf{E}(F | \mathcal{F}_t)) |\mathbf{E}((DF)_t | \mathcal{F}_t)|^2 dt \right).$$

Now, by Jensen inequality for the bivariate convex function  $(u, v) \mapsto \Phi(u)'' v^2$ , we get as in the semi-group proof of Theorem 2.1:

$$\Phi''(\mathbf{E}(F | \mathcal{F}_t)) |\mathbf{E}((DF)_t | \mathcal{F}_t)|^2 \leq \mathbf{E} \left( \Phi''(F) |(DF)_t|^2 | \mathcal{F}_t \right),$$

which gives the desired result. Here again, one can use alternatively the concavity of  $1/\Phi''$  and Cauchy-Schwarz inequality.  $\square$

Notice that the proof is a replica at paths space level of the semi-group proof for diffusions in Theorem 2.1, where  $\mathbf{P}_t$  is “replaced” by  $\mathbf{E}(F | \mathcal{F}_t)$  and the diffusion property by Itô formula. This analogy is not formal indeed, since the diffusion semi-group gives the solution of Stroock-Varadhan martingale problem associated to the related elliptic diffusion operator, see for example [KS91, Chap. 5, Sect. 4]. Such a proof can be extended to paths spaces on manifolds as what was already done for the logarithmic Sobolev inequality in [CHL97, Wan96] (see also [Hsu02, Sect. 8.3] and references therein), as stated in the following Theorem.

**Theorem 4.3** ( $\Phi$ -Sobolev for Brownian Motion on a Manifold). *Let  $(\mathcal{M}, g)$  be a smooth complete and connected Riemannian manifold equipped with the Levi-Civita connection. Let  $x \in \mathcal{M}$  and  $W_x(\mathcal{M})$  be the space of continuous paths  $\gamma : [0, 1] \rightarrow \mathcal{M}$  with  $\gamma(0) = x$ . Assume that the Ricci curvature of  $\mathcal{M}$  is uniformly bounded by the real number  $K$ . Assume that  $\Phi$  fulfils  $(\mathcal{H}1)$ , then for any smooth non-negative random variable  $F$  on  $W_x(\mathcal{M})$ :*

$$(4.5) \quad \mathbf{Ent}^{\Phi}(F) \leq \frac{1}{2} e^K \mathbf{E} \left( \Phi''(F) |DF|_{\mathbb{H}}^2 \right).$$

When  $\mathcal{M}$  is the standard  $d$ -dimensional Euclidean space  $\mathbb{R}^d$ , one has  $K = 0$  and we recover Theorem 4.2. We believe that  $\Phi$ -Sobolev inequalities under  $(\mathcal{H}1)$  still hold on the paths space of diffusions on manifolds with Driver total antisymmetry condition, as what was done in [CHL97] for the logarithmic Sobolev inequality. We have to mention that this beautiful martingale method for Wiener measure over Riemannian manifold appeared for the first time in [Fan94] for the Poincaré inequality. We will use roughly the same method under  $(\mathcal{H}2)$  on Poisson space in Section 5.1 page 353.

### 5. Phi-Sobolev for pure jumps Lévy processes and Poisson space

It is tempting to try to establish  $\Phi$ -Sobolev inequalities on Poisson space. In particular, Lévy processes have i.i.d. increments like Brownian Motion. Such processes are not diffusions, and the lack of the chain rule (1.10) forbids the

direct use of the Bakry-Emery semi-group proof. More precisely, for all Markov processes,  $\Gamma \geq 0$ , and the  $\text{CD}(\rho, \infty)$  criterion is equivalent to  $\mathbf{I}_2 \geq \rho \Gamma$ . But the equivalent form in term of strong commutation (2.3) between the semi-group and  $\Gamma$  is available only for diffusions. Nevertheless, for Lévy processes, one has  $\mathbf{I}_2 \geq 0$  and indeed  $\mathbf{I}_2 \geq \Gamma \sqrt{\Gamma}$ , which gives Brownian Motion like commutation formulas and then permits to derive respectively Poincaré and log-Sobolev inequalities, see [AL00] for the simple Poisson point process and [CM02] for more general Lévy processes. More general approaches are presented in [Wu00] (generic Poisson space) and [Pri00] (normal martingales), but here again only for Poincaré and log-Sobolev inequalities.

