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Global in time behavior of viscous surface waves: horizontally periodic motion

By

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1. Introduction

This paper is concerned with the equations of motion of a viscous incompressible fluid bounded above by an atmosphere of constant pressure and below by a fixed plane extending horizontally. The flow is governed by the Navier-Stokes equations with appropriate boundary conditions (see [17]). The gravity is the only external force. This problem is treated in [3]. There, by including the effect of surface tension on the upper free surface, Beale showed that there exists a unique global solution to the problem for a sufficiently small initial data with certain compatibility conditions. In the report [4] Beale and Nishida announced that the above solution in [3] decays in time with an algebraic decay rate. Tani showed in [15] the local in time existence result for arbitrary initial data. For the compressible case Tani and Tanaka showed the global in time existence result for small initial data in [14]. For other results see [9], [15], [14] and their references.

In this paper we assume that the motion of fluid is horizontally periodic and that spatial mean of the motion of unknown free surface over the space period is equal to zero. Under these assumptions we show that global in time solution to this problem with sufficiently small initial data decays exponentially. This problem is treated by Padula and Solonnikov in [10]. They showed the decay in time of the L^2 norm of the solutions to the problem under the assumptions that the global in time solutions exist and are bounded in time for certain norms.

To formulate our problem we take the mean depth $b > 0$ as the unit of length. We take $U_0 = \sqrt{gb}$ and $T_0 = b/U_0$ as units of velocity and time respectively. Here g is the acceleration of gravity. We denote the velocity field by $u(x, y, t)$ and the scalar pressure by $\bar{p}(x, y, t)$. Then the equations of motion of fluid is now written as follows:

$$\partial_t u + (u, \nabla)u - \nu\Delta u + \nabla\bar{p} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix},$$
$$\operatorname{div} u = 0 \qquad \text{in } \Omega(t),\ t > 0,$$

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where $\Omega(t) = \{(x', y);\ x' \in \mathbb{T}^2,\ -1 < y < \eta(x', t)\}$. Here $\mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$. The coefficient ν is the reciprocal of the Reynolds number. The unknown free surface is denoted by $y = \eta(x', t)$, $x' \in \mathbb{T}^2$, $t > 0$. On the free surface we impose the kinematic boundary condition

$$\partial_t \eta = u_3 - (\partial_1 \eta) u_1 - (\partial_2 \eta) u_2 \quad \text{on } y = \eta(x', t),\ x' \in \mathbb{T}^2,\ t > 0.$$

The balance of stress tensor at the free surface is the following

$$\bar{p} n_j - \nu \left(\partial_j u_k + \partial_k u_j\right) n_k + \sigma \nabla_F \left(\left(1 + |\nabla_F \eta|^2\right)^{-\frac{1}{2}} \nabla_F \eta \right) n_j = p_{atm} n_j.$$

Here $n = (n_1, n_2, n_3) = \left(1 + |\nabla_F \eta|^2\right)^{-\frac{1}{2}} (-\partial_1 \eta, -\partial_2 \eta, 1)$ is the outward unit normal to the free surface and σ is the nondimensionalized coefficient of surface tension. See [17] for this condition. It is understood to take a sum over repeated indeces. p_{atm} is the atmospheric pressure assumed to be constant. $\nabla_F = (\partial_1, \partial_2)$ denotes the horizontal gradient. The condition on the bottom is

$$u = 0 \quad \text{on } y = -1.$$

If we assume $u(x', y, t) \equiv 0$ and $\eta \equiv 0$, then the equations and the boundary conditions are satisfied by setting $\bar{p} = p_{atm} - y$. To consider perturbations from this equilibrium state we write the pressure as

$$\bar{p}(x', y, t) = p_{atm} - y + p(x', y, t).$$

The equations of motion become

$$\begin{aligned} &\partial_t u + (u, \nabla) u - \nu \Delta u + \nabla p = 0, \\ &\operatorname{div} u = 0 \qquad \text{in } \Omega(t),\ t > 0, \end{aligned}$$

and the balance of stress tensor at the free surface becomes

$$\begin{aligned} &(-\eta(x', t) + p) n_j - \nu (\partial_j u_k + \partial_k u_j) n_k \\ &\quad + \sigma \nabla_F \left(\left(1 + |\nabla_F \eta|^2\right)^{-\frac{1}{2}} \nabla_F \eta \right) n_j = 0 \quad \text{on} \quad y = \eta(x', t). \end{aligned}$$

A solution is uniquely determined by specifying the initial data

$$\eta(x', 0) = \eta_0(x'),\ x' \in \mathbb{T}^2 \text{ and } u(x', y, 0) = u_0(x', y),\ -1 < y < \eta_0(x')$$

subject to certain compatibility conditions and smallness assumption.

As in [3], to show existence and decay of global in time solutions, we transform the problem to the one on the equilibrium domain $\Omega = \{(x', x_3); x' \in \mathbb{T}^2,\ -1 < x_3 < 0\}$ using the unknown free surface $\eta(x', t)$. For each $t \geq 0$ we define $\Theta\ : \Omega \to \Omega(t)$ by

$$\Theta(x_1, x_2, x_3 : t) = (x_1, x_2, (\tilde{\eta} + 1) x_3 + \tilde{\eta}), \quad -1 < x_3 < 0.$$

Here $\tilde\eta$ is an extension of η to $\mathbb{T}^2 \times (-\infty, 0)$ defined by

$$\tilde\eta(x', x_3, t) = \sum_{\xi' \in \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{\eta^{(\xi')}}{1 + (|\xi'| x_3)^2} \exp(i\xi' \cdot x'), \tag{1.1}$$

where $\eta^{(\xi')}$ is the coefficient of the Fourier series expansion. The velocity u on $\Omega(t)$ is given by $u_\alpha = \dfrac{1}{J}\theta_{\alpha\beta} v_\beta$ in terms of v defined on Ω. $(\theta_{\alpha\beta})$ is the Jacobian matrix of Θ and J is the Jacobian $\det(\theta_{\alpha\beta})$. It is easily seen that u satisfies the solenoidal condition in $\Omega(t)$ if and only if v satisfies the same condition in Ω. We set $q(x,t) = p(\Theta(x,t))$. From these definitions of v and q we derive the equations for η, v and q:

$$\partial_t \eta - v_3 = 0 \quad \text{on } S_F = \{(x', 0) \in \overline{\Omega}\ ;\ x' \in \mathbb{T}^2\},\ t > 0, \tag{1.2}$$

$$\partial_t v - \nu\, \Delta v + \nabla q = F_0 + Q\nabla q, \quad \operatorname{div} v = 0 \quad \text{in} \quad \Omega,\ t > 0. \tag{1.3}$$

The α-th component (α=1,2,3) of the right hand side of (1.3) is written as follows:

(1.4)

$$\begin{aligned}
F_{0,\alpha} = {} & \frac{1}{J}\partial_3((x_3+1)\tilde\eta_t)v_\alpha - \frac{\delta_{\alpha 3}}{J}\partial_k((x_3+1)\tilde\eta_t)v_k + \frac{1}{J}((x_3+1)\tilde\eta_t)\partial_3 v_\alpha \\
& - \frac{1}{J^2}((x_3+1)\tilde\eta_t)\partial_3^2((x_3+1)\tilde\eta)v_\alpha + \frac{\delta_{\alpha 3}}{J^2}((x_3+1)\partial_t\tilde\eta)\partial_3\partial_k((x_3+1)\tilde\eta)v_k \\
& + \nu\Bigg[-\frac{2}{J}\partial_c((1+x_3)\tilde\eta)\partial_3\partial_c v_\alpha + \frac{1}{J^2}\partial_c((1+x_3)\tilde\eta)\partial_c((1+x_3)\tilde\eta)\partial_3^2 v_\alpha \\
& - 2\frac{\zeta_{ce}\zeta_{de}}{J}\partial_c\partial_3((1+x_3)\tilde\eta)\partial_d v_\alpha + 2\zeta_{\alpha 3}\zeta_{ce}\zeta_{de}\partial_c\partial_k((1+x_3)\tilde\eta)\partial_d v_k \\
& - \zeta_{ce}\partial_c\left(\frac{1}{J}\partial_e((1+x_3)\tilde\eta)\right)\partial_3 v_\alpha + \zeta_{ce}\partial_c\left(\frac{1}{J}\partial_e((1+x_3)\tilde\eta)\right) \\
& \times \frac{1}{J}\partial_3^2((1+x_3)\tilde\eta)v_\alpha + \zeta_{\alpha 3}\zeta_{ce}\partial_c\zeta_{de}\partial_d\partial_k((1+x_3)\tilde\eta)v_k \\
& + \zeta_{ce}\zeta_{de}\Bigg(-\frac{1}{J}\partial_c\partial_d\partial_3((1+x_3)\tilde\eta)\ \delta_{\alpha k} + \frac{2}{J^2}\partial_c\partial_3((1+x_3)\tilde\eta) \\
& \times \partial_d\partial_3((1+x_3)\tilde\eta)\delta_{\alpha k} - \frac{2\zeta_{\alpha 3}}{J}\partial_d\partial_3((1+x_3)\tilde\eta)\partial_c\partial_k((1+x_3)\tilde\eta) \\
& + \zeta_{\alpha 3}\partial_c\partial_d\partial_k((1+x_3)\tilde\eta)\Bigg)v_k\Bigg] - \frac{1}{J}v_\gamma\partial_\gamma v_\alpha - v_c v_d \zeta_{\alpha e}\partial_c\left(\frac{1}{J}\theta_{ed}\right),
\end{aligned}$$

(1.5)

$$\begin{aligned}
(Q\nabla q)_\alpha &= (\delta_{\alpha e} - J\zeta_{\alpha c}\zeta_{ec})\partial_e q \\
&= \Bigg(-\partial_3((1+x_3)\tilde\eta)\delta_{\alpha e} + \delta_{\alpha 3}\partial_e((1+x_3)\tilde\eta) \\
&\qquad + \delta_{e3}\partial_\alpha((1+x_3)\tilde\eta) - \frac{1}{J}\delta_{\alpha 3}\delta_{e3}\partial_c((1+x_3)\tilde\eta)\partial_c((1+x_3)\tilde\eta)\Bigg)\partial_e q,
\end{aligned}$$

where $(\zeta_{\alpha k})$ is the inverse of the Jacobian matrix $d\Theta = (\theta_{\alpha k})$. Since (1.2) must hold, the terms $\partial_t \eta$ in (1.4) are replaced by the restriction of v_3 to S_F.

The boundary condition on the bottom is

$$v = 0 \quad \text{on} \quad S_B = \{(x', -1) \in \overline{\Omega}\ ;\ x' \in \mathbb{T}^2\},\ t > 0. \tag{1.6}$$

The conditions on the upper boundary $S_F = \left\{(x', 0) \in \overline{\Omega}\ ;\ x' \in \mathbb{T}^2\right\}$ are written as follows

$$\partial_\alpha v_3 + \partial_3 v_\alpha = F_\alpha, \quad \alpha = 1, 2, \tag{1.7}$$

$$q - 2\nu\, \partial_3 v_3 - \eta + \sigma \Delta_F\, \eta = F_3 \quad \text{on } S_F,\ t > 0, \tag{1.8}$$

where

$$\begin{aligned}
F_1 = {} & \Bigg(\delta_{k1}\delta_{3\ell} \frac{1}{J} \partial_3((1+x_3)\tilde{\eta}) + \delta_{k3}\delta_{3\ell} \frac{1}{J^2} \partial_1((1+x_3)\tilde{\eta}) \\
& - \delta_{k1} \frac{1}{J} \partial_\ell((1+x_3)\tilde{\eta}) + \delta_{k3} \frac{1}{J^2} \partial_1((1+x_3)\tilde{\eta}) \partial_\ell((1+x_3)\tilde{\eta}) \Bigg) \partial_k v_\ell \\
& + \partial_3((1+x_3)\tilde{\eta}) \frac{1+J}{J^2} \partial_3 v_1 + \frac{\zeta_{k3}}{J^2} \partial_k \partial_3((1+x_3)\tilde{\eta}) v_1 \\
& - \zeta_{k1} \left(-\frac{1}{J^2} \partial_k \partial_3((1+x_3)\tilde{\eta})(\delta_{3\ell} + \partial_\ell((1+x_3)\tilde{\eta})) + \frac{1}{J} \partial_k \partial_\ell((1+x_3)\tilde{\eta}) \right) v_\ell \\
& + 2\partial_1 \eta \left(\zeta_{k1} \partial_k \left(\frac{1}{J} \theta_{1\ell} v_\ell \right) - \zeta_{k3} \partial_k \left(\frac{1}{J} \theta_{3\ell} v_\ell \right) \right) \\
& + \partial_2 \eta \left(\zeta_{k1} \partial_k \left(\frac{1}{J} \theta_{2\ell} v_\ell \right) + \zeta_{k2} \partial_k \left(\frac{1}{J} \theta_{1\ell} v_\ell \right) \right) \\
& + (\partial_1 \eta)^2 \left(\zeta_{k1} \partial_k \left(\frac{1}{J} \theta_{3\ell} v_\ell \right) + \zeta_{k3} \partial_k \left(\frac{1}{J} \theta_{1\ell} v_\ell \right) \right) \\
& + (\partial_1 \eta)(\partial_2 \eta) \left(\zeta_{k2} \partial_k \left(\frac{1}{J} \theta_{3\ell} v_\ell \right) + \zeta_{k3} \partial_k \left(\frac{1}{J} \theta_{2\ell} v_\ell \right) \right),
\end{aligned}$$

(1.9)

$$\begin{aligned}
F_2 = {} & \Bigg(\delta_{k2}\delta_{3\ell} \frac{1}{J} \partial_3\left((1+x_3)\tilde{\eta}\right) + \delta_{k3}\delta_{3\ell} \frac{1}{J^2} \partial_2((1+x_3)\tilde{\eta}) \\
& - \delta_{k2} \frac{1}{J} \partial_\ell((1+x_3)\tilde{\eta}) + \delta_{k3} \frac{1}{J^2} \partial_2((1+x_3)\tilde{\eta}) \partial_\ell((1+x_3)\tilde{\eta}) \Bigg) \partial_k v_\ell \\
& + \partial_3((1+x_3)\tilde{\eta}) \frac{1+J}{J^2} \partial_3 v_2 + \frac{\zeta_{k3}}{J^2} \partial_k \partial_3((1+x_3)\tilde{\eta}) v_2 \\
& - \zeta_{k2} \left(-\frac{1}{J^2} \partial_k \partial_3((1+x_3)\tilde{\eta})(\delta_{3\ell} + \partial_\ell((1+x_3)\tilde{\eta})) + \frac{1}{J} \partial_k \partial_\ell((1+x_3)\tilde{\eta}) \right) v_\ell \\
& + 2\partial_2 \eta \left(\zeta_{k2} \partial_k \left(\frac{1}{J} \theta_{2\ell} v_\ell \right) - \zeta_{k3} \partial_k \left(\frac{1}{J} \theta_{3\ell} v_\ell \right) \right) \\
& + \partial_1 \eta \left(\zeta_{k1} \partial_k \left(\frac{1}{J} \theta_{2\ell} v_\ell \right) + \zeta_{k2} \partial_k \left(\frac{1}{J} \theta_{1\ell} v_\ell \right) \right)
\end{aligned}$$

$$
+ (\partial_2\eta)^2 \left(\zeta_{k2}\partial_k \left(\frac{1}{J}\theta_{3\ell}v_\ell \right) + \zeta_{k3}\partial_k \left(\frac{1}{J}\theta_{2\ell}v_\ell \right) \right)
$$
$$
+ (\partial_1\eta)(\partial_2\eta) \left(\zeta_{k1}\partial_k \left(\frac{1}{J}\theta_{3\ell}v_\ell \right) + \zeta_{k3}\partial_k \left(\frac{1}{J}\theta_{1\ell}v_\ell \right) \right)
$$

and

(1.10)

$$
\begin{aligned}
F_3 = {} & \frac{2\nu}{1+|\nabla_F\,\eta|^2}\Bigg\{ - |\nabla_F\,\eta|^2\, \partial_3 v_3 \\
& + \Bigg(-\delta_{k3}\delta_{3\ell}\frac{1}{J}\partial_3((1+x_3)\tilde\eta) - \delta_{k3}\delta_{3\ell}\frac{1}{J^2}\partial_3((1+x_3)\tilde\eta) \\
& + \frac{1}{J}\delta_{k3}\partial_\ell((1+x_3)\tilde\eta) - \delta_{k3}\frac{1}{J^2}\partial_3((1+x_3)\tilde\eta)\partial_\ell((1+x_3)\tilde\eta) \Bigg)\, \partial_k v_\ell \\
& + \zeta_{k3} \left(-\frac{1}{J^2}\partial_k\partial_3((1+x_3)\tilde\eta)(\delta_{3\ell} + \partial_\ell((1+x_3)\tilde\eta)) + \frac{1}{J}\partial_k\partial_\ell((1+x_3)\tilde\eta) \right) v_\ell \\
& + (\partial_1\eta)^2\zeta_{k1}\partial_k \left(\frac{1}{J}\theta_{1\ell}v_\ell \right) + (\partial_2\eta)^2\zeta_{k2}\partial_k \left(\frac{1}{J}\theta_{2\ell}v_\ell \right) \\
& + (\partial_1\eta)(\partial_2\eta) \left(\zeta_{k1}\partial_k \left(\frac{1}{J}\theta_{2\ell}v_\ell \right) + \zeta_{k2}\partial_k \left(\frac{1}{J}\theta_{1\ell}v_\ell \right) \right) \\
& - \partial_1\eta \left(\zeta_{k1}\partial_k \left(\frac{1}{J}\theta_{3\ell}v_\ell \right) + \zeta_{k3}\partial_k \left(\frac{1}{J}\theta_{1\ell}v_\ell \right) \right) \\
& - \partial_2\eta \left(\zeta_{k2}\partial_k \left(\frac{1}{J}\theta_{3\ell}v_\ell \right) + \zeta_{k3}\partial_k \left(\frac{1}{J}\theta_{2\ell}v_\ell \right) \right) \Bigg\} - \sigma \left\{ \frac{1}{\sqrt{1+|\nabla_F\,\eta|^2}} - 1 \right\} \\
& \times \varDelta_F\,\eta + \frac{\sigma}{\left(1+|\nabla_F\,\eta|^2\right)^{\frac{3}{2}}} \left\{ (\partial_1\eta)^2\partial_1^2\eta + 2(\partial_1\eta)(\partial_2\eta)\partial_1\partial_2\eta + (\partial_2\eta)^2\partial_2^2\eta \right\}.
\end{aligned}
$$

In deriving the nonlinear terms above we have used the explicit forms of the Jacobian matrix and its inverse

$$
\theta_{\alpha\beta} = \delta_{\alpha\beta} + \delta_{3\alpha}\partial_\beta((1+x_3)\tilde\eta), \quad \zeta_{\alpha\beta} = \delta_{\alpha\beta} - \frac{\delta_{3\alpha}}{J}\partial_\beta((1+x_3)\tilde\eta).
$$

We now state our results in this paper.

Theorem 1.1. *Suppose* $3 < \ell < \dfrac{7}{2}$. *There exists a* $\delta > 0$ *such that, if the initial data* η_0, v_0 *satisfy*

(1.11)
$$
\int_{\mathbb{T}^2} \eta_0(x')dx' = 0,
$$
$$
|v_0|_{H^{\ell-1}(\Omega)} + |\eta_0|_{H_0^{\ell-\frac{1}{2}}(\mathbb{T}^2)} \le \delta,
$$

and the compatibility conditions

$$v_0 = 0 \quad \text{on } S_B,\ \operatorname{div} v_0 = 0 \quad \text{in } \Omega, \tag{1.12}$$

$$\partial_\alpha v_{0,3} + \partial_3 v_{0,\alpha} = F_\alpha(\eta_0, v_0) \quad \text{on } S_F, \alpha = 1, 2, \tag{1.13}$$

then there is a unique solution η, v, q *to the problem* (1.2)*,* (1.3)*,* (1.6)*–*(1.8) *with* $v(0) = v_0$*,* $\eta(0) = \eta_0$*. Moreover, they satisfy*

$$\begin{aligned}
&\eta \in K_0^{\ell+\frac{1}{2}}\left(\mathbb{T}^2 \times (0,\infty)\right),\\
&v \in K^{\ell}(\Omega \times (0,\infty)),\\
&\nabla q \in K^{\ell-2}(\Omega \times (0,\infty)),\ \ q|_{S_F} \in K^{\ell-\frac{3}{2}}\left(\mathbb{T}^2 \times (0,\infty)\right).
\end{aligned}$$

We give in the next section the definition of function spaces used in this paper. To prove this theorem we follow [3], but, to obtain the regularity of η we use different method that will be discussed in Section 3. We also have the higher regularity for the above solution similarly as in [3].

Theorem 1.2. *Take* $T_1 > 0$ *and an integer* $k > 0$ *arbitrarily. Then there exists* $\delta_0 > 0$ *such that, if the initial data* η_0*,* v_0 *satisfy the assumptions in Theorem* 1.1 *and, further,*

$$|v_0|_{H^{\ell-1}(\Omega)} + |\eta_0|_{H_0^{\ell-\frac{1}{2}}(\mathbb{T}^2)} \le \delta_0,$$

then it holds that

$$\begin{aligned}
&\eta \in K_0^{\ell+k+\frac{1}{2}}\left(\mathbb{T}^2 \times (T_1,\infty)\right),\ \ v \in K^{\ell+k}(\Omega \times (T_1,\infty)),\\
&\nabla q \in K^{\ell-2+k}(\Omega \times (T_1,\infty)),\ \ q|_{S_F} \in K^{\ell+k-\frac{3}{2}}\left(\mathbb{T}^2 \times (T_1,\infty)\right).
\end{aligned}$$

As in [3] the full nonlinear problem can be written in the form $L\boldsymbol{z} = F(\boldsymbol{z})$, where $\boldsymbol{z} = (\eta, v, q)$, $F = (0, F_0, F_1, F_2, F_3)$ and L is a linear operator consisting of the left hand sides of (1.2), (1.3), (1.7) and (1.8). The proof of existence result is given by the usual fixed point argument $\boldsymbol{z} = L^{-1}F(\boldsymbol{z})$. So we have to solve the linearized problem with given right hand sides and to obtain the global in time estimate for the solution of the linearized problem.

In Section 2 we introduce the function spaces and the notations used in this paper and state preliminary lemmas and proposition. In Section 3 we study the auxiliary linear problems in the half space $\Omega_\infty = \mathbb{T}^2 \times (0,\infty)$ with the several boundary conditions following the method in [13]. In Section 4 we consider the model problem in the half space including the unknown free surface. The solvability of the linear nonstationary problem is shown in Section 5. There we construct the resolvent operator and obtain its uniform estimate, which is crucial for the solvability of the full nonlinear problem. In Section 6, deriving the energy inequality, we show

Theorem 1.3. *Let* η, v, q *be the global in time solution obtained in Theorem* 1.2*. Then there exist the constants* $C > 0$ *and* $\gamma > 0$ *such that*

$$|v(t)|_{H^2(\Omega)} + |\eta(t)|_{H^3(\mathbb{T}^2)} \le C \exp(-\gamma t), \quad t > 0.$$

Our results described above are concerned with the problem transformed to the equilibrium domain Ω by use of the unknown free surface η. As noted in page 310 in [3], η obtained in Theorem 1.1 belongs to $C^1\left(\mathbb{T}^2\right)$ for each $t > 0$, so that we can convert the solution of Theorem 1.1 into the one in the domain $\Omega(t)$ depending on η.