Let us start with the simplest pure jumps Lévy process  $(X_t)_{t \geq 0}$  which is the simple Poisson point process with intensity  $\lambda > 0$ , for which

$$(5.1) \quad (\mathbf{L}f)(x) := \lambda(f(x + 1) - f(x)) =: \lambda(D_1f)(x),$$

where  $(D_yf)(x) := f(x + y) - f(x)$ . If  $g := \mathbf{P}_{t-s}(f)$ , one has:

$$\partial_s \mathbf{P}_s(\Phi(g)) := \mathbf{P}_s(\mathbf{L}\Phi(g) - \Phi'(g) \mathbf{L}g) = \lambda \mathbf{P}_s(\Psi(g, D_1g)),$$

where  $\Psi$  is defined by (1.19). But now, under  $(\mathcal{H}2)$ ,  $\Psi$  is bivariate convex on  $\mathcal{I}^{(2)}$ . It was already observed for the simple cases  $\Phi(x) = x^2$  and  $\Phi(x) = x \log x$  in the pretty paper [Wu00]. Hence, by the bivariate Jensen inequality and the commutativity property between  $D_1$  and  $\mathbf{P}_{t-s}$  (due to the i.i.d. nature of the increments), one gets the following result.

**Theorem 5.1** (Local  $\Phi$ -Sobolev for the simple Poisson point proc.).

Let  $(X_t)_{t \geq 0}$  be the simple Poisson point process on  $\mathbb{R}^d$  with intensity  $\lambda > 0$ . Assume that  $\Phi$  fulfils  $(\mathcal{H}2)$ , then for any  $t > 0$  and any smooth  $f : \mathbb{R}^d \rightarrow \mathcal{I}$ :

$$(5.2) \quad \mathbf{Ent}_{\mathbf{P}_t}^\Phi(f) \leq \lambda t \mathbf{P}_t(\Psi(f, D_1f)),$$

where  $\mathbf{P}_t(f)(x) := \mathbf{E}(f(X_t) | X_0 = x)$  is the associated semi-group as in (1.4).

In particular, by taking  $t = 1$ , we get:

**Corollary 5.1** ( $\Phi$ -Sobolev for the simple Poisson measure). Let  $\mathcal{P}_\lambda$  be the Poisson measure  $\mathcal{P}_\lambda$  of mean  $\lambda > 0$ , then, under  $(\mathcal{H}2)$ :

$$(5.3) \quad \mathbf{Ent}_{\mathcal{P}_\lambda}^\Phi(f) \leq \lambda \mathbf{E}_{\mathcal{P}_\lambda}(\Psi(f, D_1f)).$$

Such a result remains true for pure jumps Lévy processes (i.e. without Brownian part) by replacing  $D_1$  by the appropriate jump integral. The Brownian part may be added at final stage via tensorisation when possible. Namely, we can state the following.

**Theorem 5.2** (Local  $\Phi$ -Sobolev for a pure jumps Lévy process). Let

$(X_t)_{t \geq 0}$  be a pure jump Lévy process with infinitesimal generator of the form:

$$(5.4) \quad (\mathbf{L}f)(x) := \lambda \int_{\mathbb{R}^d} \left[ D_yf(x) - \theta \frac{y}{1 + |y|^2} \cdot \nabla f \right] d\nu(y),$$

where  $(\lambda, \theta) \in \mathbb{R}_+^* \times \mathbb{R}^d$  and where  $\nu$  is a Lévy measure on  $\mathbb{R}^d$ . Assume that  $\Phi$  fulfils  $(\mathcal{H}2)$ . Then for any  $t > 0$  and any smooth  $f : \mathbb{R}^d \rightarrow \mathcal{I}$ :

$$(5.5) \quad \mathbf{Ent}_{\mathbf{P}_t}^\Phi(f) \leq \lambda t \mathbf{P}_t \left( \int_{\mathbb{R}^d} \Psi(f, D_y f) d\nu(y) \right),$$

where  $\mathbf{P}_t(f)(x) := \mathbf{E}(f(X_t) | X_0 = x)$  is the associated semi-group as in (1.4).

Taking  $t = 1$  in (5.5) gives the same functional inequality for the infinitely divisible law  $\mathcal{L}(X_1)$ , exactly like in Corollary 5.1 for the simple Poisson measure. Theorem 5.1 for the simple Poisson point process can be recovered by taking  $(\theta, \nu) = (0, \delta_1)$  in Theorem 5.2.

One can state the following multi-times version of Theorem 5.2 via the so called Lu-Yau-Stroock-Zegarliński Markov decomposition method.

**Theorem 5.3** (Multi-times  $\Phi$ -Sobolev for pure jumps Lévy proc.). *Let  $(X_t)_{t \geq 0}$  be a pure jumps Lévy process on  $\mathbb{R}^d$  as in Theorem 5.2. Assume that  $\Phi$  fulfils  $(\mathcal{H}2)$ . Then for any increasing sequence of times  $t_0 := 0 < t_1 < \dots < t_n$  and any smooth  $F : \mathbb{R}^n \rightarrow \mathcal{I}$ :*

$$(5.6) \quad \mathbf{Ent}^\Phi(F(X_{t_1}, \dots, X_{t_n})) \leq \lambda \mathbf{E}(\mathcal{D}^\Phi(F(X_{t_1}, \dots, X_{t_n}))),$$

where

$$\mathcal{D}^\Phi(F) := \int_{\mathbb{R}^d} \sum_{i=1}^n (t_i - t_{i-1}) \Psi(F, D_y^{i, \dots, n} F) d\nu(y),$$

where for any  $x \in \mathbb{R}^n$

$$D_y^{i, \dots, n} F(x) := F \circ \tau_i(y)(x) - F(x),$$

and where for any  $i \in \{1, \dots, n\}$  and  $x, y \in \mathbb{R}^d$

$$F \circ \tau_i(y)(x) := F(x_1, \dots, x_{i-1}, x_i + y, \dots, x_n + y).$$

*Proof.* By induction on  $n$ , we can restrict the problem to the case  $n = 2$ . Let  $s$  and  $t$  be two distinct times with  $s < t$ . We assume that for any  $u > 0$ , a  $\Phi$ -Sobolev inequality holds for  $\mathcal{L}(X_u)$  (i.e. for  $\mathbf{P}_u$ ) with constant  $\lambda u$ . We would like to obtain a similar inequality for  $\mathcal{L}(X_s, X_t)$ . Let  $F : \mathbb{R}^2 \rightarrow \mathcal{I}$  be a smooth bivariate functional. We start with the following conditional decomposition of the  $\Phi$ -entropy (which replaces in some ways the tensorisation property):

$$\begin{aligned} \mathbf{Ent}^\Phi(F(X_s, X_t)) &= \mathbf{E}[\mathbf{E}(\Phi(F) | X_s) - \Phi(\mathbf{E}(F | X_s))] \\ &\quad + \mathbf{E}(\Phi(\mathbf{E}(F | X_s))) - \Phi(\mathbf{E}(\mathbf{E}(F | X_s))), \end{aligned}$$

where we abridged  $F(X_s, X_t)$  in  $F$  in the right hand side. In other words,

$$(5.7) \quad \mathbf{Ent}^\Phi(X) = \mathbf{E}(\mathbf{Ent}^\Phi(X | Y)) + \mathbf{Ent}^\Phi(\mathbf{E}(X | Y)),$$

where  $\mathbf{Ent}^\Phi(X | Y)$  is the conditional  $\Phi$ -entropy defined by

$$\mathbf{Ent}^\Phi(X | Y) := \mathbf{E}(\Phi(X) | Y) - \Phi(\mathbf{E}(X | Y)).$$

Now, if  $G_x(y) := F(x, x + y)$ , we have for any  $x$ :

$$\mathbf{Ent}^\Phi(G_x(X_{t-s})) \leq \lambda(t-s) \mathbf{E} \left( \int_{\mathbb{R}^d} \Psi(G_x(X_{t-s}), D_z G_x(X_{t-s})) d\nu(z) \right).$$

Therefore, since  $X_t = X_s + X_t - X_s$  and  $\mathcal{L}(X_s, X_t - X_s) = \mathcal{L}(X_s) \otimes \mathcal{L}(X_t - X_s)$ , the first term of the  $\mathbf{Ent}^\Phi$  decomposition can be bounded above as

$$\mathbf{E}[\mathbf{E}(\Phi(F) | X_s) - \Phi(\mathbf{E}(F | X_s))] \leq \lambda(t-s) \mathbf{E} \left( \int_{\mathbb{R}^d} \Psi(F, D_z^2 F) d\nu(z) \right),$$

where the 2 exponent in  $D_z^2$  means that the  $z$  translation is done on the second variable of  $F$  in the definition of  $D_z$ . On the other hand, if we define  $H_{t-s}$  by

$$H_{t-s}(x) := \mathbf{E}(F(x, x + X_{t-s})),$$

we have:

$$\mathbf{Ent}^\Phi(H_{t-s}(X_s)) \leq \lambda s \mathbf{E} \left( \int_{\mathbb{R}^d} \Psi(H_{t-s}(X_s), D_z H_{t-s}(X_s)) d\nu(z) \right).$$

Now, by the commutation formula  $D_z H_{t-s}(x) := \mathbf{E}(D_z(x \mapsto F(x, x + X_{t-s})))$  and the bivariate Jensen inequality, the last term of the  $\mathbf{Ent}^\Phi$  decomposition can be bounded above as follows:

$$\mathbf{E}(\Phi(\mathbf{E}(F | X_s))) - \Phi(\mathbf{E}(\mathbf{E}(F | X_s))) \leq \lambda s \mathbf{E} \left( \int_{\mathbb{R}^d} \Psi(F, D_z^{1,2} F) d\nu(z) \right),$$

where this time the translation is done in both variables. □

In some sense, the Markovian decomposition used in the proof of Theorem 5.3 replaces the tensorisation property (3.3) when  $(\mathcal{H}2)$  holds instead of  $(\mathcal{H}1)$ . It was introduced in Statistical Mechanics for Poincaré and logarithmic Sobolev inequalities for finite volume Boltzmann-Gibbs measures related to spins systems, cf. for example [GZ03] and [Roy99].