2. Notations and preliminaries

Let $\mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$ be the two-dimensional torus and set $\Omega = \mathbb{T}^2 \times (-1,0)$. Let $\ell \geq 0$. We denote by $H^\ell(\Omega)$ the Sobolev space of functions that are periodic in $x' = (x_1, x_2)$ with period 2π and whose derivatives in the sense of distribution to order ℓ are in L^2 if ℓ is a nonnegative integer. Otherwise we adopt the usual generalization. See, for example, [7]. $H^\ell\left(\mathbb{T}^2\right)$ denotes the Sobolev space of the order ℓ consisting of 2π periodic functions. Identifying $S_F = \partial\Omega \cap \{x_3 = 0\}$ with $\mathbb{T}^2$, we regard $H^\ell(S_F)$ as $H^\ell\left(\mathbb{T}^2\right)$. We introduce the space

$$K^\ell(\Omega \times (0,T)) = H^0\left(0,T; H^\ell(\Omega)\right) \cap H^{\frac{\ell}{2}}\left(0,T; H^0(\Omega)\right).$$

The spaces on the right hand side are Sobolev spaces of $H^\ell(\Omega)$ valued and $H^0(\Omega)$ valued functions on an interval $(0,T)$. We write $(0,\infty)$ as $\mathbb{R}^+$. We use the same notations for the spaces of $\mathbb{C}$ valued and $\mathbb{C}^3$ valued functions. From the context it should be clear which valued function spaces we refer to.

To treat (1.2) we need the following function space

$$K_0^{\ell+\frac{1}{2}}\left(\mathbb{T}^2 \times (0,T)\right) = H^0\left(0,T; H_0^{\ell+\frac{1}{2}}\left(\mathbb{T}^2\right)\right) \cap H^{\frac{\ell+\frac{1}{2}}{2}}\left(0,T; H_0^0\left(\mathbb{T}^2\right)\right),$$

where $\ell \geq 1$. We set, for $s \geq 0$,

$$H_0^s(\mathbb{T}^2) = \left\{\phi \in H^s(\mathbb{T}^2); \int_{\mathbb{T}^2} \phi(x')dx' = 0\right\}.$$

To avoid the complicated case for the compatibility at $t = 0$ we assume that ℓ is not a half integer throughout this paper. As a norm of $K^\ell(\Omega \times (0,T))$ we can adopt the one given by

$$|f|_{K^\ell} = \left(\int_0^T \left(|f(t)|_\ell^2 + \left|\frac{\partial^{[\frac{\ell}{2}]} f(t)}{\partial t^{[\frac{\ell}{2}]}}\right|_0^2\right) dt + \left|\frac{\partial^{[\frac{\ell}{2}]} f}{\partial t^{[\frac{\ell}{2}]}}\right|^2_{\frac{\ell}{2}-[\frac{\ell}{2}];T}\right)^{\frac{1}{2}},$$

where

$$\left|\frac{\partial^{[\frac{\ell}{2}]} f}{\partial t^{[\frac{\ell}{2}]}}\right|^2_{\frac{\ell}{2}-[\frac{\ell}{2}];T} = \int_0^T \int_0^T \frac{\left|\frac{\partial^{[\frac{\ell}{2}]} f(t_1)}{\partial t^{[\frac{\ell}{2}]}} - \frac{\partial^{[\frac{\ell}{2}]} f(t_2)}{\partial t^{[\frac{\ell}{2}]}}\right|_0^2}{|t_1 - t_2|^{1+2\left(\frac{\ell}{2}-[\frac{\ell}{2}]\right)}} dt_1 dt_2.$$

We define the norm of $K^\ell\left(\mathbb{T}^2 \times (0,T)\right)$ similarly. See [7] for other norms equivalent to these.

We review some properties of these function spaces.
If $u \in K^{\ell}(\Omega \times (0,T))$, then $\partial_t^k \partial_x^\alpha u \in K^{\ell-2k-|\alpha|}(\Omega \times (0,T))$ for $|\alpha| + 2k < \ell$, where $\partial_x^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3}$ and $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$. For convenience we sometimes use y to denote x_3 and set $\partial_y = \partial_3$. For $0 \le 2k < \ell - 1$, we have traces $\partial_t^k u(\cdot, 0) \in H^{\ell-2k-1}(\Omega)$ and $K^{\ell}(\Omega \times (0,T)) \subseteq C^k\left(0,T; H^{\ell-2k-1}(\Omega)\right)$. See Chapter 4 in [7].
We denote by $K^{\ell}_{(0)}(\Omega \times (0,T))$ the subspace of functions $u \in K^{\ell}(\Omega \times (0,T))$ with $\partial_t^k u(\cdot, 0) = 0$ for $2k < \ell - 1$. $K^{\ell+\frac{1}{2}}_{0,(0)}\left(\mathbb{T}^2 \times (0,T)\right)$ is defined similarly.

Lemma 2.1. *If $f \in K^{\ell}(\Omega \times (0,T))$ with $\ell > \frac{5}{2}$ and $g \in K^{s}(\Omega \times (0,T))$ with $\ell \ge s \ge 0$, then $fg \in K^{s}(\Omega \times (0,T))$ and $|fg|_{K^s} \le C\,|f|_{K^\ell}\,|g|_{K^s}$.*

We can prove this lemma in just the same way as in Lemma 5.1 in [3] using the Fourier series expansion instead of the Fourier transform. In order to show the exponential decay in time of solutions we need

Lemma 2.2. i) *Suppose $\ell > \frac{3}{2}$ and $\ell \ge s \ge 0$. If $f \in H^{\ell}(\Omega)$ and $g \in H^{s}(\Omega)$, then $fg \in H^{s}(\Omega)$ and $|fg|_{H^s} \le C\,|f|_{H^\ell}\,|g|_{H^s}$.*
ii) *Suppose $\ell > 1$ and $\ell \ge s \ge 0$. If $f \in H^{\ell}(\mathbb{T}^2)$ and $g \in H^{s}(\mathbb{T}^2)$, then $fg \in H^{s}(\mathbb{T}^2)$ and $|fg|_{H^s} \le C\,|f|_{H^\ell}\,|g|_{H^s}$.*

See Lemma 2.5 in [3] or [11] for the proof of this lemma.

Later we formulate our problem in the form of an evolution equation in some function spaces by applying to the equations of motion the projection orthogonal to the space of gradients as in [16]. Since the boundary conditions on S_F is different from the adherence condition, we introduce the projection orthogonal to the space

$$\mathcal{G}^0 = \left\{\nabla\phi\ ;\ \phi \in H^1(\Omega),\ \phi = 0 \text{ on } S_F\right\}.$$

Define the orthogonal projection by P^0 from $L^2(\Omega)$ to $\left(\mathcal{G}^0\right)^{\perp}$. We briefly review the properties of P^0. See Lemma 3.1 in [2].

Lemma 2.3. *Let $\ell \ge 0$. P^0 is a bounded operator on $H^{\ell}(\Omega)$ and $K^{\ell}(\Omega \times (0,T))$. If $\phi \in H^{\ell+1}(\Omega)$, then $P^0(\nabla\phi) = \nabla\psi$, where ψ satisfies*

$$(2.1)\qquad \Delta\psi = 0 \quad \text{in } \Omega, \quad \psi = \phi \quad \text{on } S_F, \quad \partial_n\psi = 0 \quad \text{on } S_B.$$

Proof. Suppose $v \in H^s(\Omega)$, $s \ge 1$. Then $(I - P^0)v = \nabla\varpi$, where ϖ is a weak solution of

$$(2.2)\qquad \Delta\varpi = \operatorname{div} v \quad \text{in } \Omega, \quad \varpi = 0 \quad \text{on } S_F, \quad \partial_n\varpi = v\cdot n \quad \text{on } S_B.$$

By the regularity theory of the elliptic boundary value problem, it holds that

$$|\varpi|_{H^{s+1}} \le C\left\{|\operatorname{div} v|_{H^{s-1}} + |v\cdot n|_{H^{s-\frac{1}{2}}}\right\}.$$

By interpolation we see the boundedness of P^0 on $H^{\ell}(\Omega)$ and $K^{\ell}(\Omega \times (0,T))$, $\ell \ge 0$. (2.1) follows by setting $v = \nabla\phi$ and $\psi = \phi - \varpi$. □

In the study of the Navier-Stokes equations in the fixed domain we need the projection $P : L^2(\Omega) \to \mathcal{G}^\perp$, where $\mathcal{G} = \{\nabla\phi \; ; \; \phi \in H^1(\Omega)\}$. Since $\mathcal{G}^\perp \subsetneq (\mathcal{G}^0)^\perp$, $PP^0 = P$ and there is the complement in $P^0 L^2(\Omega)$ orthogonal to $\mathcal{G}^\perp$. In fact,

Lemma 2.4. *$P^0 L^2(\Omega) = \mathcal{G}^\perp \oplus \mathbb{G}$, where*

$$\mathbb{G} = \{\nabla\phi \; ; \; \Delta\phi = 0 \quad \text{in } \Omega, \; \partial_n\phi = 0 \quad \text{on } S_B\}.$$

Proof. Suppose $\nabla\phi \in \mathcal{G}$ orthogonal to $\mathcal{G}^0$. Then

$$0 = \int_\Omega \nabla\phi \cdot \nabla\psi \, dx \quad \text{for any } \nabla\psi \in \mathcal{G}^0$$

It is easily seen that the orthogonal decomposition follows from this. □

To discuss the linear problem with inhomogeneous right hand sides and homogeneous boundary conditions, we begin with the following integral identity: Suppose $v, u \in H^2(\Omega)$, $q \in H^1(\Omega)$ and $\operatorname{div} v = 0$. Then integration by parts gives

$$\text{(2.3)} \quad \int_\Omega (-\nu\,\Delta v + \nabla q)\, u^* \, dx = \langle v, u\rangle + \int_{\partial\Omega} n_j S_{jk}(v,q) u_k^* dS - \int_\Omega q \operatorname{div} u^* dx,$$

where

$$\langle v, u\rangle = \frac{\nu}{2} \sum_{j,k} \int_\Omega (\partial_k v_j + \partial_j v_k)(\partial_k u_j + \partial_j u_k)^* dx$$

and $S_{jk}(v,q) = q\delta_{jk} - \nu\,(\partial_k v_j + \partial_j v_k)$. Here and hereafter $\{\cdot\}^*$ denotes the complex conjugate of $\{\cdot\}$. For the solvability of the linear problem with homogeneous boundary conditions we use the lemma below.

Lemma 2.5. *For $u \in H^1(\Omega)$ with $u = 0$ on S_B, we have*

$$|u|^2_{H^1(\Omega)} \le C\langle u, u\rangle$$

with $C > 0$ independent of u.

See Lemma 2.7 in [2] and [5] for the proof of this lemma.

We next formulate the problem linearized at the equilibrium state

$$\text{(2.4)} \quad \partial_t\eta - v_3 = 0 \quad \text{on } S_F,$$

$$\text{(2.5)} \quad \partial_t v - \nu\Delta v + \nabla q = f_0 \quad \text{in } \Omega,$$

$$\text{(2.6)} \quad \operatorname{div} v = 0 \quad \text{in } \Omega,$$

$$\text{(2.7)} \quad v = 0 \quad \text{on } S_B,$$

$$\text{(2.8)} \quad \partial_3 v_j + \partial_j v_3 = f_j, \qquad j = 1\ 2, \quad \text{on } S_F,$$

$$\text{(2.9)} \quad q - 2\nu\partial_3 v_3 - (1 - \sigma\Delta_F)\eta = f_3 \quad \text{on } S_F.$$

Here f_0 is given in $K^{\ell-2}(\Omega\times\mathbb{R}^+)$ and f_j in $K^{\ell-3/2}(\mathbb{T}^2\times\mathbb{R}^+)$, $j=1,2,3$ for some $\ell\geq 2$. Applying P^0 to (2.5) we have

$$\partial_t v-\nu P^0\Delta v+P^0\nabla q=P^0 f_0.$$

Using Lemma 2.3 and the boundary condition (2.9), we write this as

$$\partial_t v-\nu P^0\Delta v+\nabla q_1+\nabla q_2+\nabla q_3=P^0 f_0,$$

where

$$\Delta q_j=0\quad\text{in }\Omega,\ \partial_3 q_j=0\quad\text{on }S_B, j=1,2,3,$$
$$q_1=2\nu\partial_3 v_3,\ q_2=(1-\sigma\Delta_F)\eta,\ q_3=f_3\quad\text{on}\quad S_F.$$

As noted in page 315 in [3] and [4], each ∇q_j can written as

$$\nabla q_1=R^*(2\nu\partial_3 v_3),\quad \nabla q_2=R^*(1-\sigma\Delta_F)\eta,\quad \nabla q_1=R^* f_3,$$

where R^* is the formal adjoint with respect to the L^2 inner product of the restriction

$$R: v\to v_3|_{S_F},\ v\in P^0L^2(\Omega). \tag{2.10}$$

We now introduce the operator A defined by

$$Av=-\nu P^0\Delta v+\nabla q_1\equiv-\nu P^0\Delta v+R^*(2\nu\partial_3 v_3). \tag{2.11}$$

The equations for η and v can be written as follows

$$\partial_t\eta-Rv=0,\qquad\text{on}\quad S_F \tag{2.12}$$
$$\partial_t v+Av+R^*(1-\sigma\Delta_F)\eta=P^0f_0-R^*f_3,\qquad\text{in}\quad\Omega. \tag{2.13}$$

We first consider the case of zero initial data

$$v=0,\ \eta=0\quad\text{at}\quad t=0$$

and $f=P^0f_0-R^*f_3$ is assumed to belong to $K^{\ell-2}_{(0)}(\Omega\times\mathbb{R}^+)$. Let us extend f to be zero for $t<0$. By the Laplace transform we derive the problem for

$$\hat v=\int_{-\infty}^{\infty}e^{-\lambda t}v(t)dt,\quad \hat\eta=\int_{-\infty}^{\infty}e^{-\lambda t}\eta(t)dt.$$

Transforming (2.12), (2.13) in t, we have the linear stationary problem with a parameter $\lambda\in\mathbb{C}$,

$$\lambda\hat\eta-R\hat v=0,\qquad\text{on}\quad S_F \tag{2.14}$$
$$\lambda\hat v+A\hat v+R^*(1-\sigma\Delta_F)\hat\eta=\hat f,\qquad\text{in}\quad\Omega. \tag{2.15}$$

This problem is closely related to the equation of the resolvent for η and v

$$\lambda \begin{pmatrix} \eta \\ v \end{pmatrix} - G \begin{pmatrix} \eta \\ v \end{pmatrix} = \begin{pmatrix} g_0 \\ f \end{pmatrix} \tag{2.16}$$

for g_0 and f given in suitable function spaces (we omit the hat ˆ). G is the matrix of operators given by

$$G \begin{pmatrix} \eta \\ v \end{pmatrix} = \begin{pmatrix} 0 & R \\ -R^*(1-\sigma\Delta_F) & -A \end{pmatrix} \begin{pmatrix} \eta \\ v \end{pmatrix}.$$

In the rest of this section we reduce the second equation of (2.16) to the one with homogeneous right hand side. To do this we need the solution to the the following boundary value problem with the homogeneous boundary data and inhomogeneous right hand sides.

$$\lambda v^{(0)} - \nu \varDelta v^{(0)} + \nabla q^{(0)} = f \qquad \text{in} \quad \Omega, \tag{2.17}$$

$$\operatorname{div} v^{(0)} = 0 \qquad \text{in} \quad \Omega, \tag{2.18}$$

$$v^{(0)} = 0 \qquad \text{on} \quad S_B, \tag{2.19}$$

$$\partial_3 v_j^{(0)} + \partial_j v_3^{(0)} = 0, \qquad j = 1, 2, \qquad \text{on} \quad S_F, \tag{2.20}$$

$$q^{(0)} - 2\nu \partial_3 v_3^{(0)} = 0 \qquad \text{on} \quad S_F, \tag{2.21}$$

where f is given in $P^0 H^{\ell-2}(\Omega)$. By the integral identity (2.3) it holds

$$\lambda \left(v^{(0)}, v \right)_{L^2} + \langle v^{(0)}, v \rangle = (f, v)_{L^2} \tag{2.22}$$

for any $v \in H^1(\Omega)$ satisfying $\operatorname{div} v = 0$ in Ω and $v = 0$ on S_B. By Lemma 2.5 we see that the real part of the left hand side of (2.22) with v replaced by v_0 is positive definite for any $\operatorname{Re}\lambda \geq 0$. By the Lax–Milgram's lemma we first obtain a unique weak solution to the above problem. Since the boundary conditions (2.19) and (2.20), (2.21) satisfy the complementary condition of [1], we obtain the higher regularity of the weak solution. Thus we have

Proposition 2.1. *Suppose $\ell \geq 2$ and $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geq 0$. For a given $f \in P^0 H^{\ell-2}(\Omega)$ there is a unique solution $v^{(0)}, q^{(0)}$ of (2.17)–(2.21), which satisfies*

$$\left| v^{(0)} \right|_{H^\ell(\Omega)} + |\lambda|^{\frac{\ell}{2}} \left| v^{(0)} \right|_{H^0(\Omega)} \leq C \left(|f|_{H^{\ell-2}(\Omega)} + |\lambda|^{\frac{\ell-2}{2}} |f|_{H^0(\Omega)} \right),$$

$$\left| \nabla q^{(0)} \right|_{H^{\ell-2}(\Omega)} + |\lambda|^{\frac{\ell-2}{2}} \left| \nabla q^{(0)} \right|_{H^0(\Omega)} + \left| q^{(0)}{}_{|S_F} \right|_{H^{\ell-3/2}(\mathbb{T}^2)}$$
$$+ |\lambda|^{\frac{\ell-3/2}{2}} \left| q^{(0)}{}_{|S_F} \right|_{H^0(\mathbb{T}^2)} \leq C \left(|f|_{H^{\ell-2}(\Omega)} + |\lambda|^{\frac{\ell-2}{2}} |f|_{H^0(\Omega)} \right).$$

For the details of Proposition 2.1, see Lemma 3.3 of [3]. To see how to recover $q^{(0)}$ we refer to Section 3 of [2] and [12]. Since $v^{(0)}$ and $q^{(0)}$ obtained in

Proposition 2.1 satisfy (2.21), we can derive $\lambda v^{(0)} + A v^{(0)} = f$ by applying P^0 to (2.17). Setting $\hat{v} = v^{(1)} + v^{(0)}$ in (2.14) and (2.15), we obtain the equations for $v^{(1)}$ and η

$$\lambda\eta - R v^{(1)} = R v^{(0)} \quad \text{on} \quad S_F, \tag{2.23}$$

$$\lambda v^{(1)} + A v^{(1)} + R^*(1 - \sigma \Delta_F)\eta = 0 \quad \text{in} \quad \Omega. \tag{2.24}$$

After preparing the results of the auxiliary problems in the next section, we discuss the solvability of the problem above in Section 4.

3. The auxiliary linear problems

We are concerned in this section with the linear stationary problem with a parameter λ

$$\lambda u - \nu \Delta u + \nabla p = 0, \quad \operatorname{div} u = 0 \tag{3.1}$$

in the half space $\Omega_\infty = \mathbb{T}^2 \times (0, \infty)$. The equations are supplemented with the one of the boundary conditions:

$$\text{I)} \quad \begin{aligned} &\partial_j u_3 + \partial_3 u_j = a_j, \quad j = 1, 2, \\ &-p + 2\nu \partial_3 u_3 = a_3, \end{aligned} \tag{3.2}$$

$$\text{II)} \quad u_j = b_j, \quad j = 1, 2, \quad u_3 = 0 \tag{3.3}$$

on the boundary $\partial\Omega_\infty = \{(x', 0); x' \in \mathbb{T}^2\}$. We identify this boundary with $\mathbb{T}^2$. The boundary data a_j and b_j are given arbitrarily in $H^{\frac{1}{2}}(\mathbb{T}^2)$ and $H^{\frac{3}{2}}(\mathbb{T}^2)$, respectively.

These boundary value problems have been discussed by several authors in the various context (see, for example [13], [8]). For our later use we need the results in the following form.

Proposition 3.1. *Let γ be an arbitrarily fixed positive constant. Let a_j be given in $H^{\frac{1}{2}}(\mathbb{T}^2)$, $j = 1, 2$ and a_3 in $H_0^{\frac{1}{2}}(\mathbb{T}^2)$. Then, for* $\operatorname{Re}\lambda \geq \gamma$ *there is a unique solution u, p to* (3.1), (3.2) *satisfying*

$$\begin{aligned} &|u|_{H^2(\Omega_\infty)} + |\lambda||u|_{H^0(\Omega_\infty)} + |\nabla p|_{H^0(\Omega_\infty)} \\ &\quad \leq C \sum_{j=1}^{3} \left(|a_j|_{H^{\frac{1}{2}}(\mathbb{T}^2)} + |\lambda|^{\frac{1}{4}} |a_j|_{H^0(\mathbb{T}^2)} \right), \\ &\int_{\Omega_\infty} p\, dx' dy = 0. \end{aligned} \tag{3.4}$$

The constant C is bounded for λ with $\operatorname{Re}\lambda \geq \gamma$.