As for Brownian Motion, one can use a cylindrical method letting  $n$  tends to  $+\infty$  in (5.6) in order to obtain a  $\Phi$ -Sobolev inequality on paths space, as expressed in the following Theorem.

**Theorem 5.4** ( $\Phi$ -Sobolev on paths space of pure jumps Lévy proc.).

Let  $(X_t)_{t \geq 0}$  be a pure jump Lévy process as in Theorem 5.2. Assume that  $\Phi$  fulfils  $(\mathcal{H}2)$ . Then, for any suitable function  $F$  of  $(X_t)_{t \geq 0}$  taking its values in  $\mathcal{I}$  and any  $T > 0$ :

$$(5.8) \quad \mathbf{Ent}_{\mathcal{L}((X_t)_{0 \leq t \leq T})}^\Phi(F) \leq \lambda \mathbf{E} \left( \int_0^T \int_{\mathbb{R}^d} \Psi(F, D_y^t F) d\nu(y) dt \right),$$

where

$$D_y^t F((x_s)_{0 \leq s \leq T}) := F((x_s + y \mathbf{I}_{[t, T]}(s))_{0 \leq s \leq T}) - F((x_s)_{0 \leq s \leq T}).$$

Obviously, Theorem 5.4 is stronger than Theorem 5.2 since the latter can be recovered by simply considering in (5.8) functions  $F$  of the form

$$F((x)_{0 \leq s \leq T}) = G(x_{t_1}, \dots, x_{t_n})$$

where  $T$  is chosen bigger than  $t_n$ . Inequality (5.6) is similar to the one obtained for Brownian Motion (4.1) via tensorisation. Actually, one can use Lu-Yau-Stroock-Zegarliński Markovian method for Brownian Motion. Generic Lévy processes – and thus Brownian Motion – are particular examples of normal martingales and we believe that the approach used in [Pri00] for logarithmic Sobolev inequalities remains valid for  $\Phi$ -Sobolev inequalities, but one has to precise the condition on  $\Phi$ .

As explained below,  $\Phi$ -Sobolev inequalities under  $(\mathcal{H}2)$  can be established on paths space for generic Poisson space via a Clark-Ocône-Haussmann formula.

**5.1. Phi-Sobolev inequalities on Poisson space**

Let us explain finally how one can recover the  $\Phi$ -Sobolev inequality on Poisson space via a Clark-Ocône-Haussmann formula like what is done in [Wu00] for the logarithmic Sobolev and Poincaré inequalities (see also [Pri00]). Following closely [Wu00], let  $(E, \mathcal{B}, \nu)$  be a measured space where  $\nu$  is a  $\sigma$ -finite measure and  $(W, \mathcal{F}, \mathcal{P})$  the associated Poisson space with compensation measure  $\nu$ . Let also  $(\Omega, \mathcal{C}, \mathbf{P})$  be the Poisson space associated to

$$([0, 1] \times E, \mathcal{B}([0, 1]) \otimes \mathcal{B}, dt \otimes d\nu(z)),$$

with compensation measure  $d\mu(t, z) := dt \otimes d\nu(z)$ . One can then define the difference operators  $D_z : L^0(W, \mathcal{P}) \rightarrow L(E \times W, \nu \otimes \mathcal{P})$  and

$$D_{t,z} : L^0([0, 1] \times E, \mu) \rightarrow L^0([0, 1] \times E \times \Omega, \mu \otimes \mathbf{P})$$

by:

$$D_z F(\omega) := F(\omega + \delta_z) - F(\omega) \quad \text{and} \quad D_{t,z} \widehat{F}(\omega) := \widehat{D}_z \widehat{F}(\omega),$$

where  $\widehat{F}(\omega) := F(\omega([0, 1], dz))$ . For any  $t \in [0, 1]$ , we define

$$\mathcal{C}_t := \sigma(\omega(A); A \in \mathcal{B}([0, t]) \otimes \mathcal{B}).$$