Proposition 3.2. *Let λ be as above. Let b_j be given in $H^{\frac{3}{2}}(\mathbb{T}^2)$, $j =$*

1, 2. *Then there is a unique solution* u, p *to* (3.1), (3.3) *satisfying*

$$
\begin{aligned}
(3.5)\qquad & |u|_{H^2(\Omega_\infty)} + |\lambda||u|_{H^0(\Omega_\infty)} + |\nabla p|_{H^0(\Omega_\infty)} \\
& \leq C \sum_{j=1}^{2} \left(|b_j|_{H^{\frac{3}{2}}(\mathbb{T}^2)} + |\lambda|^{\frac{3}{4}} |b_j|_{H^0(\mathbb{T}^2)} \right), \\
& \int_{\Omega_\infty} p dx' dy = 0.
\end{aligned}
$$

We first construct and estimate the solutions to the problem (3.1), (3.2). We expand $u(x', y)$ and $p(x', y)$ in the Fourier series in $x' \in \mathbb{T}^2$:

$$
u(x', y) = \sum_{\xi' \in \mathbb{Z}^2} u^{(\xi')}(y) e^{i\xi' \cdot x'}, \quad p(x', y) = \sum_{\xi' \in \mathbb{Z}^2} p^{(\xi')}(y) e^{i\xi' \cdot x'}.
$$

For each mode $\xi' = (\xi_1, \xi_2) \in \mathbb{Z}^2$ we obtain the system of ordinary differential equations

$$
(3.6)\qquad \lambda u_j^{(\xi')} - \nu \left(\left(\frac{d}{dy} \right)^2 - |\xi'|^2 \right) u_j^{(\xi')} + i\xi_j p^{(\xi')} = 0, \quad j = 1, 2,
$$

$$
(3.7)\qquad \lambda u_3^{(\xi')} - \nu \left(\left(\frac{d}{dy} \right)^2 - |\xi'|^2 \right) u_3^{(\xi')} + \frac{dp^{(\xi')}}{dy} = 0,
$$

$$
(3.8)\qquad i\xi_1\, u_1^{(\xi')} + i\xi_2\, u_2^{(\xi')} + \frac{du_3^{(\xi')}}{dy} = 0, \qquad \text{in} \quad y > 0.
$$

We follow the arguments in [13], Section 2. For a while we assume $\xi' \neq (0, 0)$. The solution to (3.6)–(3.8) which decays as y tends to ∞ can be written as

$$
\begin{aligned}
& u^{(\xi')}(y) = \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \frac{1}{r} (i\xi_1 \Phi_1 + i\xi_2 \Phi_2) \end{pmatrix} e^{-ry} + \phi \begin{pmatrix} i\xi_1 \\ i\xi_2 \\ -|\xi'| \end{pmatrix} e^{-|\xi'|y}, \\
(3.9)\qquad & p^{(\xi')}(y) = -\lambda \phi e^{-|\xi'|y},
\end{aligned}
$$

where $|\xi'| = \sqrt{\xi_1^2 + \xi_2^2}$, $r = \sqrt{\frac{\lambda}{\nu} + |\xi'|^2}$, $|\arg r| < \frac{\pi}{4}$. Substituting (3.9) into the boundary condition (3.2), we derive the linear algebraic system for Φ_1, Φ_2, ϕ. Using the results by Solonnikov in [13], Sections 2 and 3, we have the explicit form of $u^{(\xi')}$, $p^{(\xi')}$

$$
\begin{aligned}
(3.10)\quad u_j^{(\xi')}(y) &= -\frac{a_j^{(\xi')}}{r} (1 - \delta_{j3}) e_0(y) \\
&+ \frac{e_0(y)}{r(r + |\xi'|)\mathcal{D}_0} \sum_{k=1}^{3} U_{jk}^0 a_k^{(\xi')} + \frac{e_1(y)}{(r + |\xi'|)\mathcal{D}_0} \sum_{k=1}^{3} V_{jk}^0 a_k^{(\xi')}, \ j = 1, 2, 3, \\
p^{(\xi')}(y) &= \frac{\lambda}{\mathcal{D}_0} \left(2r \left(i\xi_1 a_1^{(\xi')} + i\xi_2 a_2^{(\xi')} \right) - \frac{1}{\nu} \left(r^2 + |\xi'|^2 \right) a_3^{(\xi')} \right) e^{-|\xi'|y},
\end{aligned}
$$

where $a_j^{(\xi')}$ is the Fourier coefficient of a_j, $(j = 1,2,3)$. U^0_{jk} and V^0_{jk} are components of the matrices

$$\mathcal{U}^0 = \begin{pmatrix} (3r-|\xi'|)\frac{\lambda}{\nu}\xi_1^2 & (3r-|\xi'|)\frac{\lambda}{\nu}\xi_1\xi_2 & i\,\xi_1 r(r-|\xi'|)\frac{\lambda}{\nu^2} \\ (3r-|\xi'|)\frac{\lambda}{\nu}\xi_1\xi_2 & (3r-|\xi'|)\frac{\lambda}{\nu}\xi_2^2 & i\,\xi_2 r(r-|\xi'|)\frac{\lambda}{\nu^2} \\ -i\,\xi_1 r(r-|\xi'|)\frac{\lambda}{\nu} & -i\,\xi_2 r(r-|\xi'|)\frac{\lambda}{\nu} & -|\xi'|r(r+|\xi'|)\frac{\lambda}{\nu^2} \end{pmatrix},$$

$$\mathcal{V}^0 = \begin{pmatrix} -\frac{2}{\nu}\lambda r\xi_1^2 & -\frac{2}{\nu}\lambda r\xi_1\xi_2 & -i\xi_1\frac{\lambda}{\nu^2}\left(r^2+|\xi'|^2\right) \\ -\frac{2}{\nu}\lambda r\xi_1\xi_2 & -\frac{2}{\nu}\lambda r\xi_2^2 & -i\xi_2\frac{\lambda}{\nu^2}\left(r^2+|\xi'|^2\right) \\ -i\xi_1\frac{2|\xi'|}{\nu}\lambda r & -i\xi_2\frac{2|\xi'|}{\nu}\lambda r & \frac{\lambda|\xi'|}{\nu^2}\left(r^2+|\xi'|^2\right) \end{pmatrix},$$

and $e_0(y) = e^{-ry}$, $e_1(y) = \frac{e^{-ry}-e^{-|\xi'|y}}{r-|\xi'|}$, $\mathcal{D}_0 = (r^2+|\xi'|^2)^2 - 4r|\xi'|^3$. The next lemma was proved in Lemma 2.5 of [13].

Lemma 3.1. *If* $\operatorname{Re}\lambda \ge \gamma$, $\xi' \in \mathbb{Z}^2$, *then*

$$|\mathcal{D}_0| \ge \frac{\gamma^2}{\nu^2}, \quad |\mathcal{D}_0| \ge \frac{2}{\nu}|\lambda||\xi'|^2, \quad 2|\mathcal{D}_0| \ge \left|\frac{\lambda}{\nu}\right|^2.$$

For the fundamental solutions $e_0(y), e_1(y)$ we have the estimates

Lemma 3.2. *For any* $\xi' \in \mathbb{Z}^2 \setminus \{(0,0)\}$, $\operatorname{Re}\lambda \ge \gamma$, *we have*

$$\int_0^\infty |e_1(y)|^2\,dy \le \frac{1}{|r|^2|\xi'|},$$

$$\int_0^\infty \left|\frac{d^j e_1(y)}{dy^j}\right|^2 dy \le C\frac{|r|^{2j-1}+|\xi'|^{2j-1}}{|r|^2},\ j=1,2,3,\dots,$$

$$\int_0^\infty \left|\frac{d^j e_0(y)}{dy^j}\right|^2 dy \le \frac{1}{\sqrt{2}}|r|^{2j-1},\ j=0,1,2,\dots.$$

For these estimates, see Lemma 3.1 in [13]. To obtain the desired estimate we start with the inequality derived from (3.10)

$$(3.11)\qquad |r|^4\int_0^\infty \left|u_j^{(\xi')}(y)\right|^2 dy \le 3(1-\delta_{j3})|r|^4\left|\frac{a_j^{(\xi')}}{r}\right|^2\int_0^\infty |e_0(y)|^2\,dy$$
$$+\frac{3|r|^4}{|r(r+|\xi'|)\mathcal{D}_0|^2}\left|\sum_{k=1}^3 U^0_{jk}a_k^{(\xi')}\right|^2\int_0^\infty |e_0(y)|^2\,dy$$
$$+\frac{3|r|^4}{|(r+|\xi'|)\,\mathcal{D}_0|^2}\left|\sum_{k=1}^3 V^0_{jk}a_k^{(\xi')}\right|^2\int_0^\infty |e_1(y)|^2\,dy,$$
$$j=1,2,3.$$

Using Lemma 3.2, we estimate the right hand side of (3) term by term. We see that, for $j, k = 1, 2$

$$|r|^4 \int_0^\infty |e_0(y)|^2 \, dy \frac{1}{|r\,(r+|\xi'|)\,\mathcal{D}_0|^2} \left| U^0_{jk} a^{(\xi')}_k \right|^2 \le \frac{9}{\sqrt{2}} \frac{1}{|\mathcal{D}_0|^2} \left| \frac{\lambda}{\nu} \right|^2 |\xi'|^4 \, |r| |a^{(\xi')}_k|^2,$$

$$|r|^4 \int_0^\infty |e_1(y)|^2 \, dy \frac{1}{|(r+|\xi'|)\,\mathcal{D}_0|^2} \left| V^0_{jk} a^{(\xi')}_k \right|^2 \le \frac{4}{|\mathcal{D}_0|^2} \left| \frac{\lambda}{\nu} \right|^2 |\xi'|^3 \, |r| |r| |a^{(\xi')}_k|^2$$
$$\le \frac{4}{|\mathcal{D}_0|^2} \left| \frac{\lambda}{\nu} \right|^2 |\xi'|^3 \left(\left| \frac{\lambda}{\nu} \right|^{\frac{1}{2}} + |\xi'| \right) |r| |a^{(\xi')}_k|^2$$
$$= \frac{4}{|\mathcal{D}_0|^2} \left(\left(\left| \frac{\lambda}{\nu} \right| |\xi'|^2 \right)^{\frac{3}{2}} \left| \frac{\lambda}{\nu} \right| + \left| \frac{\lambda}{\nu} \right|^2 |\xi'|^4 \right) |r| |a^{(\xi')}_k|^2$$

and, for $j = 1, 2,\ k = 3$

$$|r|^4 \int_0^\infty |e_1(y)|^2 \, dy \frac{1}{|(r+|\xi'|)\,\mathcal{D}_0|^2} \left| V^0_{j3} a^{(\xi')}_3 \right|^2$$
$$\le \frac{1}{|\mathcal{D}_0|^2} |r| \left| \frac{\lambda}{\nu^2} \right|^2 |\xi'| \left| r^2 + |\xi'|^2 \right| |r| |a^{(\xi')}_3|^2$$
$$\le \frac{4}{\nu^2 \, |\mathcal{D}_0|^2} \left(\left| \frac{\lambda}{\nu} \right|^{\frac{7}{2}} |\xi'| + \left| \frac{\lambda}{\nu} \right|^2 |\xi'|^4 \right) |r| |a^{(\xi')}_3|^2$$
$$= \frac{4}{\nu^2 \, |\mathcal{D}_0|^2} \left(\left(\left| \frac{\lambda}{\nu} \right| |\xi'|^2 \right)^{\frac{1}{2}} \left| \frac{\lambda}{\nu} \right|^3 + \left| \frac{\lambda}{\nu} \right|^2 |\xi'|^4 \right) |r| |a^{(\xi')}_3|^2.$$

We can estimate other terms in (3) in a similar way. By Lemma 3.1 we see that the right hand side of (3) is bounded by

$$C|r| \left(|a^{(\xi')}_1|^2 + |a^{(\xi')}_2|^2 + |a^{(\xi')}_3|^2 \right),$$

where the constant C is independent of λ and ξ'. The quantity $|\xi'|^2 |p^{(\xi')}(y)|^2$ can be estimated in a same way as above where e^{-ry} is replaced by $e^{-|\xi'|y}$. For the constant mode $(0,0)$ we obtain

$$\lambda u^{(0,0)}_j - \nu \left(\frac{d}{dy} \right)^2 u^{(0,0)}_j = 0, \quad j = 1, 2, \tag{3.12}$$

$$\lambda u^{(0,0)}_3 - \nu \left(\frac{d}{dy} \right)^2 u^{(0,0)}_3 + \frac{dp^{(0,0)}}{dy} = 0, \tag{3.13}$$

$$\frac{du^{(0,0)}_3}{dy} = 0 \qquad \text{in} \quad y > 0. \tag{3.14}$$

The boundary conditions become

$$\frac{du_j^{(0,0)}}{dy}(0) = a_j\ , \quad j = 1,2, \tag{3.15}$$

$$-p^{(0,0)} + 2\nu\frac{du_3^{(0,0)}}{dy}(0) = 0. \tag{3.16}$$

The solution for the constant mode which is bounded in $y > 0$ is

$$\begin{gathered} u_j^{(0,0)}(y) = -a_j^{(0,0)}\sqrt{\frac{\nu}{\lambda}}e^{-y\sqrt{\frac{\lambda}{\nu}}}, \quad j = 1,2, \\ u_3^{(0,0)}(y) = 0,\ p^{(0,0)}(y) = 0. \end{gathered} \tag{3.17}$$

The fact that $\int_{\Omega_\infty} p\,dx'dy = 0$ follows from this. Since $\operatorname{Re}\lambda > 0$, it holds that $\operatorname{Re}\sqrt{\frac{\lambda}{\nu}} \geq \frac{1}{\sqrt{2}}\sqrt{\frac{|\lambda|}{\nu}}$. Using this, we can estimate $u_j^{(0,0)}$, $j = 1,2$ as follows:

$$\left|\frac{\lambda}{\nu}\right|^2 \int_0^\infty \left|u_j^{(0,0)}(y)\right|^2 dy \leq \frac{1}{\sqrt{2}}\sqrt{\frac{|\lambda|}{\nu}}\left|a_j^{(0,0)}\right|^2.$$

Collecting these estimates and taking sum in $\xi' \in \mathbb{Z}^2$, we can prove Proposition 3.1.

We next give a brief outline of the proof of Proposition 3.2. In the same way as the proof of Proposition 3.1 we decompose the unknowns into the Fourier mode. For $\xi' \in \mathbb{Z}^2 \setminus \{(0,0)\}$ we derive the same system (3.6), (3.7) and (3.8) supplemented with the boundary condition

$$u_j^{(\xi')}(0) = b_j^{(\xi')}, \quad j = 1,2, \quad u_3^{(\xi')}(0) = 0 \quad \text{on } y = 0.$$

Here $b_j^{(\xi')}$ is the Fourier coefficient of the boundary data. The explicit form of the solution which decays as y tends to ∞ is the following

$$u^{(\xi')}(y) = \begin{pmatrix} b_1^{(\xi')} \\ b_2^{(\xi')} \\ 0 \end{pmatrix} e_0(y) + \left(i\xi_1 b_1^{(\xi')} + i\xi_2 b_2^{(\xi')}\right)\begin{pmatrix} -\dfrac{i\xi_1}{|\xi'|} \\ -\dfrac{i\xi_2}{|\xi'|} \\ 1 \end{pmatrix} e_1(y), \tag{3.18}$$

$$p^{(\xi')}(y) = -\frac{\nu}{|\xi'|}\left(i\xi_1 b_1^{(\xi')} + i\xi_2 b_2^{(\xi')}\right)(r + |\xi'|)e^{-|\xi'|y}. \tag{3.19}$$

Using Lemma 3.2, we estimate each term in (3.18) and (3.19)

$$\begin{gathered} |r|^4 \int_0^\infty \left|b_j^{(\xi')}e_0(y)\right|^2 dy \leq \frac{1}{\sqrt{2}}|r|^3\left|b_j^{(\xi')}\right|^2, \quad j = 1,2, \\ |r|^4 \int_0^\infty \left|\left(i\xi_1 b_1^{(\xi')} + i\xi_2 b_2^{(\xi')}\right)e_1(y)\right|^2 dy \leq |r|^3\left(\left|b_1^{(\xi')}\right|^2 + \left|b_2^{(\xi')}\right|^2\right), \\ |\xi'|^2 \int_0^\infty \left|p^{(\xi')}\right|^2 dy \leq 2\nu^2|r|^3\left(\left|b_1^{(\xi')}\right|^2 + \left|b_2^{(\xi')}\right|^2\right). \end{gathered}$$

For the constant mode $\xi' = (0,0)$ we obtain the bounded solution

$$u_j^{(0,0)}(y) = b_j^{(0,0)} e^{-y\sqrt{\frac{\lambda}{\nu}}}, \quad j = 1,2, \qquad u_3^{(0,0)}(y) = 0,\ p^{(0,0)}(y) = 0, \tag{3.20}$$

which can be estimated as above. Collecting these estimates and taking sum in $\xi' \in \mathbb{Z}^2$, we can prove Proposition 3.2.

By using two previous propositions above we show

Proposition 3.3. *Let $a_j \in H^{\frac{1}{2}}(\mathbb{T}^2)$, $j = 1,2$, $a_3 \in H_0^{\frac{1}{2}}(\mathbb{T}^2)$, $b_j \in H^{\frac{3}{2}}\left(\mathbb{T}^2\right)$, $j = 1,2$ be given arbitrarily. Then there is a positive constant γ_0 such that, if $\operatorname{Re}\lambda \geq \gamma_0$, then there is a unique solution u, p to the problem*

$$\lambda u - \nu\Delta u + \nabla p = 0, \quad \operatorname{div} u = 0 \quad \text{in } \Omega, \tag{3.21}$$

$$\partial_j u_3 + \partial_3 u_j = a_j,\ j = 1,2,\ -p + 2\nu\partial_3 u_3 = a_3 \quad \text{on } S_F, \tag{3.22}$$

$$u_j = b_j, \quad j = 1,2, \quad u_3 = 0 \quad \text{on } S_B, \tag{3.23}$$

$$\int_\Omega p\, dx = 0,$$

which satisfies

$$\begin{aligned} &|u|_{H^2(\Omega)} + |\lambda||u|_{H^0(\Omega)} + |\nabla p|_{H^0(\Omega)} \\ &\leq C\left(\sum_{j=1}^{3}\left(|a_j|_{H^{\frac{1}{2}}(\mathbb{T}^2)} + |\lambda|^{\frac{1}{4}}|a_j|_{H^0(\mathbb{T}^2)}\right) + \sum_{j=1}^{2}\left(|b_j|_{H^{\frac{3}{2}}(\mathbb{T}^2)} + |\lambda|^{\frac{3}{4}}|b_j|_{H^0(\mathbb{T}^2)}\right)\right). \end{aligned} \tag{3.24}$$

Here the constant C does not depend on the boundary data and is bounded for $\operatorname{Re}\lambda \geq \gamma_0$.

By Proposition 3.2 we can construct u^0 and p^0 such that

$$\lambda u^0 - \nu\Delta u^0 + \nabla p^0 = 0, \quad \operatorname{div} u^0 = 0 \quad \text{in } \mathbb{T}^2 \times \{y > -1\}, \qquad u_j^0 = b_j, \quad j = 1,2, \quad u_3^0 = 0 \quad \text{on } y = -1. \tag{3.25}$$

Since estimate (3.5) holds and

$$\operatorname{div} u^0 = 0, \quad \int_{\mathbb{T}^2 \times \{y > -1\}} p^0 dx' dy = 0,$$

it follows that, on S_F

$$\partial_j u_3^0 + \partial_3 u_j^0 \in H^{\frac{1}{2}}(\mathbb{T}^2), \quad j = 1,2, \quad -p^0 + 2\nu\partial_3 u_3^0 \in H_0^{\frac{1}{2}}(\mathbb{T}^2).$$

Hence we can reduce the boundary value problem (3.22) to the case $b_j = 0$, $j = 1, 2$.

To construct a solution in Proposition 3.3 we first take u^f, p^f which is the solution to the problem

$$\lambda u^f - \nu \Delta u^f + \nabla p^f = 0, \quad \operatorname{div} u^f = 0 \quad \text{in } \Omega_{-\infty} = \mathbb{T}^2 \times \{y < 0\}, \tag{3.26}$$

$$\partial_j u_3^f + \partial_3 u_j^f = a_j,\ j = 1, 2,\ -p^f + 2\nu \partial_3 u_3^f = a_3 \quad \text{on } y = 0. \tag{3.27}$$

By Proposition 3.1 it is possible to construct such u^f, p^f. Then set $u^1 = \zeta(y) u^f$ and $p^1 = \zeta(y) p^f$. Here $\zeta(y)$ is a smooth cut off function such that

$$\zeta(y) = 1 \quad \text{for} \quad y \geq -\frac{1}{3}, \qquad \zeta(y) = 0 \quad \text{for} \quad y \leq -\frac{2}{3}.$$

To adjust the solenoidal condition we need

Lemma 3.3. *For $f \in H^{\ell-2}(\Omega)$, $b \in H^{\ell-\frac{3}{2}}(\mathbb{T}^2)$, $\ell \geq 2$, there is a unique solution φ to the problem*

$$\Delta \varphi = f \quad \text{in } \Omega, \varphi = 0 \quad \text{on } S_F,\ \partial_y \varphi = b \quad \text{on } S_B$$

and it holds that

$$|\varphi|_{H^\ell(\Omega)} \leq C \left(|f|_{H^{\ell-2}(\Omega)} + |b|_{H^{\ell-\frac{3}{2}}(\mathbb{T}^2)} \right).$$

For the proof of this lemma see Lemma 2.8 in [2]. By this lemma we can take ϕ^2 satisfying

$$\Delta \phi^2 = -\zeta'(y) u_3^f \quad \text{in } \Omega,\ \phi^2 = 0 \quad \text{on } S_F,\ \partial_y \phi^2 = 0 \quad \text{on } S_B. \tag{3.28}$$

Set $u^2 = \nabla \phi^2$ and take u^3, p^3 which solve

$$\begin{aligned} \lambda u^3 - \nu \Delta u^3 + \nabla p^3 = 0, \quad \operatorname{div} u^3 = 0 \quad &\text{in} \quad \mathbb{T}^2 \times \{y > -1\}, \\ u^3 = -\nabla \phi^2 \quad &\text{on} \quad y = -1. \end{aligned} \tag{3.29}$$

Note that $u_3^2 = \partial_y \phi^2 = 0$ on S_B. Hence we can apply Proposition 3.2 to construct such u^3, p^3. Let $\tilde{u} = u^1 + u^2 + u^3$ and $\tilde{p} = p^1 + p^3$. We see that

$$\tilde{u} = 0 \quad \text{on } S_B,\ \operatorname{div} \tilde{u} = 0 \quad \text{in } \Omega.$$

In terms of u^2, u^3, p^3 we define the operator

$$\mathcal{M}_0 : \left(H^{\frac{1}{2}}\left(\mathbb{T}^2\right) \right)^2 \times H_0^{\frac{1}{2}}(\mathbb{T}^2) \to \left(H^{\frac{1}{2}}(\mathbb{T}^2) \right)^2 \times H_0^{\frac{1}{2}}(\mathbb{T}^2)$$

by

$$\mathcal{M}_0 \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \equiv \begin{pmatrix} \left(\partial_1 u_3^2 + \partial_3 u_1^2\right)\big|_{S_F} + \left(\partial_1 u_3^3 + \partial_3 u_1^3\right)\big|_{S_F} \\ \left(\partial_2 u_3^2 + \partial_3 u_2^2\right)\big|_{S_F} + \left(\partial_1 u_3^3 + \partial_3 u_2^3\right)\big|_{S_F} \\ 2\nu\, \partial_3 u_3^2\big|_{S_F} + \left(-p^3 + 2\nu \partial_3 u_3^3\right)\big|_{S_F} \end{pmatrix}$$

in order to express the equalities for $\tilde{u}$, $\tilde{p}$ on S_F

$$\begin{pmatrix} \partial_1 \tilde{u}_3 + \partial_3 \tilde{u}_1 \\ \partial_2 \tilde{u}_3 + \partial_3 \tilde{u}_2 \\ -\tilde{p} + 2\nu \partial_3 \tilde{u}_3 \end{pmatrix} = (I + \mathcal{M}_0) \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}. \tag{3.30}$$

We next give an estimate of the operator norm of $\mathcal{M}_0$. Let $j, k = 1, 2, 3$. By the usual estimate for trace operator

$$\left| \partial_j u_k^2 |_{S_F} \right|_{H^{\frac{1}{2}}(\mathbb{T}^2)} \le C \left| \partial_j u_k^2 \right|_{H^1(\Omega)}.$$

Since $u^2 = \nabla \phi^2$ where ϕ^2 is the solution to (3.28) the right hand side of the inequality above can be estimated as follows:

$$C \left| \partial_j \partial_k \phi^2 \right|_{H^1(\Omega)} \le C_1 \left| \phi^2 \right|_{H^3(\Omega)} \le C_2 \left| u_3^f \right|_{H^1(\Omega_{-\infty})}. \tag{3.31}$$

By the convexity of Sobolev norm and by estimate (3.4) we see that

$$\begin{aligned} C_2 \left| u_3^f \right|_{H^1(\Omega_{-\infty})} &\le C_2 \left| u_3^f \right|_{H^2(\Omega_{-\infty})}^{\frac{1}{2}} \left| u_3^f \right|_{H^0(\Omega_{-\infty})}^{\frac{1}{2}} \\ &\le C_3 \frac{1}{|\lambda|^{\frac{1}{2}}} \sum_{m=1}^{3} \left(|a_m|_{H^{\frac{1}{2}}(\mathbb{T}^2)} + |\lambda|^{\frac{1}{4}} |a_m|_{H^0(\mathbb{T}^2)} \right). \end{aligned} \tag{3.32}$$

Hence

$$\left| \partial_j u_k^2 |_{S_F} \right|_{H^{\frac{1}{2}}(\mathbb{T}^2)} \le C_4 \frac{1}{|\lambda|^{\frac{1}{4}}} \sum_{m=1}^{3} |a_m|_{H^{\frac{1}{2}}(\mathbb{T}^2)}, \quad j, k = 1, 2, 3. \tag{3.33}$$