Let now  $G \in L^2(\Omega, \mathbf{P})$  and let  $g(t, z, \omega)$  be a predictable  $dt \otimes d\nu(z) \otimes \mathbf{P}(d\omega)$  version of  $\mathbf{E}_{\mathbf{P}}(D_{t,z} G | \mathcal{C}_t)$ . One has then the following Clark-Ocône type predictable representation formula:

$$(5.9) \quad G = \mathbf{E}_{\mathbf{P}}(G) + \int_0^1 \int_E g(t, z, \cdot) d\tilde{\omega}(t, z),$$

where  $\tilde{\omega} = \omega - \mu$  is the compensated Poisson point process and where the integral is taken in the sense of Itô. Let us assume for convenience that  $G > \inf(\mathcal{I})$  and let us define the right continuous martingale  $(M_t)_{t \in [0,1]}$  by  $M_t :=$

$\mathbf{E}_{\mathbf{P}}(G | \mathcal{C}_t)$ . Now, since  $M_{t-} = M_t dt \otimes \mathbf{P}$ -a.s., Itô formula for jumps processes (cf. [IW89]) gives:

$$\begin{aligned} \mathbf{Ent}_{\mathbf{P}}^{\Phi}(G) &= \mathbf{E}_{\mathbf{P}}(\Phi(M_1) - \Phi(M_0)) \\ &= \mathbf{E}_{\mathbf{P}}\left(\int_0^1 \int_E \Psi(M_t, g(t, z)) dt d\nu(z)\right). \end{aligned}$$

Now, it remains to use the bivariate convexity of  $\Psi$  which comes from  $(\mathcal{H}2)$  in order to get via Jensen inequality and (5.9) that:

$$\Psi(M_t, g(t, z)) = \Psi(\mathbf{E}_{\mathbf{P}}(G | \mathcal{C}_t), \mathbf{E}_{\mathbf{P}}(D_{t,z} G | \mathcal{C}_t)) \leq \mathbf{E}_{\mathbf{P}}(\Psi(G, D_{t,z} G) | \mathcal{C}_t),$$

which gives finally:

$$(5.10) \quad \mathbf{Ent}_{\mathbf{P}}^{\Phi}(G) \leq \mathbf{E}_{\mathbf{P}}\left(\int_0^1 \int_E \Psi(G, D_{t,z} G) dt d\nu(z)\right).$$

The time interval  $[0, 1]$  can be easily replaced by  $[0, T]$ . One can recover Theorems 5.1, 5.3, 5.4 and 5.3 as Corollaries, in exactly the same way used in [Wu00] for Poincaré and logarithmic Sobolev inequalities. Additionally, one can derive by the same method F.K.G. inequalities, as what is done in [Wu00] for Poisson space and in [Hu00] for diffusions. As we have seen, the method used here for Poisson space is a replica of the method used to establish Theorem 4.2 concerning Wiener measure. The major difference is the lack of chain rule in discrete space settings which leads to replace  $(\mathcal{H}1)$  by  $(\mathcal{H}2)$ .

**5.2. Some remarks about discrete Dirichlet forms**

We collect here few remarks about  $(\mathcal{H}2)$ ,  $(\mathcal{H}2')$ , and comparisons with standard Dirichlet forms in discrete space settings.

**Remark 8** (Comparison with classical Dirichlet forms). Assume that  $(\mathcal{H}2')$  holds. Since  $\Phi'$  is concave,  $\Phi''$  is non-increasing and therefore, when  $v \geq 0$

$$\Psi(u, v) = \frac{1}{2} \int_u^{u+v} (u + v - w) \Phi''(w) dw \leq \Phi''(u) v^2.$$

The case  $v \leq 0$  is very similar, and we get finally that under  $(\mathcal{H}2')$ :

$$(5.11) \quad \Psi(u, v) \leq \Phi''(u) v^2 \quad \text{on} \quad \mathcal{I}^{(2)}.$$

On the other hand, if we assume only  $(\mathcal{H}2)$ , one can write:

$$\Psi(u, v) = \frac{1}{2} \int_u^{u+v} (u + v - w) \Phi''(w) dw \leq v(\Phi'(u + v) - \Phi'(u)),$$

which gives that under  $(\mathcal{H}2)$ :

$$(5.12) \quad \Psi(u, v) \leq v(\Phi'(u + v) - \Phi'(u)) \quad \text{on} \quad \mathcal{I}^{(2)}.$$

Thus, under  $(\mathcal{H}2)$  (resp.  $(\mathcal{H}2')$ ), we recover for example the well known  $(Df)^2/f$  (resp.  $Df D \log f$ ) Dirichlet forms when  $\Phi(x) = x \log x$ , as in [Wu00]. When  $\Phi(x) = x^2$ , both of them give  $2(Df)^2$ .