We next estimate the traces of $\partial_j u_k^3$ and p^3 on S_F. Since we use Proposition 3.2 to construct u^3, p^3, we only have to replace y by $y+1$ in (3.18) and (3.19). Let $j = 1, 2$. The explicit form of $\partial_y u_j^3$ is written as

$$\begin{aligned} \partial_y u_j^3(x', 0) = \sum_{\xi' \in \mathbb{Z}^2} e^{ix' \cdot \xi'} \Bigg\{ &-r b_j^{(\xi')} e^{-r}. \\ &+ \frac{\xi_j}{|\xi'|} \cdot (\xi_1 b_1^{(\xi')} + \xi_2 b_2^{(\xi')}) \frac{-re^{-r} + |\xi'| e^{-|\xi'|}}{r - |\xi'|} \Bigg\}, \end{aligned}$$

where $b_j = -\partial_j \phi^2, j = 1, 2$ on S_B. We regard the second term as 0 for $\xi' = (0, 0)$ in the above sum. From this we have

$$\begin{aligned} \left| \partial_y u_j^3 |_{S_F} \right|_{H^{\frac{1}{2}}(\mathbb{T}^2)}^2 \le \sum_{\xi'} 2|\xi'| \Bigg\{ &|r|^2 \left| b_j^{(\xi')} \right|^2 \left| e^{-r} \right|^2 \\ &+ \left| \xi_1 b_1^{(\xi')} + \xi_2 b_2^{(\xi')} \right|^2 \frac{\left| -re^{-r} + |\xi'| e^{-|\xi'|} \right|^2}{\left| r - |\xi'| \right|^2} \Bigg\}. \end{aligned} \tag{3.34}$$

Note that $|r| \leq \sqrt{2}\,\mathrm{Re}\, r$ since $\mathrm{Re}\,\lambda > 0$. From this fact we have $|e^{-r}|^2 = e^{-2\,\mathrm{Re}\, r} \leq e^{-\sqrt{2}|r|}$. Using this inequality and the equality

$$\left|\frac{1}{r-|\xi'|}\right| = \left|\frac{\nu}{\lambda}\left(\sqrt{\frac{\lambda}{\nu}+|\xi'|^2}+|\xi'|\right)\right|,$$

from (3.34) we easily derive

$$\begin{aligned}\left|\partial_y u_j^3\big|_{S_F}\right|^2_{H^{\frac{1}{2}}(\mathbb{T}^2)} &\leq C\sum_{\xi'}\left(\left|b_1^{(\xi')}\right|^2+\left|b_2^{(\xi')}\right|^2\right)\\ &= C\left(\left|\partial_1\phi^2\big|_{H^0(S_B)}\right|^2+\left|\partial_2\phi^2\big|_{H^0(S_B)}\right|^2\right) \leq C\left|\nabla\phi^2\right|^2_{H^1(\Omega)}.\end{aligned}$$

Since ϕ^2 is the solution to (3.28), this can be estimated by $\left|u_3^f\right|_{H^0(\Omega_{-\infty})}$. Taking into account the estimate for u^f obtained in Proposition 3.1, we obtain

$$\left|\partial_y u_j^3\big|_{S_F}\right|_{H^{\frac{1}{2}}(\mathbb{T}^2)} \leq \frac{C}{|\lambda|^{\frac{3}{4}}}\sum_{m=1}^{3}|a_m|_{H^{\frac{1}{2}}(\mathbb{T}^2)}, \quad j=1,2. \tag{3.35}$$

In a similar way we can obtain

$$\left|\partial_j u_3^3\big|_{S_F}\right|_{H^{\frac{1}{2}}(\mathbb{T}^2)} \leq \frac{C}{|\lambda|^{\frac{3}{4}}}\sum_{m=1}^{3}|a_m|_{H^{\frac{1}{2}}(\mathbb{T}^2)}, \quad j=1,2. \tag{3.36}$$

The explicit form of $p^3\big|_{S_F}$ is as follows

$$p^3(x',0) = \sum_{\xi'\in\mathbb{Z}^2\setminus\{(0,0)\}} e^{ix'\cdot\xi'}\left(-\frac{\nu}{|\xi'|}\left(i\xi_1 b_1^{(\xi')}+i\xi_2 b_2^{(\xi')}\right)(r+|\xi'|)\right)e^{-|\xi'|}.$$

From this we have

$$\begin{aligned}&\left|p^3(x',0)\big|_{S_F}\right|^2_{H^{\frac{1}{2}}(\mathbb{T}^2)}\\ &\leq \sum_{\xi'\in\mathbb{Z}^2\setminus\{(0,0)\}}|\xi'|\frac{\nu^2}{|\xi'|^2}|\xi'|^2\left(\left|b_1^{(\xi')}\right|^2+\left|b_2^{(\xi')}\right|^2\right)|r+|\xi'||^2 e^{-2|\xi'|}\\ &\leq \sum_{\xi'\in\mathbb{Z}^2\setminus\{(0,0)\}}\frac{3\nu^2}{2}\left(\frac{|\lambda|}{\nu}|\xi'|+3|\xi'|^3\right)e^{-2|\xi'|}\left(\left|b_1^{(\xi')}\right|^2+\left|b_2^{(\xi')}\right|^2\right)\\ &\leq C(1+|\lambda|)\left(\left|\partial_1\phi^2\big|_{H^0(S_B)}\right|^2+\left|\partial_2\phi^2\big|_{H^0(S_B)}\right|^2\right).\end{aligned} \tag{3.37}$$

By the same reasoning as above we obtain the estimate

$$\left|p^3(x',0)\big|_{S_F}\right|_{H^{\frac{1}{2}}(\mathbb{T}^2)} \leq \frac{C}{|\lambda|^{\frac{1}{4}}}\sum_{m=1}^{3}|a_m|_{H^{\frac{1}{2}}(\mathbb{T}^2)}. \tag{3.38}$$

Collecting (3.33), (3.35), (3.36) and (3.38), we see that one can take a positive constant $\gamma_0 > 0$ with the following property: if $\operatorname{Re}\lambda \geq \gamma_0$, then the operator norm of $\mathcal{M}_0$ satisfies $|\mathcal{M}_0| \leq \frac{1}{2}$, hence the bounded inverse $(I+\mathcal{M}_0)^{-1}$ exists. Therefore, for such a λ we can construct $\tilde{u}$, $\tilde{p}$ for a_1, a_2, a_3 replaced by $(I+\mathcal{M}_0)^{-1}(a_1,a_2,a_3)^T$. This $(\tilde{u},\tilde{p})$ obviously satisfies the boundary condition on S_F and the homogeneous boundary condition on S_B . The equations satisfied by $(\tilde{u},\tilde{p})$ in Ω are the followings:

$$\lambda\tilde{u} - \nu\Delta\tilde{u} + \nabla\tilde{p} = -\big(2\zeta'(y)\partial_y u^f + \zeta''(y)u^f\big) + p^f\nabla\zeta(y) + \lambda u^2 - \nu\Delta u^2, \tag{3.39}$$

$$\operatorname{div}\tilde{u} = 0. \tag{3.40}$$

Here $(I+\mathcal{M}_0)^{-1}(a_1,a_2,a_3)^T$ determines the right hand side of (3.39). By Proposition 2.1 we have the solution (v^0, q^0) to the problem (2.17)–(2.21) with f in (2.17) replaced by the right hand side of (3.39). Here we note that the third component $u_3^{f,(0,0)}(y)$ and $p^{f,(0,0)}$ vanish, where $u^{f,(0,0)}(y)$ and $p^{f,(0,0)}$ are the constant mode in the Fourier expansion of u^f and p^f respectively. From this we easily see that $u_3^{2,(0,0)}$ also vanishes by its construction. Hence the third component of the constant mode in the Fourier expansion of the right hand side of (3.39) is equal to 0. From this fact, the solenoidal condition and the boundary conditions (2.19), (2.21) it follows that

$$\frac{dq^{0,(0,0)}}{dy} = 0 \quad \text{on } -1 < y < 0, \quad q^{0,(0,0)}(0) = 0.$$

Thus we see that

$$\int_\Omega q^0 dx = \int_{-1}^0 \int_{\mathbb{T}^2} q^0 dx' dy = 0.$$

By setting $u = \tilde{u} - v^0, p = \tilde{p} - q^0$, we obtain the solution (u,p) which satisfies all the requirements in Proposition 3.3.

4. The model problems

In this section we are first concerned with construction and estimation of the solutions to the equations with a parameter λ

$$\lambda u - \nu\Delta u + \nabla p = 0, \tag{4.1}$$

$$\operatorname{div} u = 0, \tag{4.2}$$

in the half space $\Omega_\infty = \big\{(x',y); x' \in \mathbb{T}^2, y > 0\big\}$ supplemented with the conditions

$$\lambda\eta - u_3 = b_0, \tag{4.3}$$

$$\partial_j u_3 + \partial_3 u_j = 0, \quad j = 1,2, \tag{4.4}$$

$$-p + 2\nu\partial_3 u_3 + \sigma\Delta_F\eta = b_3 \tag{4.5}$$

on the boundary $\partial\Omega_\infty = \mathbb{T}^2 \times \{y=0\}$. Here $\eta(x')$ is the unknown function on the boundary. The first result in this section is

Proposition 4.1. *Let $\ell \geq 2$. Let b_0, b_3 be given in $H_0^{\ell-\frac{1}{2}}(\mathbb{T}^2)$ and $H_0^{\ell-\frac{3}{2}}(\mathbb{T}^2)$, respectively. Fix $\gamma > 0$ arbitrarily. For $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geq \gamma$, there exists a unique solution η, u, p to (4.1)–(4.4) and (4.5) satisfying*

$$\int_{\Omega_\infty} p\, dx'dy = 0, \qquad \int_{\mathbb{T}^2} \eta\, dx' = 0,$$

$$\begin{aligned}(4.6)\qquad & |u|_{H^\ell(\Omega_\infty)} + |\lambda|^{\frac{\ell}{2}}|u|_{H^0(\Omega_\infty)} + |\nabla p|_{H^{\ell-2}(\Omega_\infty)} + |\lambda|^{\frac{\ell-2}{2}}|\nabla p|_{H^0(\Omega_\infty)} \\ &+ |\eta|_{H^{\ell+\frac{1}{2}}(\mathbb{T}^2)} + |\lambda|^{\frac{\ell+1/2}{2}}|\eta|_{H^0(\mathbb{T}^2)} \\ &\leq C_\gamma\Big(|b_3|_{H^{\ell-\frac{3}{2}}(\mathbb{T}^2)} + |\lambda|^{\frac{\ell-3/2}{2}}|b_3|_{H^0(\mathbb{T}^2)} \\ &\quad + |b_0|_{H^{\ell-\frac{1}{2}}(\mathbb{T}^2)} + |\lambda|^{\frac{\ell-1/2}{2}}|b_0|_{H^0(\mathbb{T}^2)}\Big).\end{aligned}$$

Further, η satisfies

$$(4.7)\qquad |\eta|_{H^{\frac{5}{2}}(\mathbb{T}^2)} \leq C_\gamma\left(|b_3|_{H^{\frac{1}{2}}(\mathbb{T}^2)} + |b_0|_{H^{\frac{3}{2}}(\mathbb{T}^2)}\right),$$

$$(4.8)\qquad |\lambda||\eta|_{H^{\frac{3}{2}}(\mathbb{T}^2)} \leq C_\gamma\left(|b_3|_{H^{\frac{1}{2}}(\mathbb{T}^2)} + |b_0|_{H^{\frac{3}{2}}(\mathbb{T}^2)}\right).$$

The constant C_γ remains bounded for $\operatorname{Re}\lambda \geq \gamma$.

We divide the proof of this proposition into several steps. As in the previous section we follow [13]. After expanding the unknowns into the Fourier series in $x' \in \mathbb{T}^2$, we obtain the system of ordinary differential equations (3.6), (3.7) and (3.8) supplemented with the boundary conditions on $y=0$

$$(4.9)\qquad \lambda\eta^{(\xi')} - u_3^{(\xi')} = b_0^{(\xi')},$$

$$(4.10)\qquad i\xi_\alpha u_3^{(\xi')} + \frac{du_\alpha^{(\xi')}}{dy} = 0, \quad \alpha = 1,2,$$

$$(4.11)\qquad -p^{(\xi')} + 2\nu\frac{du_3^{(\xi')}}{dy} - \sigma|\xi'|^2\eta^{(\xi')} = b_3^{(\xi')},$$

where $\xi' = (\xi_1,\xi_2) \in \mathbb{Z}^2$, $|\xi'| = \sqrt{\xi_1^2+\xi_2^2}$ and $b_\alpha^{(\xi')}(\alpha = 0,3)$ denotes the corresponding Fourier coefficient of the boundary data. For a while we assume $\xi' \neq (0,0)$. From (4.9) and (4.11) we can eliminate the unknown $\eta^{(\xi')}$ and replace (4.11) by

$$(4.12)\qquad -p^{(\xi')} + 2\nu\frac{du_3^{(\xi')}}{dy} - \frac{\sigma}{\lambda}|\xi'|^2u_3^{(\xi')} = b_3^{(\xi')} + \frac{\sigma}{\lambda}|\xi'|^2b_0^{(\xi')}.$$

We use (3.9) again to obtain the solution. Substituting (3.9) into (4.10) and (4.12), we derive the linear algebraic system for Φ_1, Φ_2, ϕ.

By virtue of the results in [13], Sections 2 and 3, we get the explicit form of the solution $u^{(\xi')},\ p^{(\xi')},\ \eta^{(\xi')}$

$$u_\alpha^{(\xi')}(\lambda,y) = \frac{e_0(y)}{r(r+|\xi'|)\mathcal{D}} i\xi_\alpha r(r-|\xi'|)\frac{\lambda}{\nu^2}\left(b_3^{(\xi')}+\frac{\sigma}{\lambda}\left|\xi'\right|^2 b_0^{(\xi')}\right) + \frac{e_1(y)}{(r+|\xi'|)\mathcal{D}}(-i\xi_\alpha)\frac{\lambda}{\nu^2}\left(r^2+|\xi'|^2\right)\left(b_3^{(\xi')}+\frac{\sigma}{\lambda}\left|\xi'\right|^2 b_0^{(\xi')}\right),\ \alpha=1,2, \tag{4.13}$$

$$u_3^{(\xi')}(\lambda,y) = \frac{e_0(y)}{\mathcal{D}}(-|\xi'|)\frac{\lambda}{\nu^2}\left(b_3^{(\xi')}+\frac{\sigma}{\lambda}\left|\xi'\right|^2 b_0^{(\xi')}\right) + \frac{e_1(y)}{(r+|\xi'|)\mathcal{D}}|\xi'|\frac{\lambda}{\nu^2}\left(r^2+|\xi'|^2\right)\left(b_3^{(\xi')}+\frac{\sigma}{\lambda}\left|\xi'\right|^2 b_0^{(\xi')}\right), \tag{4.14}$$

$$p^{(\xi')}(\lambda,y) = -\frac{1}{\mathcal{D}}\frac{\lambda}{\nu}\left(r^2+|\xi'|^2\right)\left(b_3^{(\xi')}+\frac{\sigma}{\lambda}\left|\xi'\right|^2 b_0^{(\xi')}\right)e^{-|\xi'|y}, \tag{4.15}$$

$$\eta^{(\xi')} = \frac{1}{\lambda}\left(1-\frac{\sigma|\xi'|^3}{\nu^2\mathcal{D}}\right)b_0^{(\xi')} - \frac{|\xi'|}{\nu^2\mathcal{D}}b_3^{(\xi')}, \tag{4.16}$$

where

$$\mathcal{D} = \left(r^2+|\xi'|^2\right)^2 - 4r|\xi'|^3 + \frac{\sigma}{\nu^2}|\xi'|^3.$$

To obtain crucial estimates we need

Lemma 4.1. *Let $\gamma>0$ be arbitrarily fixed. For $\lambda\in\mathbb{C}$ with $\operatorname{Re}\lambda\ge\gamma$ it holds that*

$$|\mathcal{D}| \ge \frac{\gamma^2}{\nu^2},\quad |\mathcal{D}| \ge \frac{2}{\nu}|\lambda||\xi'|^2, \tag{4.17}$$

$$\sigma|\xi'|^3 \le \nu^2\left(\frac{7}{2}+\frac{\sigma}{2}\sqrt{\frac{\nu}{\gamma}}\right)|\mathcal{D}|,\quad |\lambda|^2 \le \left(3\nu^2+\frac{\sigma}{2}\sqrt{\frac{\nu}{\gamma}}\right)|\mathcal{D}|. \tag{4.18}$$

This lemma was proved in [13], Lemma 2.5. We first note that

$$2^{-\frac{3}{4}}\left(\sqrt{\frac{|\lambda|}{\nu}}+|\xi|\right) \le |r| \le \sqrt{\frac{|\lambda|}{\nu}}+|\xi|. \tag{4.19}$$

We proceed to estimate $u^{(\xi')}$ term by term as in proving Proposition 3.1. The first and second terms of $u_\alpha^{(\xi')}, \alpha=1,2$ can be estimated as follows:

$$|r|^{2\ell}\int_0^\infty |e_0(y)|^2\,dy \frac{\left|i\,\xi_\alpha\right|^2\left|r\,(r-|\xi'|)\right|^2}{\left|r\,(r+|\xi'|)\,\mathcal{D}\right|^2}\left|\frac{\lambda}{\nu^2}b_3^{(\xi')}+\frac{\sigma|\xi'|^2}{\nu^2}b_0^{(\xi')}\right|^2 \le \sqrt{2}\left(\frac{|\lambda|^2}{\nu^4}\frac{|\xi'|^2|r|^2}{|\mathcal{D}|^2}|r|^{2\ell-3}\left|b_3^{(\xi')}\right|^2+\frac{\sigma^2}{\nu^4}\frac{|\xi'|^6}{|\mathcal{D}|^2}|r|^{2\ell-1}\left|b_0^{(\xi')}\right|^2\right), \tag{4.20}$$

$$|r|^{2\ell}\int_0^\infty |e_1(y)|^2\,dy \frac{\left|-i\xi_\alpha\right|^2\left|r^2+|\xi'|^2\right|^2}{\left|(r+|\xi'|)\,\mathcal{D}\right|^2}\left|\frac{\lambda}{\nu^2}b_3^{(\xi')}+\frac{\sigma\left|\xi'\right|^2}{\nu^2}b_0^{(\xi')}\right|^2 \le 4\left(\frac{|\lambda|^2}{\nu^4}\frac{|\xi'||r|^3}{|\mathcal{D}|^2}|r|^{2\ell-3}\left|b_3^{(\xi')}\right|^2+\frac{\sigma^2}{\nu^4}\frac{|r||\xi'|^5}{|\mathcal{D}|^2}|r|^{2\ell-1}\left|b_0^{(\xi')}\right|^2\right). \tag{4.21}$$

Here we have used Lemma 3.2 and the fact that

$$\frac{\left|r^2+|\xi'|^2\right|^2}{\left|r+|\xi'|\right|^2} \leq 2|r|^2.$$

In the same manner we can derive the similar inequalities for the first and second term of $u_3^{(\xi')}$. For $p^{(\xi')}$ we have

$$\begin{aligned}(4.22)\qquad & |r|^{2\ell-4}\int_0^\infty |\xi'|^2\left|p^{(\xi')}\right|^2 dy \\ &\leq\ 4\left(\frac{|\lambda|^2}{\nu^2}\frac{|\xi'||r|^3}{|\mathcal{D}|^2}|r|^{2\ell-3}\left|b_3^{(\xi')}\right|^2+\frac{\sigma^2}{\nu^2}\frac{|r||\xi'|^5}{|\mathcal{D}|^2}|r|^{2\ell-1}\left|b_0^{(\xi')}\right|^2\right).\end{aligned}$$

As a consequence of these estimates, by Lemma 4.1 we have

$$\begin{aligned}(4.23)\qquad & \sum_{\alpha=1}^{3}|r|^{2\ell}\int_0^\infty\left|u_\alpha^{(\xi')}\right|^2 dy+|r|^{2\ell-4}\int_0^\infty|\xi'|^2\left|p^{(\xi')}\right|^2 dy \\ &\leq C\left(|r|^{2\ell-1}\left|b_0^{(\xi')}\right|^2+|r|^{2\ell-3}\left|b_3^{(\xi')}\right|^2\right).\end{aligned}$$

By Lemma 3.2 the derivatives with respect to y up to the ℓ-th order can be bounded by the right hand side of (4.23) if ℓ is a positive integer. For $\xi'=(0,0)$ the solution becomes trivial, because $b_0^{(0,0)}=b_3^{(0,0)}=0$. It also follows from (4.9) that $\eta^{(0,0)}=0$. Collecting these and summing over $\xi'\in\mathbb{Z}^2$, we obtain the estimates for u and p in Proposition 4.1 for a positive integer $\ell\geq 2$. For non integer ℓ we obtain the desired estimates by interpolation.

We now start to estimate $\eta^{(\xi')}$. Note that we can rewrite (4.16) as follows

$$(4.24)\qquad \eta^{(\xi')}=\frac{r^3+|\xi'|r^2+3|\xi'|^2r-|\xi'|^3}{\nu(r+|\xi'|)\mathcal{D}}b_0^{(\xi')}-\frac{|\xi'|}{\nu^2\mathcal{D}}b_3^{(\xi')}\equiv I_0\ +\ I_3.$$

Lemma 4.2. *Let* $\operatorname{Re}\lambda\geq\gamma\,(>0)$ *and* $\ell\geq 2$. *Then*

$$(4.25)\qquad |r|^{\ell+\frac12}|\eta^{(\xi')}|\leq C\left(|r|^{\ell-\frac12}|b_0|+|r|^{\ell-\frac32}\left|b_3^{(\xi')}\right|\right),$$

$$(4.26)\qquad |\xi'|^{\frac52}|\eta^{(\xi')}|+|\lambda||\xi'|^{\frac32}|\eta^{(\xi')}|\leq C\left(|\xi'|^{\frac32}\left|b_0^{(\xi')}\right|+|\xi'|^{\frac12}\left|b_3^{(\xi')}\right|\right).$$

The constant C is uniformly bounded for $\operatorname{Re}\lambda\geq\gamma\,(>0)$.

Proof. From (4.24) and Lemma 4.1, we have

$$\begin{aligned}|r|^{\ell+\frac12}|I_0| &= |r|^{\ell+\frac12}\frac{|r^3+|\xi|r^2+3|\xi|^2r-|\xi|^3|}{|\nu||r+|\xi|||\mathcal{D}|}\left|b_0^{(\xi')}\right| \\ &\leq\frac{C}{\nu|\mathcal{D}|}\left(\sqrt{\frac{|\lambda|}{\nu}}+|\xi|\right)^3|r|^{\ell-\frac12}\left|b_0^{(\xi')}\right| \\ &\leq C|r|^{\ell-\frac12}\left|b_0^{(\xi')}\right|.\end{aligned}$$

It is easily seen from Lemma 4.1 that the constant C is uniformly bounded for $\operatorname{Re}\lambda \geq \gamma\,(>0)$. In the same way we obtain

$$|r|^{\ell+\frac{1}{2}}|I_3| \leq C|r|^{\ell-\frac{3}{2}}\left|b_3^{(\xi')}\right|.$$

Combining these two inequalities, we can show (4.25). The first term of the left hand side of (4.26) can be estimated in the same manner as above. For the second term in (4.26) we use expression (4.16).

$$|\lambda|\,|\xi'|^{\frac{3}{2}}\left|\eta^{(\xi')}\right| \leq \left|1-\frac{\sigma|\xi'|^3}{\nu^2\mathcal{D}}\right||\xi'|^{\frac{3}{2}}\left|b_0^{(\xi')}\right| + \frac{|\lambda||\xi'|^2}{\nu^2\,|\mathcal{D}|}\,|\xi'|^{\frac{1}{2}}\left|b_3^{(\xi')}\right|.$$

From this inequality and Lemma 4.1 the second estimate follows. □

This lemma completes the proof of Proposition 4.1.
From Proposition 4.1 it follows

Corollary 4.1. *Let $2 \leq \ell \leq 4$. Let b_0 be given in $H_0^{\ell-\frac{1}{2}}(\mathbb{T}^2)$. Then we can take $\gamma_1 > 0$ so that , for $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geq \gamma_1$, there exists a unique solution η, u, p to (4.1)–(4.4) and*

$$-p + 2\nu\partial_3 u_3 + (-1+\sigma\Delta_F)\,\eta = 0 \tag{4.27}$$

$$\int_{\Omega_\infty} p\,dx'dy = 0, \qquad \int_{\mathbb{T}^2} \eta\,dx' = 0,$$

which satisfies

$$\begin{aligned}
&|u|_{H^\ell(\Omega_\infty)} + |\lambda|^{\frac{\ell}{2}}|u|_{H^0(\Omega_\infty)} + |\nabla p|_{H^{\ell-2}(\Omega_\infty)} + |\lambda|^{\frac{\ell-2}{2}}\,|\nabla p|_{H^0(\Omega_\infty)}\\
&\quad + |\eta|_{H^{\ell+\frac{1}{2}}(\mathbb{T}^2)} + |\lambda|^{\frac{\ell+1/2}{2}}|\eta|_{H^0(\mathbb{T}^2)}\\
&\quad \leq C_{\gamma_1}\left(|b_0|_{H^{\ell-\frac{1}{2}}(\mathbb{T}^2)} + |\lambda|^{\frac{\ell-1/2}{2}}|b_0|_{H^0(\mathbb{T}^2)}\right).
\end{aligned}$$

Further, this solution satisfies

$$|u|_{H^2(\Omega_\infty)} + |\lambda||u|_{H^0(\Omega_\infty)} + |\nabla p|_{H^0(\Omega_\infty)} \leq C_{\gamma_1}|b_0|_{H^{\frac{3}{2}}(\mathbb{T}^2)}, \tag{4.28}$$

$$|\eta|_{H^{\frac{5}{2}}(\mathbb{T}^2)} + |\lambda||\eta|_{H^{\frac{3}{2}}(\mathbb{T}^2)} \leq C_{\gamma_1}|b_0|_{H^{\frac{3}{2}}(\mathbb{T}^2)}. \tag{4.29}$$

The constant C_{γ_1} remains bounded for $\operatorname{Re}\lambda \geq \gamma_1$.