**Remark 9** (“L-1” and “L-2” forms of log-Sobolev inequalities). In continuous settings, the equivalence between  $L^1$  and  $L^2$  forms of the logarithmic Sobolev inequality, given respectively by

$$(5.13) \quad \forall f, \mathbf{Ent}_\mu(f^2) \leq c \mathbf{E}_\mu(\Gamma f)$$

and

$$(5.14) \quad \forall f > 0, \mathbf{Ent}_\mu(f) \leq \frac{1}{4} c \mathbf{E}_\mu\left(\frac{\Gamma f}{f}\right)$$

is a consequence of the chain rule (1.10) for  $\Gamma$ , which is itself a consequence of the diffusion property (1.11) for the associated infinitesimal generator. In discrete settings, the lack of chain rule destroys this equivalence, and actually, the simple Poisson measure satisfies the  $L^1$  form but not the  $L^2$  form, cf. for example [ABC<sup>+</sup>00, Chap. 1]. The fact that the semi-group of the simple Poisson point process is not hypercontractive was noticed by Surgailis eighteen years ago in [Sur84]. The concentration of measure consequences of the  $L^1$  form are only Poisson-like in discrete settings, which is completely logical, cf. [Led99].

**Remark 10** (Poissonian  $L^1$  logarithmic Sobolev inequality). Notice that the  $\Gamma$  operator (cf. (1.5)) associated to the pure jumps Lévy process with generator  $\mathbf{L}$  given by (5.1) is:

$$(5.15) \quad (\Gamma f)(x) := \frac{\lambda}{2} \int_{\mathbb{R}^d} |D_y f(x)|^2 d\nu(y).$$

When  $\Phi(x) = x \log x$ , the Dirichlet forms comparison (5.12) yields

$$\begin{aligned} \mathbf{Ent}_{\mathbf{P}_t}(f) &\leq \lambda t \mathbf{P}_t \left( \int_{\mathbb{R}^d} f^{-1} (D_y f)^2 d\nu(y) \right) \\ &= \lambda t \mathbf{P}_t \left( f^{-1} \int_{\mathbb{R}^d} (D_y f)^2 d\nu(y) \right) \\ &= 2t \mathbf{P}_t \left( \frac{\Gamma f}{f} \right), \end{aligned}$$

which is the  $L^1$  form (5.13) for measure  $\mathbf{P}_t$  and with constant  $2t$ .

**Remark 11** (Discussion on  $(\mathcal{H}2)$  and  $(\mathcal{H}2')$ ). Let  $\zeta_1(u, v) = \Phi''(u) v^2$  and  $\zeta_2(u, v) = (\Phi'(u + v) - \Phi'(u)) v$ , for  $(u, v) \in \mathcal{I}^{(2)}$ . Such functions are non negative since  $\Phi$  is convex. One has  $\Psi \leq \zeta_1$  when  $\Phi'$  is concave, and since

$$\nabla^2 \zeta_1(u, v) = \begin{pmatrix} \Phi'''(u) v^2 & 2 \Phi'''(u) v \\ 2 \Phi'''(u) v & 2 \Phi''(u) \end{pmatrix},$$

$(\mathcal{H}1)$  is equivalent to  $\Phi''' \Phi'' \geq 2 \Phi'''^2$  which is nothing else than the concavity of  $1/\Phi''$  when  $\Phi''$  is convex. One can check that  $\Phi(x) = x \log x$  and  $\Phi(x) = x^2$  are

in some sense extremal solutions of this O.D.I., up to affine additions. Similarly, one has  $\Psi \leq \zeta_2$  when  $\Phi$  is convex, and since  $\nabla^2 \zeta_2(u, v)$  is equal to

$$\begin{pmatrix} (\Phi'''(u+v) - \Phi'''(u))v & \Phi''(u+v) - \Phi''(u) + \Phi'''(u+v)v \\ \Phi''(u+v) - \Phi''(u) + \Phi'''(u+v)v & 2\Phi''(u+v) + \Phi'''(u+v)v \end{pmatrix},$$