Proof. Let η be given in $H_0^{\frac{3}{2}}(\mathbb{T}^2)$. By Proposition 4.1 we have the solution $\check{\eta}, u, p$ to the problem (4.1)–(4.4) and

$$-p + 2\nu\partial_3 u_3 + \sigma\Delta_F\check{\eta} = \eta.$$

Then we can define the mapping

$$H_0^{\frac{3}{2}}\left(\mathbb{T}^2\right) \ni \eta \to \check{\eta} \in H_0^{\frac{3}{2}}\left(\mathbb{T}^2\right)$$

for fixed b_0. For given η_j, $j=1,2$ in $H_0^{\frac{3}{2}}\left(\mathbb{T}^2\right)$, estimate (4.8) yields

$$|\lambda|\left|\check{\eta}_1-\check{\eta}_2\right|_{H^{\frac{3}{2}}(\mathbb{T}^2)} \le C_\gamma\left|\eta_1-\eta_2\right|_{H^{\frac{1}{2}}(\mathbb{T}^2)} \le C_\gamma\left|\eta_1-\eta_2\right|_{H^{\frac{3}{2}}(\mathbb{T}^2)}.$$

We can choose $\gamma_1>0$ so that, if $\operatorname{Re}\lambda\ge\gamma_1$, then

$$\frac{C_\gamma}{|\lambda|}\le\frac{C_\gamma}{\operatorname{Re}\lambda}\le\frac{C_\gamma}{\gamma_1}\le\frac{1}{2}.$$

Hence, for λ with $\operatorname{Re}\lambda\ge\gamma_1$ we have the unique fixed point $\eta=\check{\eta}$, which solves (4.1)–(4.4) and (4.27) together with the corresponding u and p. For this η, by estimates (4.7), (4.8), it holds that

$$|\eta|_{H^{\frac{5}{2}}(\mathbb{T}^2)}+|\lambda||\eta|_{H^{\frac{3}{2}}(\mathbb{T}^2)}\le C_{\gamma_1}|b_0|_{H^{\frac{3}{2}}(\mathbb{T}^2)}. \tag{4.30}$$

The constant C_{γ_1} remains bounded for $\operatorname{Re}\lambda\ge\gamma_1$. To obtain estimate (4.28), we set $\ell=2$ in (4.20) and (4.21) with b_3 replaced by $\eta=\check{\eta}$. To estimate the first and second terms of $u_\alpha^{(\xi')}$, $\alpha=1,2$, we have

$$\begin{aligned}|r|^4\int_0^\infty|e_0(y)|^2\,dy\frac{\left|i\,\xi_\alpha\right|^2\left|r\left(r-|\xi'|\right)\right|^2}{\left|r\left(r+|\xi'|\right)\mathcal{D}\right|^2}&\left|\frac{\lambda}{\nu^2}\eta^{(\xi')}+\frac{\sigma|\xi'|^2}{\nu^2}\,b_0^{(\xi')}\right|^2\\ \le\sqrt{2}\left(\frac{1}{\nu^4}\frac{|\xi'||r|^3}{|\mathcal{D}|^2}|\lambda|^2|\xi'|\left|\eta^{(\xi')}\right|^2\right.&\left.+\frac{\sigma^2}{\nu^4}\frac{|\xi'|^3|r|^3}{|\mathcal{D}|^2}|\xi'|^3\left|b_0^{(\xi')}\right|^2\right),\end{aligned} \tag{4.31}$$

$$\begin{aligned}|r|^4\int_0^\infty|e_1(y)|^2\,dy\frac{\left|-i\xi_\alpha\right|^2\left|r^2+|\xi'|^2\right|^2}{\left|\left(r+|\xi'|\right)\mathcal{D}\right|^2}&\left|\frac{\lambda}{\nu^2}\eta^{(\xi')}+\frac{\sigma\left|\xi'\right|^2}{\nu^2}b_0^{(\xi')}\right|^2\\ \le4\left(\frac{1}{\nu^4}\frac{|r|^4}{|\mathcal{D}|^2}|\lambda|^2|\xi'|\left|\eta^{(\xi')}\right|^2\right.&\left.+\frac{\sigma^2}{\nu^4}\frac{|r|^4|\xi'|^2}{|\mathcal{D}|^2}|\xi'|^3\left|b_0^{(\xi')}\right|^2\right).\end{aligned} \tag{4.32}$$

By Lemma 4.1 we see that the right hand sides of both inequalities above can be bounded by

$$C\left(|\lambda|^2|\xi'|\left|\eta^{(\xi')}\right|^2+|\xi'|^3\left|b_0^{(\xi')}\right|^2\right)$$

where the constant C is independent of λ and ξ'. We can estimate u_3 in the same manner. Since we already have (4.30), taking sums in $\xi'\in\mathbb{Z}^2$, we can show

$$|u|_{H^2(\Omega_\infty)}+|\lambda||u|_{H^0(\Omega_\infty)}\le C_{\gamma_1}|b_0|_{H^{\frac{3}{2}}(\mathbb{T}^2)}.$$

For p we set $\ell=2$ in (4.22) with b_3 replaced by η

$$\int_0^\infty|\xi'|^2\left|p^{(\xi')}\right|^2dy\le4\left(\frac{1}{\nu^2}\frac{|r|^4}{|\mathcal{D}|^2}|\lambda|^2|\xi'|\left|\eta^{(\xi')}\right|^2+\frac{\sigma^2}{\nu^2}\frac{|r|^4|\xi'|^2}{|\mathcal{D}|^2}|\xi'|^3\left|b_0^{(\xi')}\right|^2\right)$$

and proceed as above. Combining the estimates for u and η, we obtain (4.28). □

We next prove

Proposition 4.2. *Let b_0 be given in $H_0^{\frac{3}{2}}(\mathbb{T}^2)$. There is a $\gamma_2 > 0$ such that, for $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geq \gamma_2$, the problem*

$$\lambda\eta - u_3 = b_0 \quad \text{on} \quad S_F, \tag{4.33}$$
$$\lambda u - \nu\Delta u + \nabla p = 0 \quad \text{in} \quad \Omega, \tag{4.34}$$
$$\operatorname{div} u = 0 \quad \text{in} \quad \Omega, \tag{4.35}$$
$$u = 0 \quad \text{on} \quad S_B, \tag{4.36}$$
$$\partial_\alpha u_3 + \partial_3 u_\alpha = 0, \quad \alpha = 1,2, \quad \text{on} \quad S_F, \tag{4.37}$$
$$-p + 2\nu\partial_3 u_3 + (1 - \sigma\Delta_F)\,\eta = 0 \quad \text{on} \quad S_F, \tag{4.38}$$

$$\int_\Omega p\, dx'dy = 0, \qquad \int_{\mathbb{T}^2} \eta\, dx' = 0.$$

has a unique solution η, u, p which satisfies

$$|u|_{H^2(\Omega)} + |\lambda||u|_{H^0(\Omega)} + |\nabla p|_{H^0(\Omega)} \leq C|b_0|_{H^{\frac{3}{2}}(\mathbb{T}^2)}, \tag{4.39}$$
$$|\eta|_{H^{\frac{5}{2}}(\mathbb{T}^2)} + |\lambda||\eta|_{H^{\frac{3}{2}}(\mathbb{T}^2)} \leq C|b_0|_{H^{\frac{3}{2}}(\mathbb{T}^2)}. \tag{4.40}$$

The proof of this proposition starts with the solution η, u^f, p^f to the problem in the half space

$$\lambda\eta - u_3^f = b_0 \quad \text{on} \quad \partial\Omega_{-\infty} = \mathbb{T}^2 \times \{y = 0\}, \tag{4.41}$$
$$\lambda u^f - \nu\Delta u^f + \nabla p^f = 0 \quad \text{in} \quad \Omega_{-\infty}, \tag{4.42}$$
$$\operatorname{div} u^f = 0 \quad \text{in} \quad \Omega_{-\infty}, \tag{4.43}$$
$$\partial_\alpha u_3^f + \partial_3 u_\alpha^f = 0, \; \alpha = 1,2, \quad \text{on} \quad \partial\Omega_{-\infty}, \tag{4.44}$$
$$-p^f + 2\nu\partial_3 u_3^f + (1 - \sigma\Delta_F)\,\eta = 0 \quad \text{on} \quad \partial\Omega_{-\infty}. \tag{4.45}$$

Corollary 4.1 implies the existence of such a solution. Indeed, let u', p', η' be the solution to the problem (4.1)–(4.4) and (4.27) in Ω_∞ for $-b_0$ given in Corollary 4.1. Then for $(x', y) \in \Omega_{-\infty}$

$$u^f(x', y) = (u_1'(x', -y), u_2'(x', -y), -u_3'(x', -y)), \; p^f(x', y) = p'(x', -y) \tag{4.46}$$

and $\eta(x') = -\eta'(x')$ satisfy (4.41)–(4.45). Using the cut off function $\zeta(y)$ used in Section 3, we set $u^1 = \zeta(y)u^f$, $p^1 = \zeta(y)p^f$. To adjust the solenoidal condition, we put $u^2 = \nabla\phi^2$, where ϕ^2 is the solution to the problem

$$\Delta\phi^2 = -\zeta'(y)u_3^f \quad \text{in } \Omega, \tag{4.47}$$
$$\phi^2 = 0 \quad \text{on } S_F, \; \partial_y\phi^2 = 0 \quad \text{on } S_B, \tag{4.48}$$

guaranteed by Lemma 3.3. Here we notice that the constant mode $\phi^{2,(0,0)}$ of ϕ^2 is equal to zero since $u_3^{f,(0,0)}$ is zero. To adjust the boundary conditions on S_F and S_B, we use Proposition 3.3, noting that $u_3^2 = 0$ on S_B. Let u^3, p^3 be the solution to the problem

$$\lambda u^3 - \nu \Delta u^3 + \nabla p^3 = 0, \quad \operatorname{div} u^3 = 0 \qquad \text{in} \quad \Omega, \tag{4.49}$$

$$\partial_\alpha u_3^3 + \partial_3 u_\alpha^3 = -\left(\partial_\alpha u_3^2 + \partial_3 u_\alpha^2\right), \alpha = 1, 2, \qquad \text{on} \quad S_F, \tag{4.50}$$

$$-p^3 + 2\nu \partial_3 u_3^3 = -2\nu \partial_3 u_3^2 \qquad \text{on} \quad S_F, \tag{4.51}$$

$$u^3 = -\nabla \phi^2 \qquad \text{on} \quad S_B. \tag{4.52}$$

Proposition 2.1 implies the existence of the solution u^4, p^4 to the homogeneous boundary value problem

$$\lambda u^4 - \nu \Delta u^4 + \nabla p^4 = \nu\left(2\zeta' \partial_3 u^f + \zeta'' u^f\right) - p^f \nabla \zeta - \lambda u^2 + \nu \Delta u^2, \tag{4.53}$$
$$\operatorname{div} u^4 = 0 \qquad \text{in} \quad \Omega,$$

$$\partial_\alpha u_3^4 + \partial_3 u_\alpha^4 = 0, \ \alpha = 1, 2, \qquad \text{on} \quad S_F, \tag{4.54}$$

$$-p^4 + 2\nu \partial_3 u_3^4 = 0 \qquad \text{on} \quad S_F, \tag{4.55}$$

$$u^4 = 0 \qquad \text{on} \quad S_B. \tag{4.56}$$

Here we set $\tilde{u} = u^1 + u^2 + u^3 + u^4$, $\tilde{p} = p^1 + p^3 + p^4$. Then we see that

$$\lambda \tilde{u} - \nu \Delta \tilde{u} + \nabla \tilde{p} = 0 \qquad \text{in} \quad \Omega, \tag{4.57}$$

$$\operatorname{div} \tilde{u} = 0 \qquad \text{in} \quad \Omega, \tag{4.58}$$

$$\partial_\alpha \tilde{u}_3 + \partial_3 \tilde{u}_\alpha = 0, \quad \alpha = 1, 2, \qquad \text{on} \quad S_F, \tag{4.59}$$

$$-\tilde{p} + 2\nu \partial_3 \tilde{u}_3 + (1 - \sigma \Delta_F)\eta = 0 \qquad \text{on} \quad S_F, \tag{4.60}$$

$$\tilde{u} = 0 \qquad \text{on} \quad S_B, \tag{4.61}$$

and that

$$\lambda \eta - \tilde{u}_3 = b_0 + \mathcal{M} b_0 \qquad \text{on} \quad S_F. \tag{4.62}$$

Here $\mathcal{M}$ is the linear operator defined by

$$\mathcal{M} b_0 \equiv -R(u^2 + u^3 + u^4) = -\left(u_3^2 + u_3^3 + u_3^4\right)\big|_{S_F}.$$

By virtue of the estimates of solutions in Corollary 4.1, Lemma 3.3, Proposition 3.3 and Proposition 2.1, we see that $\mathcal{M}$ is the bounded operator in $H_0^{\frac{3}{2}}\left(\mathbb{T}^2\right)$. If there exists a bounded inverse $(I + \mathcal{M})^{-1}$ in $H_0^{\frac{3}{2}}\left(\mathbb{T}^2\right)$, the proof of Proposition 4.2 is completed by solving problem (4.33)–(4.38) with b_0 replaced by $(I + \mathcal{M})^{-1} b_0$. For this we begin by estimating each $R u^j$, $j = 2, 3, 4$ in $\mathcal{M} b_0$ in terms of the norm $|b_0|_{H^{\frac{3}{2}}(\mathbb{T}^2)}$. It is clear that $\left|R\, u^j\right|_{H^{\frac{3}{2}}(\mathbb{T}^2)} \leq \left|u^j\right|_{H^2(\Omega)}$. By Proposition 2.1 the norm $\left|u^4\right|_{H^2(\Omega)}$ can be estimated in terms of the L^2 norm of the right hand sides (4.53), which is bounded as follows:

$$\begin{aligned} &\left|\nu\left(2\zeta' \partial_3 u^f + \zeta'' u^f\right) - p^f \nabla \zeta - \lambda u^2 + \nu \Delta u^2\right|_{H^0(\Omega)} \\ &\qquad \leq C\left(\left|u^f\right|_{H^1(\Omega_{-\infty})} + \left|p^f\right|_{H^0(\Omega_{-\infty})} + \left|\lambda u^2\right|_{H^0(\Omega)} + \left|u^2\right|_{H^2(\Omega)}\right). \end{aligned} \tag{4.63}$$

Since $u^2 = \nabla\phi^2$ and ϕ^2 is the solution to problem (4.47), (4.48), it holds that, for any $\psi \in H^1(\Omega)$ vanishing on S_F,

$$\left(\lambda u^2, \nabla\psi\right)_{L^2(\Omega)} = \left(-\lambda \operatorname{div}(\zeta(y)u^f), \psi\right)_{L^2(\Omega)} = \left(\lambda\zeta(y)u^f, \nabla\psi\right)_{L^2(\Omega)}.$$

From this fact it follows that

$$\left|\lambda u^2\right|_{H^0(\Omega)} \le \left|\lambda u^f\right|_{H^0(\Omega_{-\infty})},$$

whose right hand side can be estimated when we recall (4.46) and expressions (4.13), (4.14) with b_3 replaced by η. In fact, for the first and second terms of $u_\alpha^{f,(\xi')}, \alpha = 1, 2$

$$(4.64)\qquad |\lambda|^2 \int_0^\infty |e_0(y)|^2\, dy \frac{|i\,\xi_\alpha|^2\, |r\,(r-|\xi'|)|^2}{|r\,(r+|\xi'|)\,\mathcal{D}|^2} \left|\frac{\lambda}{\nu^2}\eta^{(\xi')} + \frac{\sigma|\xi'|^2}{\nu^2}\, b_0^{(\xi')}\right|^2$$
$$\le \frac{\sqrt{2}}{\nu^4}\left(\frac{|\lambda|^2}{|r|\,|\mathcal{D}|^2}|\lambda|^2|\xi'|^2\left|\eta^{(\xi')}\right|^2 + \sigma^2\frac{|\xi'|^3|\lambda|^2}{|r|\,|\mathcal{D}|^2}|\xi'|^3\left|b_0^{(\xi')}\right|^2\right),$$

$$(4.65)\qquad |\lambda|^2 \int_0^\infty |e_1(y)|^2\, dy \frac{|-i\xi_\alpha|^2\, \left|r^2+|\xi'|^2\right|^2}{|(r+|\xi'|)\,\mathcal{D}|^2} \left|\frac{\lambda}{\nu^2}\eta^{(\xi')} + \frac{\sigma\,|\xi'|^2}{\nu^2} b_0^{(\xi')}\right|^2$$
$$\le \frac{4}{\nu^4}\left(\frac{|\lambda|^2}{|\mathcal{D}|^2}|\lambda|^2|\xi'|\left|\eta^{(\xi')}\right|^2 + \sigma^2\frac{|\lambda|^2|\xi'|^2}{|\mathcal{D}|^2}|\xi'|^3\left|b_0^{(\xi')}\right|^2\right)$$

(cf. (4.20), (4.21) with $\ell = 0$). We can estimate u_3^f in the same manner. Summing in $\xi' \in \mathbb{Z}^2$ and using Lemma 4.1 and (4.29), we obtain

$$\left|\lambda u^2\right|_{H^0(\Omega)} \le \left|\lambda u^f\right|_{H^0(\Omega_{-\infty})} \le C|\lambda|^{-\frac{1}{4}}\,|b_0|_{H^{\frac{3}{2}}(\mathbb{T}^2)}.$$

The last term of (4.63) can be estimated by (4.28) as follows:

$$(4.66)\qquad \left|\Delta u^2\right|_{H^0(\Omega)} \le \left|\phi^2\right|_{H^3(\Omega)} \le C\left|u^f\right|_{H^1(\Omega_{-\infty})}$$
$$\le C\left|u^f\right|^{\frac{1}{2}}_{H^0(\Omega_{-\infty})}\left|u^f\right|^{\frac{1}{2}}_{H^2(\Omega_{-\infty})} \le C|\lambda|^{-\frac{5}{8}}\,|b_0|_{H^{\frac{3}{2}}(\mathbb{T}^2)}.$$

Here we used the convexity of Sobolev norm. To estimate $|p^f|$ we use (4.15) with b_3 replaced by η and proceed as above

$$\int_0^\infty \left|p^{\,f,(\xi')}\right|^2 dy \le \frac{4|r|^4}{\nu^2\,|\mathcal{D}|^2}\left(\frac{|\lambda|^2}{|\xi'|}\left|\eta^{(\xi')}\right|^2 + \sigma^2|\xi'|^3\left|b_0^{(\xi')}\right|^2\right).$$

By Lemma 4.1 the quantity

$$\frac{|\lambda|^2|r|^4}{|\mathcal{D}|^2}$$

is bounded from above. Hence, summing in ξ' and using estimate (4.29), we obtain

$$\left|p^f\right|_{H^0(\Omega_{-\infty})} \le \frac{C}{|\lambda|}\,|b_0|_{H^{\frac{3}{2}}(\mathbb{T}^2)}.$$

We next estimate $\left|u^3\right|_{H^0(\Omega)}$. By (3.24) in Proposition 3.3 it holds that

$$(4.67)\quad \left|u^3\right|_{H^2(\Omega)} \le C\Bigg(\left|\nabla\phi^2\right|_{H^{\frac{3}{2}}(\mathbb{T}^2)} + |\lambda|^{\frac{3}{4}} \left|\nabla\phi^2\right|_{H0(\mathbb{T}^2)} + \sum_{\alpha=1}^{3} \left(\left|\partial_\alpha u_3^2 + \partial_3 u_\alpha^2\right|_{H^{\frac{1}{2}}(\mathbb{T}^2)} + |\lambda|^{\frac{1}{4}} \left|\partial_\alpha u_3^2 + \partial_3 u_\alpha^2\right|_{H^0(\mathbb{T}^2)} \right) \Bigg).$$

Since $u^2 = \nabla\phi^2$, (4.66) yields

$$\left|\partial_\alpha u_3^2 + \partial_3 u_\alpha^2\right|_{H^{\frac{1}{2}}(\mathbb{T}^2)} = \left|2\partial_\alpha\partial_3\phi^2\right|_{H^{\frac{1}{2}}(\mathbb{T}^2)} \le C\left|\phi^2\right|_{H^3(\Omega)} \le C\left|u^f\right|_{H^1(\Omega_{-\infty})} \le \frac{C}{|\lambda|^{\frac{5}{8}}}\left|b_0\right|_{H^{\frac{3}{2}}(\mathbb{T}^2)},$$

$$|\lambda|^{\frac{1}{4}}\left|\partial_\alpha u_3^2 + \partial_3 u_\alpha^2\right|_{H^0(\mathbb{T}^2)} \le \frac{C}{|\lambda|^{\frac{3}{8}}}\left|b_0\right|_{H^{\frac{3}{2}}(\mathbb{T}^2)}, \ \alpha = 1,2,3,$$

and

$$\left|\nabla\phi^2\right|_{H^{\frac{3}{2}}(\mathbb{T}^2)} \le C\left|\phi^2\right|_{H^3(\Omega)} \le \frac{C}{|\lambda|^{\frac{5}{8}}}\left|b_0\right|_{H^{\frac{3}{2}}(\mathbb{T}^2)}.$$

We need to estimate the second term in (4.67) more carefully because of the factor $|\lambda|^{\frac{3}{4}}$. Noting that

$$\nabla\phi^2 = \left(\partial_1\phi^2, \partial_2\phi^2, 0\right) \quad \text{on } S_B,$$

we see that the norm $\left|\nabla\phi^2\right|_{H^0(\mathbb{T}^2)}$ is equivalent to the square root of

$$\sum_{\xi'\in\mathbb{Z}^2\setminus\{(0,0)\}} |\xi'|^2 \left|\phi^{2,(\xi')}(-1)\right|^2,$$

where

$$\phi^2(x',y) = \sum_{\xi'\in\mathbb{Z}^2\setminus\{(0,0)\}} \phi^{2,(\xi')}(y)\, e^{ix'\cdot\xi'}, \ (x',y)\in\Omega = \mathbb{T}^2\times(-1,0).$$

From equation (4.47) it follows that, for each $\xi'\in\mathbb{Z}^2\setminus\{(0,0)\}$,

$$(4.68)\quad -\left(\frac{d}{dy}\right)^2\phi^{2,(\xi')}(y) + |\xi'|^2\phi^{2,(\xi')}(y) = \zeta'(y)u_3^{f,(\xi')}(y) \quad \text{on } (-1,0).$$

We take the scalar product of (4.68) with $|\xi'|^2\phi^{2,(\xi')}(y)$ on the interval $(-1,0)$. Then, integrating by parts and using the boundary conditions (4.48), we obtain

$$\int_{-1}^{0}\left|\frac{d}{dy}|\xi'|\phi^{2,(\xi')}(y)\right|^2 dy + \int_{-1}^{0}\left||\xi'|^2\phi^{2,(\xi')}(y)\right|^2 dy \le \left(\int_{-1}^{0}\left|\zeta'(y)u_3^{f,(\xi')}(y)\right|^2 dy\right)^{\frac{1}{2}} \left(\int_{-1}^{0}\left||\xi'|^2\phi^{2,(\xi')}(y)\right|^2 dy\right)^{\frac{1}{2}}$$

by the Schwarz inequality. From this we can derive

$$\int_{-1}^{0} \left| \frac{d}{dy} |\xi'| \phi^{2,(\xi')}(y) \right|^2 dy \le \frac{1}{2} \int_{-1}^{0} \left| \zeta'(y) u_3^{f,(\xi')}(y) \right|^2 dy.$$

This inequality together with the equality

$$|\xi'| \phi^{2,(\xi')}(-1) = \int_{-1}^{0} \frac{d}{dy} \Big(|\xi'| \phi^{2,(\xi')}(y) \Big) \, dy$$

yields

$$\begin{aligned} |\xi'|^2 \left| \phi^{2,(\xi')}(-1) \right|^2 &\le \left(\int_{-1}^{0} \left| \frac{d}{dy} \Big(|\xi'| \phi^{2,(\xi')}(y) \Big) \right| dy \right)^2 \\ &\le \int_{-1}^{0} \left| \frac{d}{dy} |\xi'| \phi^{2,(\xi')}(y) \right|^2 dy \le \frac{1}{2} \int_{-1}^{0} \left| \zeta'(y) u_3^{f,(\xi')}(y) \right|^2 dy. \end{aligned}$$

Summing in ξ' and using estimate (4.28) we obtain

$$|\lambda|^{\frac{3}{4}} \left| \nabla \phi^2 \right|_{H^0(\mathbb{T}^2)} \le C |\lambda|^{\frac{3}{4}} \left| u^f \right|_{H^0(\Omega_{-\infty})} \le C |\lambda|^{-\frac{1}{4}} \left| b_0 \right|_{H^{\frac{3}{2}}(\mathbb{T}^2)}.$$

Collecting the estimates obtained above, we can show

$$\left| \mathcal{M} b_0 \right|_{H^{\frac{3}{2}}(\mathbb{T}^2)} \le \frac{C}{|\lambda|^{\frac{1}{4}}} \left| b_0 \right|_{H^{\frac{3}{2}}(\mathbb{T}^2)},$$

where the constant C is independent of λ. Therefore we can conclude the assertion of Proposition 4.2.