$\zeta_2$  is convex in  $u$  but not necessarily in  $v$ . Finally, let us show why  $(\mathcal{H}2')$  implies  $(\mathcal{H}2)$ . Assume that  $(\mathcal{H}2')$  holds. The function  $\Psi$  is non negative since  $\Phi$  is convex, and it is convex in each variable  $u$  and  $v$  when  $\Phi$  and  $\Phi''$  are both convex. Moreover, it is bivariate convex when  $\Phi(x) = x \log x$  or  $\Phi(x) = x^2$ , as observed in the pretty paper [Wu00]. Actually, this bi-convexity holds in much more cases since

$$\nabla^2 \Psi(u, v) := \begin{pmatrix} \Phi''(u+v) - \Phi''(u) - \Phi'''(u)v & \Phi''(u+v) - \Phi''(u) \\ \Phi''(u+v) - \Phi''(u) & \Phi''(u+v) \end{pmatrix},$$

for which  $\text{Tr } \nabla^2 \Psi(u, v) \geq 0$  when  $\Phi$  and  $\Phi''$  are convex, and

$$\text{Det } \nabla^2 \Psi(u, v) = \Phi''(u) (\Phi''(u+v) - \Phi''(u)) - \Phi''(u+v) \Phi'''(u)v,$$

which is non negative since  $\Phi$  and  $\Phi''$  are convex and  $\Phi'$  is concave. Alternatively, one can use Gershgorin-Hadamard Theorem (cf. [HJ90, Sect. 6.1]) to see directly that  $\nabla^2 \Psi$  is non negative under  $(\mathcal{H}2')$ .

### 6. Links with Boltzmann-Shannon entropy

We assume here that  $\mathcal{I} = \mathbb{R}_+$ . Let  $f$  be a probability density function on  $\mathbb{R}^d$  with respect to the Lebesgue measure. When  $\Phi$  is convex, one can define the Shannon  $\Phi$ -entropy of  $f$  by:

$$(6.1) \quad \mathbf{H}^\Phi(f) := -\mathbf{Ent}^\Phi(f \, dx \mid dx) := -\mathbf{Ent}_{dx}^\Phi(f) := - \int_{\mathbb{R}^d} \widehat{\Phi}(f)(x) \, dx,$$

where  $\widehat{\Phi}(u) := \Phi(u) - \Phi(1)u$ . For any random vector  $X$  in  $\mathbb{R}^d$  with density  $f$  with respect to the Lebesgue measure, we denote  $\mathbf{H}^\Phi(X) := \mathbf{H}^\Phi(f)$ . This functional is invariant by translations and hence does not depend on the mean of  $f$ . We recover Shannon entropy (or negentropy)  $\mathbf{H}$  when  $\Phi(x) = x \log x$ , cf. [ABC<sup>+</sup>00, Chap. 10]. Actually,  $\mathbf{S} := -\mathbf{H}$  is also known as Shannon Information in Information Theory or Boltzmann entropy in Kinetic Gases Theory, see for example [Lie75] and [Vil03]. Similarly, one can define the Shannon  $\Phi$ -entropy of a discrete probability measure  $p_1\delta_{x_1} + \dots + p_n\delta_{x_n}$  by

$$(6.2) \quad \mathbf{H}^\Phi(p_1\delta_{x_1} + \dots + p_n\delta_{x_n}) := - \sum_{i=1}^n \widehat{\Phi}(p_i).$$

Since  $x \in \mathbb{R}^n \mapsto \widehat{\Phi}(x_1) + \dots + \widehat{\Phi}(x_n)$  is convex,  $\mathbf{H}^\Phi$  is a concave functional on the simplex

$$\{(p_1, \dots, p_n) \in \mathbb{R}_+ \times \dots \times \mathbb{R}_+, p_1 + \dots + p_n = 1\}.$$



At fixed  $n$ , it achieves its minimum 0 for Dirac measures, which are the extremal points of the simplex above, and its maximum  $-n \widehat{\Phi}(1/n)$  for the uniform probability measure by convexity. Notice that the continuous version is not always non-negative.

The important sub-additivity of Shannon entropy  $\mathbf{H}$  states that for any random vector  $(X_1, \dots, X_n)$  with an absolute continuous law with respect to the Lebesgue measure:

$$(6.3) \quad \mathbf{H}((X_1, \dots, X_n)) \leq \mathbf{H}(X_1) + \dots + \mathbf{H}(X_n),$$

with equality if and only if  $X_1, \dots, X_n$  are independent. Such a property relies on the non-negativity of Kullback-Leibler relative entropy and on the basic additivity of the logarithm:  $\log(ab) = \log a + \log b$ :

$$0 \leq \mathbf{Ent}(\mathcal{L}((X_1, X_2)) | \mathcal{L}(X_1) \otimes \mathcal{L}(X_2)) = \mathbf{H}((X_1, X_2)) - \mathbf{H}(X_1) - \mathbf{H}(X_2).$$