5. Linear nonstationary problem

In this section we solve globally in time the linear nonstationary problem

(5.1)	$\partial_t \eta - v_3 = 0$	on S_F,
(5.2)	$\partial_t v - \nu \Delta v + \nabla q = f_0$	in Ω,
(5.3)	$\operatorname{div} v = 0$	in Ω,
(5.4)	$v = 0$	on S_B,
(5.5)	$\partial_3 v_j + \partial_j v_3 = 0, \quad j = 1, 2,$	on S_F,
(5.6)	$q - 2\nu \partial_3 v_3 - (1 - \sigma \Delta_F) \eta = 0$	on S_F,
(5.7)	$v = 0, \ \eta = 0$	$t = 0$

for arbitrarily given $f_0 \in K_{(0)}^{\ell-2}(\Omega \times \mathbb{R}^+)$ with $3 < \ell < \frac{7}{2}$. As seen in Section 2, we obtain the linear stationary problem (2.14), (2.15) with a parameter $\lambda \in \mathbb{C}$ by transforming in t. By applying the orthogonal projection P^0 introduced in Section 2, this stationary problem was reduced to (2.16):

$$(\lambda - G) \begin{pmatrix} \eta \\ v \end{pmatrix} = \begin{pmatrix} g_0 \\ f \end{pmatrix} \tag{5.8}$$

with $g_0 \in H_0^{\frac{3}{2}}\left(\mathbb{T}^2\right)$ and $f \in P^0 H^0(\Omega)$. G is the matrix of operators

$$G = \begin{pmatrix} 0 & R \\ -R^*(1-\sigma\Delta_F) & -A \end{pmatrix}$$

introduced in Section 2. From now on we set $X = H_0^{\frac{3}{2}}\left(\mathbb{T}^2\right) \times P^0 H^0(\Omega)$ and $Y = H_0^{\frac{5}{2}}\left(\mathbb{T}^2\right) \times P^0 H^2(\Omega)$.

Proposition 5.1. *Let $\gamma_2 > 0$ be the constant in Proposition* 4.2. *Let (g_0, f) be arbitrarily given in X. If λ satisfies $\mathrm{Re}\,\lambda \geq \gamma_2$, then there is a unique solution (η, v) to problem* (5.8) *such that*

$$\partial_j v_3 + \partial_3 v_j = 0 \quad \text{on } S_F,\ j = 1, 2, \tag{5.9}$$

$$v = 0 \quad \text{on } S_B, \tag{5.10}$$

$$\begin{aligned} |v|_{H^2(\Omega)} + |\lambda||v|_{H^0(\Omega)} + |\eta|_{H^{\frac{5}{2}}(\mathbb{T}^2)} + |\lambda||\eta|_{H^{\frac{3}{2}}(\mathbb{T}^2)} \\ \leq C\left(|f|_{H^0(\Omega)} + |g_0|_{H^{\frac{3}{2}}(\mathbb{T}^2)}\right). \end{aligned} \tag{5.11}$$

The constant $C > 0$ remains bounded for $\mathrm{Re}\,\lambda \geq \gamma_2$.

Proof. As noticed in the end of Section 2, we first take the solution $v^{(0)}, q^{(0)}$ to problem (2.17)–(2.21) obtained in Proposition 2.1. Note that $Rv^{(0)}$ belongs to $H_0^{\frac{3}{2}}\left(\mathbb{T}^2\right)$. We next take the solution $\eta, v^{(1)}, q^{(1)}$ to the problem

$$\lambda\eta - v_3^{(1)} = Rv^{(0)} + g_0 \qquad \text{on} \quad S_F, \tag{5.12}$$

$$\lambda v^{(1)} - \nu\Delta v^{(1)} + \nabla q^{(1)} = 0 \qquad \text{in} \quad \Omega, \tag{5.13}$$

$$\mathrm{div}\, v^{(1)} = 0 \qquad \text{in} \quad \Omega, \tag{5.14}$$

$$v^{(1)} = 0 \qquad \text{on} \quad S_B, \tag{5.15}$$

$$\partial_\alpha v_3^{(1)} + \partial_3 v_\alpha^{(1)} = 0, \quad \alpha = 1, 2, \qquad \text{on} \quad S_F, \tag{5.16}$$

$$-q^{(1)} + 2\nu\partial_3 v_3^{(1)} + (1 - \sigma\Delta_F)\eta = 0 \qquad \text{on} \quad S_F, \tag{5.17}$$

by using Proposition 4.2 . We apply P^0 to (2.17) and to (5.13). Using Lemma 2.3 and boundary conditions (2.21), (5.17), we have

$$\lambda v^{(0)} + Av^{(0)} = f$$

$$\lambda v^{(1)} + Av^{(1)} + R^*(1 - \sigma\Delta_F)\eta = 0 \qquad \text{in} \quad \Omega. \tag{5.18}$$

Setting $v = v^{(0)} + v^{(1)}$, we obtain the solution (η, v) to (5.8). Estimate (5.11) comes from the estimates in Propositions 2.1, 4.2. □

By this proposition we can regard G as a densely defined closed operator in X, whose domain of definition is given by

$$\begin{aligned} \mathcal{D}(G) = \Big\{(\eta, v) \in X \ ;\ \eta \in H_0^{\frac{3}{2}}(\mathbb{T}^2),\ v \in P^0 H^2(\Omega), \\ \partial_j v_3 + \partial_3 v_j = 0 \ \text{ on } S_F,\ j = 1, 2, \quad v = 0 \quad \text{on } S_B\Big\}. \end{aligned} \tag{5.19}$$

Proposition 4.2 above states that $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geq \gamma_2$ belongs to the resolvent set $\rho(G)$ and that the resolvent operator satisfies

$$\left|(\lambda - G)^{-1}\right|_X \leq \frac{C}{|\lambda|}, \tag{5.20}$$

$$\left|(\lambda - G)^{-1}\right|_{X \to Y} \leq C, \tag{5.21}$$

for $\operatorname{Re}\lambda \geq \gamma_2$. As a consequence of (5.20) and (5.21) we have

Corollary 5.1. *We can take $\theta \in \left(\frac{\pi}{2}, \pi\right)$ so that, if $|\arg(\lambda - \gamma_2)| \leq \theta$, then $\lambda \in \rho(G)$ and estimate* (5.21) *holds with a different constant C.*

For the proof see, e.g., [6], Chapter 1, Section 3. Another important feature following from (5.11) is that $(\lambda - G)^{-1}$ is a compact operator by the Rellich theorem. Hence the spectrum $\sigma(G)$ consists of eigenvalues, and if $\lambda \in \mathbb{C}$ is not an eigenvalue of G, λ belongs to $\rho(G)$. Keeping this in mind, we have

Lemma 5.1. *If* $\operatorname{Re}\lambda \geq 0$*, then λ belongs to $\rho(G)$.*

Proof. Let λ be as such. Suppose that

$$(\lambda - G)\begin{pmatrix}\eta \\ v\end{pmatrix} = \begin{pmatrix}0 \\ 0\end{pmatrix},$$

or equivalently

$$\begin{gathered}\lambda\eta - Rv = 0 \qquad \text{on} \quad S_F, \\ \lambda v + Av + R^*(1 - \sigma\Delta_F)\eta = 0 \qquad \text{in} \quad \Omega.\end{gathered}$$

Taking account of the definition of P^0 and R^*, we can recover a scalar q so that

$$\lambda\eta - v_3 = 0 \qquad \text{on} \quad S_F, \tag{5.22}$$

$$\lambda v - \nu\Delta v + \nabla q = 0 \qquad \text{in} \quad \Omega, \tag{5.23}$$

$$\operatorname{div} v = 0 \qquad \text{in} \quad \Omega, \tag{5.24}$$

$$v = 0 \qquad \text{on} \quad S_B, \tag{5.25}$$

$$\partial_j v_3 + \partial_3 v_j = 0, \quad j = 1, 2, \qquad \text{on} \quad S_F, \tag{5.26}$$

$$-q + 2\nu\partial_3 v_3 + (1 - \sigma\Delta_F)\eta = 0 \qquad \text{on} \quad S_F. \tag{5.27}$$

We take the scalar product of (5.23) with v. Employing integral identity (2.3) and using the boundary conditions (5.25)–(5.27), we have

$$\lambda(v, v)_{L^2} + \langle v, v\rangle + \int_{\mathbb{T}^2}(1 - \sigma\Delta_F)\eta v_3^* dx' = 0.$$

Substituting $v_3 = \lambda\eta$ into the boundary integral and integrating by parts, we have

$$\lambda(v, v)_{L^2} + \langle v, v\rangle + \lambda^* \int_{\mathbb{T}^2}(|\eta|^2 + \sigma\left|\nabla_F \eta\right|^2)dx' = 0.$$

Taking the real part of this identity, we see that $v \equiv 0$ by Lemma 2.5 and $\eta = 0$. Hence, if $\operatorname{Re}\lambda \geq 0$, λ belongs to $\rho(G)$. □

Since the resolvent set $\rho(G)$ is open in $\mathbb{C}$, we can extend $\rho(G)$ beyond the imaginary axis by this lemma. Combining this fact and Corollary 5.1, we can obtain

Proposition 5.2. *We can take $\gamma_0 > 0$ and $\theta_0 \in \left(\frac{\pi}{2}, \pi\right)$ so that , if $|\arg(\lambda + \gamma_0)| \leq \theta_0$, then λ belongs to $\rho(G)$ and it holds that*

$$\left|(\lambda - G)^{-1}\right|_X \leq \frac{C}{1 + |\lambda|},$$
$$\left|(\lambda - G)^{-1}\right|_{X \to Y} \leq C. \tag{5.28}$$

We state the regularity property of the solution to (5.8).

Proposition 5.3. *Let $\ell \geq 2$. Assume that λ satisfies the same condition in Proposition 5.2. Suppose $f \in P^0 H^{\ell-2}(\Omega), g_0 \in H_0^{\ell-\frac{1}{2}}(\mathbb{T}^2)$. Then the solution*

$$\begin{pmatrix} \eta \\ v \end{pmatrix} = (\lambda - G)^{-1} \begin{pmatrix} g_0 \\ f \end{pmatrix} \tag{5.29}$$

satisfies

$$\begin{aligned} &|v|_{H^\ell(\Omega)} + |\lambda|^{\frac{\ell}{2}}|v|_{H^0(\Omega)} + |\eta|_{H^{\ell+\frac{1}{2}}(\mathbb{T}^2)} + |\lambda|^{\frac{\ell}{2}+\frac{1}{4}}|\eta|_{H^0(\mathbb{T}^2)} \\ &\leq C\left(|f|_{H^{\ell-2}(\Omega)} + |\lambda|^{\frac{\ell}{2}-1}|f|_{H^0(\Omega)} + |g_0|_{H^{\ell-\frac{1}{2}}(\mathbb{T}^2)} + |\lambda|^{\frac{\ell}{2}-\frac{1}{4}}|g_0|_{H^0(\mathbb{T}^2)}\right). \end{aligned} \tag{5.30}$$

Proof. As in the proof of the previous lemma, we treat the following problem by recovering q:

$$\lambda\eta - v = g_0 \quad \text{on} \quad S_F, \tag{5.31}$$
$$\lambda v - \nu\Delta v + \nabla q = f \quad \text{in} \quad \Omega, \tag{5.32}$$
$$\operatorname{div} v = 0 \quad \text{in} \quad \Omega, \tag{5.33}$$
$$v = 0 \quad \text{on} \quad S_B, \tag{5.34}$$
$$\partial_j v_3 + \partial_3 v_j = 0, \quad j = 1, 2, \quad \text{on} \quad S_F, \tag{5.35}$$
$$-q + 2\nu\partial_3 v_3 + (1 - \sigma\Delta_F)\eta = 0 \quad \text{on} \quad S_F. \tag{5.36}$$

We begin the proof by differentiating η, v, q in the horizontal coordinates x' formally and estimating these derivatives. To do this we define the operator $\Lambda^{\ell-2}$ by

$$\Lambda^{\ell-2}\phi = \sum_{\xi' \in \mathbb{Z}^2} |\xi'|^{\ell-2}\phi^{(\xi')}e^{ix'\cdot\xi'} \qquad \text{for} \quad \phi = \sum_{\xi' \in \mathbb{Z}^2} \phi^{(\xi')}e^{ix'\cdot\xi'}.$$

Applying $\Lambda^{\ell-2}$ to (5.31)–(5.36), we see that

$$\begin{pmatrix}\Lambda^{\ell-2}\eta\\ \Lambda^{\ell-2}v\end{pmatrix} = (\lambda - G)^{-1}\begin{pmatrix}\Lambda^{\ell-2}g_0\\ \Lambda^{\ell-2}f\end{pmatrix}. \tag{5.37}$$

From Proposition 5.2 it follows that $\Lambda^{\ell-2}v \in H^2(\Omega)$, hence $\Lambda^{\ell-2}v\big|_{S_F} \in H^{\frac{3}{2}}(\mathbb{T}^2)$. This implies that $Rv \in H_0^{\ell-\frac{1}{2}}(\mathbb{T}^2)$. We regard v, q as the solution to the boundary value problem for the Stokes system

$$-\nu\Delta v + \nabla q = f - \lambda v, \quad \operatorname{div} v = 0 \qquad \text{in} \quad \Omega, \tag{5.38}$$

$$\partial_j v_3 + \partial_3 v_j = 0, \quad j = 1, 2, \qquad \text{on} \quad S_F, \tag{5.39}$$

$$v_3 = Rv \qquad \text{on} \quad S_F, \tag{5.40}$$

$$v = 0 \qquad \text{on} \quad S_B. \tag{5.41}$$

Note that the set of the boundary conditions (5.39), (5.40) satisfies the complementary condition. Therefore we obtain the estimate

$$\begin{aligned}|v|_{H^\ell(\Omega)} &\le C\left(|Rv|_{H^{\ell-\frac{1}{2}}(\mathbb{T}^2)} + |\lambda||v|_{H^{\ell-2}(\Omega)} + |f|_{H^{\ell-2}(\Omega)}\right)\\ &\le C\left(\left|\Lambda^{\ell-2}v\right|_{H^2(\Omega)} + |\lambda||v|_{H^{\ell-2}(\Omega)} + |f|_{H^{\ell-2}(\Omega)}\right).\end{aligned}$$

See Lemma 3.3 in [3] for this estimate. By Proposition 5.2 we have

$$\begin{aligned}\left|\Lambda^{\ell-2}v\right|_{H^2(\Omega)} &\le C\left(\left|\Lambda^{\ell-2}f\right|_{H^0(\Omega)} + \left|\Lambda^{\ell-2}g_0\right|_{H^{\frac{3}{2}}(\mathbb{T}^2)}\right)\\ &\le C\left(|f|_{H^{\ell-2}(\Omega)} + |g_0|_{H^{\ell-\frac{1}{2}}(\mathbb{T}^2)}\right).\end{aligned}$$

By the interpolation inequality of Sobolev norms we can bound the term $|\lambda||v|_{H^{\ell-2}(\Omega)}$ as follows:

$$|\lambda||v|_{H^{\ell-2}(\Omega)} \le |\lambda||v|^{1-\frac{2}{\ell}}_{H^\ell(\Omega)}|v|^{\frac{2}{\ell}}_{H^0(\Omega)} \le \varepsilon|v|_{H^\ell(\Omega)} + C_\varepsilon|\lambda|^{\frac{\ell}{2}}|v|_{H^0(\Omega)}.$$

Choosing $\varepsilon > 0$ suitably, we obtain

$$|v|_{H^\ell(\Omega)} \le C\left(|\lambda|^{\frac{\ell}{2}}|v|_{H^0(\Omega)} + |g_0|_{H^{\ell-\frac{1}{2}}(\mathbb{T}^2)} + |f|_{H^{\ell-2}(\Omega)}\right).$$

The first term in the right hand side can be estimated as follows

$$\begin{aligned}|\lambda|^{\frac{\ell}{2}}|v|_{H^0(\Omega)} &\le C|\lambda|^{\frac{\ell}{2}-1}\left(|f|_{H^0(\Omega)} + |g_0|_{H^{\frac{3}{2}}(\mathbb{T}^2)}\right)\\ &\le C\left(|\lambda|^{\frac{\ell}{2}-1}|f|_{H^0(\Omega)} + |\lambda|^{\frac{\ell}{2}-1}|g_0|^{\frac{3}{2\ell-1}}_{H^{\ell-\frac{1}{2}}(\mathbb{T}^2)}|g_0|^{\frac{2\ell-4}{2\ell-1}}_{H^0(\mathbb{T}^2)}\right)\\ &\le C\left(|\lambda|^{\frac{\ell}{2}-1}|f|_{H^0(\Omega)} + |g_0|_{H^{\ell-\frac{1}{2}}(\mathbb{T}^2)} + |\lambda|^{\frac{\ell}{2}-\frac{1}{4}}|g_0|_{H^0(\mathbb{T}^2)}\right).\end{aligned} \tag{5.42}$$

By Proposition 5.2 we obtain

$$\left|\Lambda^{\ell-2}\eta\right|_{H^{\frac{5}{2}}(\mathbb{T}^2)} \le C\left(\left|\Lambda^{\ell-2}f\right|_{H^0(\Omega)} + \left|\Lambda^{\ell-2}g_0\right|_{H^{\frac{3}{2}}(\mathbb{T}^2)}\right),$$

hence

$$|\eta|_{H^{\ell+\frac{1}{2}}(\mathbb{T}^2)} \le C\left(|f|_{H^{\ell-2}(\Omega)} + |g_0|_{H^{\ell-\frac{1}{2}}(\mathbb{T}^2)}\right).$$

We next estimate $|\lambda|^{\frac{\ell}{2}+\frac{1}{4}}|\eta|_{H^0(\mathbb{T}^2)}$. Since we already show the boundedness of the resolvent operator in Proposition 5.2, we can assume $|\lambda| \ge 1$. Using (5.31), we have

$$|\lambda|^{\frac{\ell}{2}+\frac{1}{4}}|\eta|_{H^0(\mathbb{T}^2)} \le C|\lambda|^{\frac{\ell}{2}-\frac{3}{4}}\left(|Rv|_{H^0(\mathbb{T}^2)} + |g_0|_{H^0(\mathbb{T}^2)}\right).$$

We can bound $|Rv|_{H^0(\mathbb{T}^2)}$ by $|v|_{H^1(\Omega)}$ so that Proposition 5.2 yields,

$$\begin{aligned}|Rv|_{H^0(\mathbb{T}^2)} \le C|v|_{H^1(\Omega)} &\le C|v|^{\frac{1}{2}}_{H^0(\Omega)}|v|^{\frac{1}{2}}_{H^2(\Omega)}\\ &\le C|\lambda|^{-\frac{1}{2}}\left(|f|_{H^0(\Omega)} + |g_0|_{H^{\frac{3}{2}}(\mathbb{T}^2)}\right).\end{aligned}$$

Hence, as in (5.42) we obtain

$$\begin{aligned}|\lambda|^{\frac{\ell}{2}+\frac{1}{4}}|\eta|_{H^0(\mathbb{T}^2)} &\le C\left(|\lambda|^{\frac{\ell}{2}-\frac{5}{4}}\left(|f|_{H^0(\Omega)} + |g_0|_{H^{\frac{3}{2}}(\mathbb{T}^2)}\right) + |\lambda|^{\frac{\ell}{2}-\frac{3}{4}}|g_0|_{H^0(\mathbb{T}^2)}\right)\\ &\le C\left(|\lambda|^{\frac{\ell-2}{2}}|f|_{H^0(\Omega)} + |g_0|_{H^{\ell-\frac{1}{2}}(\mathbb{T}^2)} + |\lambda|^{\frac{\ell}{2}-\frac{1}{4}}|g_0|_{H^0(\mathbb{T}^2)}\right).\end{aligned}$$

Combining these estimates we get (5.30). □

We now consider the problem (5.1)–(5.7).

Theorem 5.1. *Let $\ell > 2$ be not a half integer. Suppose that the inhomogeneous term f_0 in (5.2) is given in $K^{\ell-2}_{(0)}(\Omega \times \mathbb{R}^+)$. Then there is a unique solution (η, v, q) to problem (5.1)–(5.7), with*

$$\eta \in K^{\ell+\frac{1}{2}}_{0,(0)}(\mathbb{T}^2 \times \mathbb{R}^+),\ v \in K^{\ell}_{(0)}(\Omega \times \mathbb{R}^+),$$

$$\nabla q \in K^{\ell-2}_{(0)}(\Omega \times \mathbb{R}^+),\ q|_{S_F} \in K^{\ell-\frac{3}{2}}_{(0)}(\mathbb{T}^2 \times \mathbb{R}^+). \tag{5.43}$$

This solution satisfies

$$\begin{aligned}&|\eta|_{K^{\ell+\frac{1}{2}}_{0,(0)}(\mathbb{T}^2\times\mathbb{R}^+)} + |v|_{K^{\ell}_{(0)}(\Omega\times\mathbb{R}^+)} + |\nabla q|_{K^{\ell-2}_{(0)}(\Omega\times\mathbb{R}^+)} + \left|q|_{S_F}\right|_{K^{\ell-\frac{3}{2}}_{(0)}(\mathbb{T}^2\times\mathbb{R}^+)}\\ &\quad \le C|f_0|_{K^{\ell-2}_{(0)}(\Omega\times\mathbb{R}^+)}.\end{aligned} \tag{5.44}$$

Proof. We give the outline of the proof since the argument is almost in the same line as in [3], Section 3. Extending f_0 to be zero for $t < 0$, and transforming this extension in t, we have

$$\hat{f}_0(\tau) = \int_0^\infty e^{-i\tau t} f_0(t)dt \qquad \text{for} \quad \tau \in \mathbb{R}.$$

By setting $\lambda = i\tau$, we see that this $\hat{f}_0(-i\lambda)$ has an analytic extension to the half-plane $\operatorname{Re}\lambda \ge 0$. By Proposition 5.3 we can find $\hat{\eta}(-i\lambda)$ and $\hat{v}(-i\lambda)$ such that

$$\begin{pmatrix}\hat{\eta}(-i\lambda)\\ \hat{v}(-i\lambda)\end{pmatrix} = (\lambda - G)^{-1}\begin{pmatrix}0\\ P^0\hat{f}_0(-i\lambda)\end{pmatrix}. \tag{5.45}$$

From the estimate (5.30) it follows that

(5.46)
$$|\hat{v}(-i\lambda)|_{H^\ell(\Omega)} + |\lambda|^{\frac{\ell}{2}}|\hat{v}(-i\lambda)|_{H^0(\Omega)} + |\hat{\eta}(-i\lambda)|_{H^{\ell+\frac{1}{2}}(\mathbb{T}^2)} + |\lambda|^{\frac{\ell}{2}+\frac{1}{4}}|\hat{\eta}(-i\lambda)|_{H^0(\mathbb{T}^2)}$$
$$\le C\left(|\hat{f}_0(-i\lambda)|_{H^{\ell-2}(\Omega)} + |\lambda|^{\frac{\ell}{2}-1}|\hat{f}_0(-i\lambda)|_{H^0(\Omega)}\right).$$

Since the right hand side of (5.45) is analytic in λ in the half-plane $\operatorname{Re}\lambda \ge 0$, $\hat{\eta}(-i\lambda)$ and $\hat{v}(-i\lambda)$ are also analytic there. We see that $\hat{v}(-i\lambda)$ on the line $\operatorname{Re}\lambda = k$ with a fixed $k > 0$ is the transform of $e^{-kt}v(t)$ which, by (5.46), is in $K^\ell(\Omega\times\mathbb{R}^+)$. Then Paley–Wiener theorem implies that $e^{-kt}v(t) \in K^\ell_{(0)}(\Omega\times\mathbb{R}^+)$. As we see in Proposition 5.2, the imaginary axis $\operatorname{Re}\lambda = 0$ belongs to the resolvent set $\rho(G)$. Hence we can convert the path of the inverse transform from $\operatorname{Re}\lambda = k$ to $\operatorname{Re}\lambda = 0$, so that $v(t) \in K^\ell_{(0)}(\Omega\times\mathbb{R}^+)$. By the same argument we have $\eta(t) \in K^{\ell+\frac{1}{2}}_{(0)}\left(\mathbb{T}^2\times\mathbb{R}^+\right)$. We can recover q from the definition of P^0, whose estimate follows from (5.1) and (5.6). □

As a corollary of this theorem we can show

Corollary 5.2. *Let $\ell > 2$ be not a half integer. Suppose that f_0 and f_j $(j = 1, 2)$ are given in $K^{\ell-2}_{(0)}(\Omega\times\mathbb{R}^+)$ and $K^{\ell-\frac{3}{2}}_{(0)}\left(\mathbb{T}^2\times\mathbb{R}^+\right)$, respectively. Then there is a unique solution (η, v, q) to problem (5.1)–(5.7) where (5.5) is replaced by*

$$\partial_3 v_j + \partial_j v_3 = f_j, \qquad j = 1, 2 \quad \text{on } S_F.$$

This solution satisfies

(5.47)
$$|\eta|_{K^{\ell+\frac{1}{2}}_{(0)}(\mathbb{T}^2\times\mathbb{R}^+)} + |v|_{K^\ell_{(0)}(\Omega\times\mathbb{R}^+)} + |\nabla q|_{K^{\ell-2}_{(0)}(\Omega\times\mathbb{R}^+)} + \left|q|_{S_F}\right|_{K^{\ell-\frac{3}{2}}_{(0)}(\mathbb{T}^2\times\mathbb{R}^+)}$$
$$\le C\left(|f_0|_{K^{\ell-2}_{(0)}(\Omega\times\mathbb{R}^+)} + |f_1|_{K^{\ell-\frac{3}{2}}_{(0)}(\mathbb{T}^2\times\mathbb{R}^+)} + |f_2|_{K^{\ell-\frac{3}{2}}_{(0)}(\mathbb{T}^2\times\mathbb{R}^+)}\right)$$

with the indicated norms.

The outline of the proof of Corollary. We again follow the proof of Theorem 2 in [3]. By Theorem 4.2.3 of [7] we can choose $z \in K^{\ell+1}_{(0)}(\Omega\times\mathbb{R}^+)$ so that

$$z = 0,\ \partial_y z = 0,\ \partial_y^2 z = (f_2, -f_1, 0) \quad \text{on } S_F$$

and z vanishes near S_B. Setting $v^{(1)} = \nabla\times z$ and $v = v^{(1)} + v^{(2)}$, we can reduce the problem to the one for $(\eta, v^{(2)}, q)$ in the previous theorem. □

6. Decay estimates for full nonlinear problem

In this section we give the proof of Theorem 1.3. As noted in Theorem 1.2, after a finite instant $T_1 > 0$, the solution η, v, q to the problem (1.2), (1.3), (1.6)–(1.8) and

$$\eta(0) = \eta_0, \quad v(0) = v_0$$

belongs to the more regular class:

$$\eta \in K^{\ell+2+\frac{1}{2}}\left((T_1,\infty)\times\mathbb{T}^2\right), \quad v \in K^{\ell+2}\left((T_1,\infty)\times\Omega\right),$$
$$\nabla q \in K^{\ell}\left((T_1,\infty)\times\Omega\right), \quad q|_{S_F} \in K^{\ell+\frac{1}{2}}\left((T_1,\infty)\times\mathbb{T}^2\right).$$

Since $3 < \ell < \dfrac{7}{2}$ implies

$$K^{\ell+2+\frac{1}{2}}\left((T_1,\infty)\times\mathbb{T}^2\right) \subseteq C^k\left((T_1,\infty);H^{\ell+1+\frac{1}{2}-2k}\left(\mathbb{T}^2\right)\right)$$

for $\ell+1+\dfrac{1}{2}-2k>0$ and

$$K^{\ell+2}\left((T_1,\infty)\times\Omega\right) \subseteq C^k\left((T_1,\infty);H^{\ell+1-2k}(\Omega)\right) \text{ for } \quad \ell+1-2k>0$$

with continuous imbedding, it holds that

$$|v(t)|_{H^{\ell+1}(\Omega)} + |\eta(t)|_{H_0^{\ell+\frac{3}{2}}(\mathbb{T}^2)} \le C\delta_0 \qquad \text{for} \quad t \ge T_1, \tag{6.1}$$

where C is independent of the solution and $\delta_0 > 0$ is given in Theorem 1.2. Employing these facts, we derive the energy inequality in the following.

Since we are interested in the real-valued initial data, we can assume that any scalar or vector functions in the sequel are real valued.

Step 1. We take the $L^2(\Omega)$ scalar product of the first equation of (1.3) with v to get by integrating by parts

$$\frac{1}{2}\frac{d}{dt}|v|_\Omega^2 + \langle v,v\rangle + \int_{\partial\Omega} n_j S_{jk}(v,q)v_k dx' = (F_0,v)_\Omega + (Q\nabla q, v)_\Omega. \tag{6.2}$$

From the first equation of (1.3) we have

$$(I-Q)\nabla q = -\partial_t v + \nu\Delta v + F_0.$$

Since $I-Q = J\left(\zeta_{jc}\zeta_{jk}\right)$ is symmetric positive definite if (ζ_{ij}) is nonsingular, we can replace ∇q in the right hand side of (6.2) by

$$\nabla q = (I-Q)^{-1}\left(-\partial_t v + \nu\Delta v + F_0\right), \tag{6.3}$$

so that

$$\begin{aligned}(Q\nabla q, v)_\Omega &= \left(Q(I-Q)^{-1}\left(-\partial_t v + \nu\Delta v + F_0\right), v\right)_\Omega \\ &= -\frac{1}{2}\frac{d}{dt}\left(Q(I-Q)^{-1}v,v\right)_\Omega + \frac{1}{2}\left(\partial_t\left(Q(I-Q)^{-1}\right)v,v\right)_\Omega \\ &\quad + \left(Q(I-Q)^{-1}\left(\nu\Delta v + F_0\right), v\right)_\Omega.\end{aligned} \tag{6.4}$$

On the boundary $\partial\Omega = S_B \cup S_F$ (1.6)–(1.8) hold, hence the boundary terms in (6.2) can be written as follows:

$$\begin{aligned}\int_{\partial\Omega} n_j S_{jk}(v,q)v_k dx' &= -\nu\left(F_1,v_1\right)_{\mathbb{T}^2} - \nu\left(F_2,v_2\right)_{\mathbb{T}^2} \\ &\quad + \left(\left(1-\sigma\Delta_F\right)\eta, v_3\right)_{\mathbb{T}^2} + \left(F_3,v_3\right)_{\mathbb{T}^2}.\end{aligned} \tag{6.5}$$

Here we identify S_F with $\mathbb{T}^2$. From (1.2) it follows that

$$((1-\sigma\Delta_F)\,\eta, v_3)_{\mathbb{T}^2} = ((1-\sigma\Delta_F)\,\eta, \partial_t\eta)_{\mathbb{T}^2} = \frac{1}{2}\frac{d}{dt}\left\{|\eta|^2_{H^0(\mathbb{T}^2)} + \sigma\,|\nabla_F\,\eta|^2_{H^0(\mathbb{T}^2)}\right\}. \tag{6.6}$$

Collecting these, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|v|^2_{H^0(\Omega)} &+ \langle v, v\rangle + \frac{1}{2}\frac{d}{dt}\left\{|\eta|^2_{H^0(\mathbb{T}^2)} + \sigma\,|\nabla_F\,\eta|^2_{H^0(\mathbb{T}^2)}\right\} \\ &+ \frac{1}{2}\frac{d}{dt}\left(Q(I-Q)^{-1}v, v\right)_\Omega \\ &= \nu\,(F_1, v_1)_{\mathbb{T}^2} + \nu\,(F_2, v_2)_{\mathbb{T}^2} - (F_3, v_3)_{\mathbb{T}^2} \\ &\quad + (F_0, v)_\Omega + \frac{1}{2}\left(\partial_t\left(Q(I-Q)^{-1}\right)v, v\right)_\Omega \\ &\quad + \left(Q(I-Q)^{-1}\left(\nu\Delta v + F_0\right), v\right)_\Omega.\end{aligned} \tag{6.7}$$

Step 2. Differentiate the first equation of (1.3) with respect to x_α ($\alpha = 1, 2$) and take the inner product with $\partial_\alpha v$. In the same manner as in Step 1 we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\partial_\alpha v|^2_{H^0(\Omega)} &+ \langle\partial_\alpha v, \partial_\alpha v\rangle + \int_{\mathbb{T}^2}(1-\sigma\mathit{\Delta}_F)\partial_\alpha\eta\partial_\alpha v_3 dx' \\ &= \nu\,(\partial_\alpha F_1, \partial_\alpha v_1)_{\mathbb{T}^2} + \nu\,(\partial_\alpha F_2, \partial_\alpha v_2)_{\mathbb{T}^2} - (\partial_\alpha F_3, \partial_\alpha v_3)_{\mathbb{T}^2} \\ &\quad + (\partial_\alpha F_0, \partial_\alpha v)_\Omega + (Q\partial_\alpha\nabla q, \partial_\alpha v)_\Omega + (\partial_\alpha Q\nabla q, \partial_\alpha v)_\Omega.\end{aligned}$$

We use (1.2) again to get

$$((1-\sigma\Delta_F)\,\partial_\alpha\eta, \partial_\alpha v_3)_{\mathbb{T}^2} = ((1-\sigma\Delta_F)\,\partial_\alpha\eta, \partial_t\partial_\alpha\eta)_{\mathbb{T}^2} = \frac{1}{2}\frac{d}{dt}\left\{|\partial_\alpha\eta|^2_{H^0(\mathbb{T}^2)} + \sigma\,|\nabla_F\,\partial_\alpha\eta|^2_{H^0(\mathbb{T}^2)}\right\}. \tag{6.8}$$

Replacing ∇q by (6.3), we have

(6.9)

$$\begin{aligned}(Q\partial_\alpha\nabla q, \partial_\alpha v)_\Omega &= \left(Q\partial_\alpha\left((I-Q)^{-1}\left(-\partial_t v + \nu\Delta v + F_0\right)\right), \partial_\alpha v\right)_\Omega \\ &= \left(Q(I-Q)^{-1}\left(-\partial_t\partial_\alpha v + \nu\Delta\partial_\alpha v + \partial_\alpha F_0\right), \partial_\alpha v\right)_\Omega \\ &\quad + \left(Q\partial_\alpha(I-Q)^{-1}\left(-\partial_t v + \nu\Delta v + F_0\right), \partial_\alpha v\right)_\Omega \\ &= -\frac{1}{2}\frac{d}{dt}\left(Q(I-Q)^{-1}\partial_\alpha v, \partial_\alpha v\right)_\Omega \\ &\quad + \frac{1}{2}\left(\partial_t\left(Q(I-Q)^{-1}\right)\partial_\alpha v, \partial_\alpha v\right)_\Omega \\ &\quad + \left(Q(I-Q)^{-1}\left(\nu\Delta\partial_\alpha v + \partial_\alpha F_0\right), \partial_\alpha v\right)_\Omega \\ &\quad + \left(Q\partial_\alpha(I-Q)^{-1}\left(-\partial_t\partial_\alpha v + \nu\Delta\partial_\alpha v + \partial_\alpha F_0\right), \partial_\alpha v\right)_\Omega.\end{aligned}$$

Collecting these, we obtain

$$
\begin{aligned}
(6.10)\quad & \frac{1}{2}\frac{d}{dt}|\partial_\alpha v|^2_{H^0(\Omega)} + \langle \partial_\alpha v, \partial_\alpha v\rangle + \frac{1}{2}\frac{d}{dt}\left\{|\partial_\alpha \eta|^2_{H^0(\mathbb{T}^2)} + \sigma\,|\nabla_F\, \partial_\alpha \eta|^2_{H^0(\mathbb{T}^2)}\right\} \\
& + \frac{1}{2}\frac{d}{dt}\left(Q(I-Q)^{-1}\partial_\alpha v, \partial_\alpha v\right)_\Omega \\
& = \nu\,(\partial_\alpha F_1, \partial_\alpha v_1)_{\mathbb{T}^2} + \nu\,(\partial_\alpha F_2, \partial_\alpha v_2)_{\mathbb{T}^2} - (\partial_\alpha F_3, \partial_\alpha v_3)_{\mathbb{T}^2} \\
& \quad + (\partial_\alpha F_0, \partial_\alpha v)_\Omega + \frac{1}{2}\left(\partial_t\left(Q(I-Q)^{-1}\right)\partial_\alpha v, \partial_\alpha v\right)_\Omega \\
& \quad + \left(Q(I-Q)^{-1}\left(\nu\Delta\partial_\alpha v + \partial_\alpha F_0\right), \partial_\alpha v\right)_\Omega \\
& \quad + \left(Q\partial_\alpha (I-Q)^{-1}\left(-\partial_t v + \nu\Delta v + F_0\right), \partial_\alpha v\right)_\Omega \\
& \quad + \left(\partial_\alpha Q(I-Q)^{-1}\left(-\partial_t v + \nu\Delta v + F_0\right), \partial_\alpha v\right)_\Omega .
\end{aligned}
$$

Step 3. We next derive the similar equality for $\partial^2_{\alpha\beta} v$ and $\partial^2_{\alpha\beta}\eta$, $\alpha, \beta = 1, 2$. To do this we apply $\partial^2_{\alpha\beta}$ to the first equation of (1.3) and take the inner product with $\partial^2_{\alpha\beta} v$. In a similar way as above we obtain

$$
\begin{aligned}
(6.11)\quad & \frac{1}{2}\frac{d}{dt}|\partial^2_{\alpha\beta} v|^2_{H^0(\Omega)} + \langle \partial^2_{\alpha\beta} v, \partial^2_{\alpha\beta} v\rangle \\
& + \frac{1}{2}\frac{d}{dt}\left\{|\partial^2_{\alpha\beta}\eta|^2_{H^0(\mathbb{T}^2)} + \sigma\left|\nabla_F\, \partial^2_{\alpha\beta}\eta\right|^2_{H^0(\mathbb{T}^2)}\right\} \\
& + \frac{1}{2}\frac{d}{dt}\left(Q(I-Q)^{-1}\partial^2_{\alpha\beta} v, \partial^2_{\alpha\beta} v\right)_\Omega \\
& = \nu\left(\partial^2_{\alpha\beta} F_1, \partial^2_{\alpha\beta} v_1\right)_{\mathbb{T}^2} + \nu\left(\partial^2_{\alpha\beta} F_2, \partial^2_{\alpha\beta} v_2\right)_{\mathbb{T}^2} - \left(\partial^2_{\alpha\beta} F_3, \partial^2_{\alpha\beta} v_3\right)_{\mathbb{T}^2} \\
& \quad + (\partial^2_{\alpha\beta} F_0, \partial^2_{\alpha\beta} v)_\Omega + \frac{1}{2}\left(\partial_t\left(Q(I-Q)^{-1}\right)\partial^2_{\alpha\beta} v, \partial^2_{\alpha\beta} v\right)_\Omega \\
& \quad + \left(Q(I-Q)^{-1}\left(\nu\Delta\partial^2_{\alpha\beta} v + \partial^2_{\alpha\beta} F_0\right), \partial^2_{\alpha\beta} v\right)_\Omega \\
& \quad + \left(\partial_\alpha\left(Q(I-Q)^{-1}\right)\partial_\beta\left(-\partial_t v + \nu\Delta v + F_0\right), \partial^2_{\alpha\beta} v\right)_\Omega \\
& \quad + \left(\partial_\beta\left(Q(I-Q)^{-1}\right)\partial_\alpha\left(-\partial_t v + \nu\Delta v + F_0\right), \partial^2_{\alpha\beta} v\right)_\Omega \\
& \quad + \left(\partial^2_{\alpha\beta}\left(Q(I-Q)^{-1}\right)\left(-\partial_t v + \nu\Delta v + F_0\right), \partial^2_{\alpha\beta} v\right)_\Omega .
\end{aligned}
$$

Step 4. We next derive the similar equality for $\partial_t v$ and $\partial_t\eta$. Differentiate the first equation of (1.3) in t and take the inner product with $\partial_t v$. Similarly we have

$$
\begin{aligned}
(6.12)\quad & \frac{1}{2}\frac{d}{dt}|\partial_t v|^2_{H^0(\Omega)} + \langle \partial_t v, \partial_t v\rangle + \frac{1}{2}\frac{d}{dt}\left\{|\partial_t \eta|^2_{H^0(\mathbb{T}^2)} + \sigma\,|\nabla_F\, \partial_t \eta|^2_{H^0(\mathbb{T}^2)}\right\} \\
& + \frac{1}{2}\frac{d}{dt}\left(Q(I-Q)^{-1}\partial_t v, \partial_t v\right)_\Omega \\
& = \nu\,(\partial_t F_1, \partial_t v_1)_{\mathbb{T}^2} + \nu\,(\partial_t F_2, \partial_t v_2)_{\mathbb{T}^2} - (\partial_t F_3, \partial_t v_3)_{\mathbb{T}^2} \\
& \quad + (\partial_t F_0, \partial_t v)_\Omega + \frac{1}{2}\left(\partial_t\left(Q(I-Q)^{-1}\right)\partial_t v, \partial_t v\right)_\Omega \\
& \quad + \left(Q(I-Q)^{-1}\left(\nu\Delta\partial_t v + \partial_t F_0\right), \partial_t v\right)_\Omega
\end{aligned}
$$

$$
\begin{aligned}
&+ \left(Q\partial_t (I-Q)^{-1} \left(-\partial_t v + \nu \Delta v + F_0\right), \partial_t v\right)_\Omega \\
&+ \left(\partial_t Q (I-Q)^{-1} \left(-\partial_t v + \nu \Delta v + F_0\right), \partial_t v\right)_\Omega .
\end{aligned}
$$