Notice that this sub-additivity is different from the one related to Kullback-Leibler relative entropy (3.2) since Shannon entropy is opposite in sign and based on the Lebesgue measure which is not a probability measure. This fact was explained in [ABC<sup>+</sup>00, Chap. 10]. Actually, such a sub-additivity property is not related to the Lebesgue measure, as noticed in [Cha03], and one can show that for any positive measure  $\mu = \otimes_{i=1}^n \mu_i$  on a product space and any non-negative real valued integrable function  $f$ :

$$(6.4) \quad \mathbf{Ent}_\mu(f) \geq \mathbf{Ent}_{\mu_1} \left( \int f d\mu_{\setminus 1} \right) + \dots + \mathbf{Ent}_{\mu_n} \left( \int f d\mu_{\setminus n} \right),$$

where  $\mu_{\setminus i} := \mu_1 \times \dots \times \mu_{i-1} \times \mu_{i+1} \times \dots \times \mu_n$ , with equality if and only if  $f$  is a tensor product function. Beware that (3.2) and (6.4) are opposite. We ignore if the sub-additivity property (6.3) of Shannon entropy can be generalised to any convex  $\Phi$ . Actually, the functional  $f \mapsto \mathbf{H}^\Phi(f)$  is concave and formally, the Fréchet derivatives are given by:

$$(D\mathbf{H}^\Phi)(f)(h) = - \int_{\mathbb{R}^d} \Phi'(f) h dx \quad \text{and} \quad (D^2\mathbf{H}^\Phi)(f)(h, h) = - \int_{\mathbb{R}^d} \Phi''(f) h^2 dx.$$

Therefore, Shannon like  $\Phi$ -entropy  $\mathbf{H}^\Phi$  achieves its maximum under the linear constraint  $\mathbf{E}(W(X)) = \int_{\mathbb{R}^d} W(x) f(x) dx = c$  for probability densities functions of the form:

$$f_W := (\widehat{\Phi})'^{-1}(-\lambda - \beta W),$$

where  $(\lambda, \beta) \in \mathbb{R}^2$  is chosen in such a way that the constraint is fulfilled and that  $f_W$  is a probability density function with respect to the Lebesgue measure. Notice that  $(\widehat{\Phi})'^{-1}$  is the derivative of the Young conjugate of  $\widehat{\Phi}$ . The Gaussian maximum of Shannon entropy  $\mathbf{H}$  at fixed covariance appears as a particular case, for which  $\Phi(x) = x \log x$ ,  $W(x) = |x|^2$ ,  $(\widehat{\Phi})'^{-1}(y) = \exp(y - 1)$ ,  $\beta = d/(2c)$  and  $\lambda = Z_{c,d}$  is the Gaussian normalising constant. More generally, we recovered as a particular case the famous “principle of maximum entropy”

which states that Boltzmann-Shannon entropy is maximised under linear constraint by Boltzmann-Gibbs measures. It is tempting and quite natural to ask if  $\mathbf{H}^\Phi$  shares more common properties with Shannon entropy  $\mathbf{H}$ . As we have seen, the convex conjugate functional will play a role. This question is partly answered in [TV93], [BTT86], [BR82a] and [BR82c] and references therein. Entropy like measure of information are still actively explored, and one can find recent results in [Dra04] and in Flemming Topsøe papers for example.

**Final words.** One may retain that the classical relative entropy  $\mathbf{Ent}(\nu | \mu)$  where  $\mu$  and  $\nu$  are positive Borel measures possesses a lot of properties coming from the very particular base function  $\Phi(x) = \Theta(x) := x \log x$ :

- $\Theta$  is strictly convex, and thus  $\mathbf{Ent}_\mu(f) = 0$  iff  $f$  is constant  $\mu$ -a.s.;
- $\Theta(0) = \Theta(1) = 0$  and thus  $\mathbf{H}(X)$  vanishes when  $X$  is constant;
- $1/\Theta''$  is affine and hence concave and thus  $f \mapsto \mathbf{Ent}_\mu(f)$  is convex;
- $\Theta(ab) = b\Theta(a) + a\Theta(b)$  and thus  $f \mapsto \mathbf{Ent}_\mu(f)$  is 1-homogeneous;
- The Young conjugate  $\Theta^*(u) = e^{u-1}$  is monotone and  $\Theta^{*'} = \Theta^*$ .

Recall that  $\Theta^*(u) := \int_0^u \Theta'^{-1}(x) dx$ . Some of these properties are well imitated by  $x \mapsto |x|^p$  with  $p \in (1, 2]$ , which is exactly the family of simple power convex functions between  $x \mapsto \Theta(x)$  and  $x \mapsto x^2$  and the latter is for some aspects the “simplest” one. Some results involving  $\mathbf{Ent}_\mu$  rely only on few properties of  $\Theta$  whereas other ones rely on all of them, and this fact obviously puts some limits on the possible generalisations.

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