We now set

$$
\begin{aligned}
\mathcal{E}(t) &= |\, v(t)\,|^2_{H^0(\Omega)} + \sum_{\alpha=1}^{2} |\partial_\alpha v(t)|^2_{H^0(\Omega)} + \sum_{\alpha,\beta=1}^{2} |\partial^2_{\alpha\beta} v(t)|^2_{H^0(\Omega)} \\
&\quad + |\partial_t v(t)|^2_{H^0(\Omega)}, \\
\mathcal{F}(t) &= \langle v, v\rangle + \sum_{\alpha=1}^{2} \langle \partial_\alpha v, \partial_\alpha v\rangle + \sum_{\alpha,\beta=1}^{2} \langle \partial^2_{\alpha\beta} v, \partial^2_{\alpha\beta} v\rangle \\
&\quad + \langle \partial_t v, \partial_t v\rangle, \\
\mathcal{H}(t) &= |\eta|^2_{H^0(\mathbb{T}^2)} + \sigma\, |\nabla_F\, \eta|^2_{H^0(\mathbb{T}^2)} + \sum_{\alpha=1}^{2} \left(|\partial_\alpha \eta|^2_{H^0(\mathbb{T}^2)} + \sigma\, |\nabla_F\, \partial_\alpha \eta|^2_{H^0(\mathbb{T}^2)} \right) \\
&\quad + \sum_{\alpha,\beta=1}^{2} \left(|\partial^2_{\alpha\beta} \eta|^2_{H^0(\mathbb{T}^2)} + \sigma \left|\nabla_F\, \partial^2_{\alpha\beta} \eta\right|^2_{H^0(\mathbb{T}^2)} \right) \\
&\quad + |\partial_t \eta|^2_{H^0(\mathbb{T}^2)} + \sigma\, |\nabla_F\, \partial_t \eta|^2_{H^0(\mathbb{T}^2)}, \\
\mathcal{Q}(t) &= \left(Q(I-Q)^{-1} v, v\right)_\Omega + \sum_{\alpha=1}^{2} \left(Q(I-Q)^{-1} \partial_\alpha v, \partial_\alpha v\right)_\Omega \\
&\quad + \sum_{\alpha,\beta=1}^{2} \left(Q(I-Q)^{-1} \partial^2_{\alpha\beta} v, \partial^2_{\alpha\beta} v\right)_\Omega + \left(Q(I-Q)^{-1} \partial_t v, \partial_t v\right)_\Omega, \\
\mathcal{N}_B(t) &= \nu\, (F_1, v_1)_{\mathbb{T}^2} + \nu\, (F_2, v_2)_{\mathbb{T}^2} - (F_3, v_3)_{\mathbb{T}^2} \\
&\quad + \sum_{\alpha=1}^{2} \left(\nu\, (\partial_\alpha F_1, \partial_\alpha v_1)_{\mathbb{T}^2} + \nu\, (\partial_\alpha F_2, \partial_\alpha v_2)_{\mathbb{T}^2} - (\partial_\alpha F_3, \partial_\alpha v_3)_{\mathbb{T}^2}\right) \\
&\quad + \sum_{\alpha,\beta=1}^{2} \left(\nu \left(\partial^2_{\alpha\beta} F_1, \partial^2_{\alpha\beta} v_1\right)_{\mathbb{T}^2} + \nu \left(\partial^2_{\alpha\beta} F_2, \partial^2_{\alpha\beta} v_2\right)_{\mathbb{T}^2} - \left(\partial^2_{\alpha\beta} F_3, \partial^2_{\alpha\beta} v_3\right)_{\mathbb{T}^2}\right) \\
&\quad + \nu\, (\partial_t F_1, \partial_t v_1)_{\mathbb{T}^2} + \nu\, (\partial_t F_2, \partial_t v_2)_{\mathbb{T}^2} - (\partial_t F_3, \partial_t v_3)_{\mathbb{T}^2}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{N}_\Omega(t) &= (F_0, v)_\Omega + \frac{1}{2} \left(\partial_t \left(Q(I-Q)^{-1}\right) v, v\right)_\Omega \\
&\quad + \left(Q(I-Q)^{-1} \left(\nu \Delta v + F_0\right), v\right)_\Omega \\
&\quad + \sum_{\alpha=1}^{2} \Big\{ (\partial_\alpha F_0, \partial_\alpha v)_\Omega + \frac{1}{2} \left(\partial_t \left(Q(I-Q)^{-1}\right) \partial_\alpha v, \partial_\alpha v\right)_\Omega \\
&\qquad + \left(Q(I-Q)^{-1} \left(\nu \Delta \partial_\alpha v + \partial_\alpha F_0\right), \partial_\alpha v\right)_\Omega \\
&\qquad + \left(Q\partial_\alpha (I-Q)^{-1} \left(-\partial_t v + \nu \Delta v + F_0\right), \partial_\alpha v\right)_\Omega
\end{aligned}
$$

$$
\begin{aligned}
&+ \left(\partial_\alpha Q(I-Q)^{-1}\left(-\partial_t v + \nu\Delta v + F_0\right), \partial_\alpha v\right)_\Omega \Big\} \\
&+ \sum_{\alpha,\beta=1}^{2} \Big\{ (\partial^2_{\alpha\beta} F_0, \partial^2_{\alpha\beta} v)_\Omega + \frac{1}{2}\left(\partial_t\left(Q(I-Q)^{-1}\right)\partial^2_{\alpha\beta} v, \partial^2_{\alpha\beta} v\right)_\Omega \\
&\quad + \left(Q(I-Q)^{-1}\left(\nu\Delta\partial^2_{\alpha\beta} v + \partial^2_{\alpha\beta} F_0\right), \partial^2_{\alpha\beta} v\right)_\Omega \\
&\quad + \left(\partial_\alpha\left(Q(I-Q)^{-1}\right)\partial_\beta\left(-\partial_t v + \nu\Delta v + F_0\right), \partial^2_{\alpha\beta} v\right)_\Omega \\
&\quad + \left(\partial_\beta\left(Q(I-Q)^{-1}\right)\partial_\alpha\left(-\partial_t v + \nu\Delta v + F_0\right), \partial^2_{\alpha\beta} v\right)_\Omega \\
&\quad + \left(\partial^2_{\alpha\beta}\left(Q(I-Q)^{-1}\right)\left(-\partial_t v + \nu\Delta v + F_0\right), \partial^2_{\alpha\beta} v\right)_\Omega \Big\} \\
&+ (\partial_t F_0, \partial_t v)_\Omega + \frac{1}{2}\left(\partial_t\left(Q(I-Q)^{-1}\right)\partial_t v, \partial_t v\right)_\Omega \\
&+ \left(Q(I-Q)^{-1}\left(\nu\Delta\partial_t v + \partial_t F_0\right), \partial_t v\right)_\Omega \\
&+ \left(Q\partial_t(I-Q)^{-1}\left(-\partial_t v + \nu\Delta v + F_0\right), \partial_t v\right)_\Omega \\
&+ \left(\partial_t Q(I-Q)^{-1}\left(-\partial_t v + \nu\Delta v + F_0\right), \partial_t v\right)_\Omega .
\end{aligned}
$$

Summing the equalities obtained in Steps $1 \sim 4$, we get

$$
\frac{1}{2}\frac{d}{dt}\left(\mathcal{E}(t) + \mathcal{H}(t) + \mathcal{Q}(t)\right) + \mathcal{F}(t) = \mathcal{N}_B(t) + \mathcal{N}_\Omega(t). \tag{6.13}
$$

Proposition 6.1. *Let η, v, q be the solution given in Theorem 1.2. If $\delta_0 > 0$ in Theorem 1.2 is small enough, it holds that*

$$
\mathcal{H}(t) \le C\mathcal{F}(t) \qquad \text{for} \quad t > T_1, \tag{6.14}
$$

with a constant $C > 0$ independent of the solution.

For the proof of this proposition and the estimates of terms in $\mathcal{N}_B(t)$, $\mathcal{N}_\Omega(t)$ and $\mathcal{Q}(t)$, we need the following lemmas.

Lemma 6.1. *If $\delta_0 > 0$ in Theorem 1.2 is small enough, then the normal derivatives of the velocity components $\partial_3 v_j, j = 1, 2, 3$ on S_F can be written in terms of the tangential derivatives of the velocity components on S_F for $t > T_1$, with T_1 given in Theorem 1.2.*

Proof. From the solenoidal condition, it holds

$$
\partial_3 v_3 = -\partial_1 v_1 - \partial_2 v_2. \tag{6.15}
$$

Note that η and v satisfy on S_F. Using (6.15), we can rewrite (1.6), (1.7) as follows:

$$
\begin{aligned}
\partial_3 v_1 + \partial_1 v_3 &= C_{11}(\eta, \nabla\tilde{\eta})\partial_3 v_1 + C_{12}(\eta, \nabla\tilde{\eta})\partial_3 v_2 \\
&\quad + F_1'\left(\eta, \nabla\tilde{\eta}, \nabla^2\tilde{\eta}, v, \nabla_F v\right), \\
\partial_3 v_2 + \partial_2 v_3 &= C_{21}(\eta, \nabla\tilde{\eta})\partial_3 v_1 + C_{22}(\eta, \nabla\tilde{\eta})\partial_3 v_2 \\
&\quad + F_2'\left(\eta, \nabla\tilde{\eta}, \nabla^2\tilde{\eta}, v, \nabla_F v\right).
\end{aligned}
$$

Take $\delta_0 > 0$ so small that

$$\det \begin{pmatrix} 1 - C_{11}(\eta, \nabla\tilde{\eta}) & -C_{12}(\eta, \nabla\tilde{\eta}) \\ -C_{21}(\eta, \nabla\tilde{\eta}) & 1 - C_{22}(\eta, \nabla\tilde{\eta}) \end{pmatrix} \geq \frac{1}{2} \qquad \text{on} \quad S_F.$$

Then we can solve the above system for $\partial_3 v_1$and $\partial_3 v_2$. □

Lemma 6.2. *Let u and p satisfy*

$$-\nu\Delta u + \nabla p = f_0, \quad \operatorname{div} u = 0 \qquad \text{in} \quad \Omega, \tag{6.16}$$

$$u = 0 \qquad \text{on} \quad S_B, \tag{6.17}$$

$$\partial_j u_3 + \partial_3 u_j = f_j \qquad \text{on} \quad S_F, \quad j = 1, 2, \tag{6.18}$$

$$u_3 = f_3 \qquad \text{on} \quad S_F, \tag{6.19}$$

then it holds for any $\ell \geq 0$

$$\begin{aligned} &|u|_{H^{\ell+2}(\Omega)} + |\nabla p|_{H^\ell(\Omega)} \\ &\quad \leq C\left(|f_0|_{H^\ell(\Omega)} + |f_1|_{H^{\ell+1-\frac{1}{2}}(\mathbb{T}^2)} + |f_2|_{H^{\ell+1-\frac{1}{2}}(\mathbb{T}^2)} + |f_3|_{H^{\ell+2-\frac{1}{2}}(\mathbb{T}^2)}\right). \end{aligned} \tag{6.20}$$

This estimate comes from the facts that the Stokes system is elliptic in the sense of [1], and that the boundary conditions satisfy the complementary condition.

Lemma 6.3. *Take $\ell \geq 0$. Let f be given in $H^\ell(\mathbb{T}^2)$. If ϕ defined on $\mathbb{T}^2$ satisfies*

$$(1 - \sigma\Delta_F)\phi = f \qquad \text{on} \quad \mathbb{T}^2,$$

then it holds that

$$|\phi|_{H^{\ell+2}(\mathbb{T}^2)} \leq C|f|_{H^\ell(\mathbb{T}^2)}. \tag{6.21}$$

This can be seen by Fourier series expansion.

The proof of Proposition 6.1. Since η, v and q satisfy for $t > T_1$

$$\begin{aligned} &-\nu\Delta v + \nabla q = -\partial_t v + F_0 + Q\nabla q && \text{in} \quad \Omega, \\ &\operatorname{div} v = 0 && \text{in} \quad \Omega, \\ &v = 0 && \text{on} \quad S_B, \\ &\partial_j v_3 + \partial_3 v_j = F_j && \text{on} \quad S_F, \ j = 1, 2, \\ &v_3 = v_3 && \text{on} \quad S_F, \end{aligned}$$

by Lemma 6.2 it holds that

$$\begin{aligned} &|v(t)|_{H^3(\Omega)} + |\nabla q(t)|_{H^1(\Omega)} \\ &\quad \leq C\Big(|\partial_t v(t)|_{H^1(\Omega)} + |F_0(t)|_{H^1(\Omega)} + |Q\nabla q|_{H^1(\Omega)} \\ &\qquad + |F_1(t)|_{H^{\frac{3}{2}}(\mathbb{T}^2)} + |F_2(t)|_{H^{\frac{3}{2}}(\mathbb{T}^2)} + |v_3(t)|_{H^{\frac{5}{2}}(\mathbb{T}^2)}\Big). \end{aligned} \tag{6.22}$$

Taking account of the explicit forms (1.4) and (1.5) of F_0 and $Q\nabla q$ and estimate (6.1), we can estimate the second and third terms in the right hand side as follow:

$$(6.23)\quad |F_0|_{H^1(\Omega)} + |Q\nabla q|_{H^1(\Omega)} \le C\delta_0 \left(|v(t)|_{H^3(\Omega)} + |\nabla q(t)|_{H^1(\Omega)} + \left|{\nabla_F}^3 \eta(t)\right|_{H^{\frac{1}{2}}(\mathbb{T}^2)} \right).$$

Remind that ∇_F denotes the horizontal gradient (see page 272). In a similar way we have

$$(6.24)\quad |F_1(t)|_{H^{\frac{3}{2}}(\mathbb{T}^2)} + |F_2(t)|_{H^{\frac{3}{2}}(\mathbb{T}^2)} \le C\delta_0 \left(|v(t)|_{H^3(\Omega)} + \left|{\nabla_F}^3 \eta(t)\right|_{H^{\frac{1}{2}}(\mathbb{T}^2)} \right).$$

From the solenoidal condition we see that the trace of v_3 onto S_F belongs to $H_0^{\frac{5}{2}}(\mathbb{T}^2)$. Thus the last term in the right hand side of (6.22) can be estimated as

$$(6.25)\qquad |v_3(t)|_{H^{\frac{5}{2}}(\mathbb{T}^2)} \le C|{\nabla_F}^2 v_3(t)|_{H^{\frac{1}{2}}(\mathbb{T}^2)} \le |{\nabla_F}^2 v_3(t)|_{H^1(\Omega)}.$$

Collecting these and taking $\delta_0 > 0$ sufficiently small we get

$$(6.26)\quad |v(t)|_{H^3(\Omega)} + |\nabla q(t)|_{H^1(\Omega)} \le C \left(|\partial_t v(t)|_{H^1(\Omega)} + \delta_0 \left|{\nabla_F}^3 \eta(t)\right|_{H^{\frac{1}{2}}(\mathbb{T}^2)} + |{\nabla_F}^2 v_3(t)|_{H^1(\Omega)} \right).$$

We now have to estimate the norm of $\eta(t)$ in terms of the norms of $v(t)$. We differentiate the boundary condition (1.8) in $\partial_j,\ j = 1, 2$ to get

$$(6.27)\quad -(1 - \sigma\Delta_F)\,\partial_j \eta = -\partial_j q + 2\nu \partial_j \partial_3 v_3 + \partial_j F_3 \text{ on } S_F,\ t \ge T_1,\ j = 1, 2.$$

By use of the solenoidal condition and by Lemma 6.3 we obtain

$$(6.28)\quad |\nabla_F\, \eta(t)|_{H^{\frac{5}{2}}(\mathbb{T}^2)} \le C \Big(|\nabla_F\, q(t)|_{H^{\frac{1}{2}}(\mathbb{T}^2)} + |\nabla_F\, (\partial_1 v_1 + \partial_2 v_2)\,(t)|_{H^{\frac{1}{2}}(\mathbb{T}^2)} + |\nabla_F\, F_3(t)|_{H^{\frac{1}{2}}(\mathbb{T}^2)} \Big).$$

Taking the explicit form (1.9) of F_3 and estimate (6.1) into account we see that

$$(6.29)\qquad |\nabla_F\, F_3(t)|_{H^{\frac{1}{2}}(\mathbb{T}^2)} \le C\delta_0\, |\nabla_F\, \eta(t)|_{H^{\frac{5}{2}}(\mathbb{T}^2)}.$$

Hence we obtain

$$(6.30)\qquad |\nabla_F\, \eta(t)|_{H^{\frac{5}{2}}(\mathbb{T}^2)} \le C \Big(|\nabla_F\, q(t)|_{H^1(\Omega)} + \left|{\nabla_F}^2 v\right|_{H^1(\Omega)} + \delta_0\, |\nabla_F\, \eta(t)|_{H^{\frac{5}{2}}(\mathbb{T}^2)} \Big).$$

From this and (6.26) we can derive

$$|\nabla_F\,\eta(t)|_{H^{\frac{5}{2}}(\mathbb{T}^2)} \le C\left(|\partial_t v(t)|_{H^1(\Omega)} + \delta_0\left|\nabla_F{}^3\eta(t)\right|_{H^{\frac{1}{2}}(\mathbb{T}^2)} + \left|\nabla_F{}^2 v\right|_{H^1(\Omega)} + \delta_0\left|\nabla_F\,\eta(t)\right|_{H^{\frac{5}{2}}(\mathbb{T}^2)}\right). \tag{6.31}$$

Taking $\delta_0 > 0$ sufficiently small, we get

$$|\nabla_F\,\eta(t)|_{H^{\frac{5}{2}}(\mathbb{T}^2)} \le C\left(|\partial_t v(t)|_{H^1(\Omega)} + \left|\nabla_F{}^2 v\right|_{H^1(\Omega)}\right). \tag{6.32}$$

Since equation (1.2) holds, the norms of the time derivative $\partial_t\eta$ in $\mathcal{H}(t)$ can be estimated as follows:

$$\begin{aligned} |\nabla_F\,\partial_t\eta|_{H^0(\mathbb{T}^2)} &= |\nabla_F\,v_3|_{H^0(\mathbb{T}^2)} \le C\,|\nabla_F\,v|_{H^1(\Omega)}\,, \\ |\partial_t\eta|_{H^0(\mathbb{T}^2)} &= |v_3|_{H^0(\mathbb{T}^2)} \le C\,|v|_{H^1(\Omega)}\,. \end{aligned} \tag{6.33}$$

From (6.32) and (6.33), we can derive (6.14) by use of Lemma 2.5. □

We next give the bounds for $\mathcal{N}_B(t), \mathcal{N}_\Omega(t)$ and $\mathcal{Q}(t)$ by $\mathcal{F}(t)$ in the following propositions. The proof of these propositions is elementary, but lengthy. Therefore we explain only how to get the bounds for the terms containing the highest order derivatives and the time derivatives. To get the estimates for nonlinear terms we frequently use Lemma 2.2.

Proposition 6.2. *Let η, v, q be the solution given in Theorem* 1.1. *Then it holds that*

$$|\mathcal{N}_B(t)| \le C\delta_0\mathcal{F}(t) \qquad \textit{for} \quad t > T_1 \tag{6.34}$$

with a constant $C > 0$ independent of the solution.

To show this proposition we use

Lemma 6.4. *Let $\varphi \in H^1\left(\mathbb{T}^2\right)$ and let $\psi \in H^{\frac{1}{2}}\left(\mathbb{T}^2\right)$. Then it holds that*

$$|(\partial_\alpha\varphi,\psi)_{\mathbb{T}^2}| \le C\,|\varphi|_{H^{\frac{1}{2}}(\mathbb{T}^2)}\,|\psi|_{H^{\frac{1}{2}}(\mathbb{T}^2)}\,, \quad \alpha = 1,2.$$

This lemma can be easily proved by Fourier series expansion.

The outline of the proof of Proposition 6.2. We first treat the term of the form

$$\left(\partial^2_{\alpha\beta}\left(\frac{\partial_k\left((1+x_3)\,\tilde{\eta}\right)}{J}\partial_\ell v_m\right), \partial^2_{\alpha\beta}v_n\right)_{\mathbb{T}^2} \equiv I_1(t), \quad \alpha,\beta,\ell = 1,2$$

in $\left(\partial^2_{\alpha\beta}F_j, \partial^2_{\alpha\beta}v_j\right)_{\mathbb{T}^2}$. By Lemma 6.4 we have

$$\begin{aligned} |I_1(t)| &\le C\left|\frac{\partial_k\left((1+x_3)\,\tilde{\eta}\right)}{J}\partial_\ell v_m\right|_{H^{\frac{3}{2}}(\mathbb{T}^2)}\left|\partial^2_{\alpha\beta}v_j\right|_{H^{\frac{1}{2}}(\mathbb{T}^2)} \\ &\le C\,|\eta|_{H^3(\mathbb{T}^2)}\,|\partial_\ell v_n|_{H^{\frac{3}{2}}(\mathbb{T}^2)}\left|\partial^2_{\alpha\beta}v_j\right|_{H^{\frac{1}{2}}(\mathbb{T}^2)} \\ &\le C\,|\eta(t)|_{H^3(\mathbb{T}^2)}\sum_{\alpha,\beta=1}^{2}\left|\partial^2_{\alpha\beta}v_j(t)\right|^2_{H^{\frac{1}{2}}(\mathbb{T}^2)}\,. \end{aligned}$$

By the trace theorem and by Lemma 2.5 we have

$$\sum_{\alpha,\beta=1}^{2} \left|\partial_{\alpha\beta}^2 v_j(t)\right|_{H^{\frac{1}{2}}(\mathbb{T}^2)}^2 \leq C \sum_{\alpha,\beta=1}^{2} \langle \partial_{\alpha\beta}^2 v(t), \partial_{\alpha\beta}^2 v(t)\rangle.$$

If ℓ is 3, by Lemma 6.1 we can replace $\partial_3 v_n$ by a linear combination of tangential derivatives $\partial_a v$ with coefficients written in terms of η and its derivatives. Hence we can get the same estimate as for the case $\ell = 1, 2$. We next deal with the term

$$\begin{aligned}
&\left(\partial_{\alpha\beta}^2\left(\sigma\left\{\frac{1}{\sqrt{1+|\nabla_F\,\eta|^2}}-1\right\}\varDelta_F\,\eta\right),\partial_{\alpha\beta}^2 v_n\right)_{\mathbb{T}^2}\\
&=\left(\partial_{\alpha\beta}^2\left(\sigma\frac{-|\nabla_F\,\eta|^2}{\sqrt{1+|\nabla_F\,\eta|^2}\left(1+\sqrt{1+|\nabla_F\,\eta|^2}\right)}\varDelta_F\,\eta\right),\partial_{\alpha\beta}^2 v_n\right)_{\mathbb{T}^2}\\
&\equiv I_2(t), \quad \alpha,\ \beta = 1,2
\end{aligned}$$

in $\left(\partial_{\alpha\beta}^2 F_3,\ \partial_{\alpha\beta}^2 v_3\right)_{\mathbb{T}^2}$. By Lemma 6.4 we have

$$\begin{aligned}
&|I_2(t)|\\
&\quad\leq C\left|\partial_\beta\left(\sigma\frac{-|\nabla_F\,\eta|^2}{\sqrt{1+|\nabla_F\,\eta|^2}\left(1+\sqrt{1+|\nabla_F\,\eta|^2}\right)}\varDelta_F\,\eta\right)\right|_{H^{\frac{1}{2}}(\mathbb{T}^2)}\left|\partial_{\alpha\beta}^2 v_j\right|_{H^{\frac{1}{2}}(\mathbb{T}^2)}\\
&\quad\leq C\,|\nabla_F\,\eta(t)|_{H^{\frac{3}{2}}(\mathbb{T}^2)}\,|\eta(t)|_{H^{\frac{7}{2}}(\mathbb{T}^2)}\left|\partial_{\alpha\beta}^2 v_j(t)\right|_{H^{\frac{1}{2}}(\mathbb{T}^2)}.
\end{aligned}$$

Since $\eta(t) \in H_0^4\left(\mathbb{T}^2\right)$ for $t \geq T_1$, by the Poincaré inequality and by estimate (6.32), we have

$$|\eta(t)|_{H^{\frac{7}{2}}(\mathbb{T}^2)} \leq C\left(|\partial_t v(t)|_{H^1(\Omega)} + \left|\nabla_F{}^2 v\right|_{H^1(\Omega)}\right).$$

From this, by use of Lemma 2.5 we can derive

$$|I_2(t)| \leq C\delta_0\left(\sum_{\alpha,\beta=1}^{2}\langle\partial_{\alpha\beta}^2 v(t),\ \partial_{\alpha\beta}^2 v(t)\rangle + \langle\partial_t v(t),\ \partial_t v(t)\rangle\right).$$

The terms in $(\partial_t F_j, \partial_t v_j)_{\mathbb{T}^2}$ can be treated similarly. The term

$$\left(\partial_t\left(\frac{\partial_k\left((1+x_3)\,\tilde{\eta}\right)}{J}\partial_\ell v_m\right),\partial_t v_n\right)_{\mathbb{T}^2} \equiv I_3(t), \quad \ell = 1,2$$

can be rewritten as follows

$$
\begin{aligned}
I_3(t) &= \left(\frac{\partial_k\left((1+x_3)\,\tilde{\eta}\right)}{J} \partial_\ell \partial_t v_m + \partial_t \left(\frac{\partial_k\left((1+x_3)\,\tilde{\eta}\right)}{J} \right) \partial_\ell v_m, \partial_t v_n \right)_{\mathbb{T}^2} \\
&= \left(\partial_\ell \left(\frac{\partial_k\left((1+x_3)\,\tilde{\eta}\right)}{J} \partial_t v_m \right) - \partial_\ell \left(\frac{\partial_k\left((1+x_3)\,\tilde{\eta}\right)}{J} \right) \partial_t v_m, \partial_t v_n \right)_{\mathbb{T}^2} \\
&\quad + \left(\partial_t \left(\frac{\partial_k\left((1+x_3)\,\tilde{\eta}\right)}{J} \right) \partial_\ell v_m, \partial_t v_n \right)_{\mathbb{T}^2} .
\end{aligned}
$$

By Lemma 6.4 and by the Schwarz inequality, we have

$$
\begin{aligned}
|I_3(t)| \le C &\left| \frac{\partial_k\left((1+x_3)\,\tilde{\eta}\right)}{J} \partial_t v_m \right|_{H^{\frac{1}{2}}(\mathbb{T}^2)} |\partial_t v_n|_{H^{\frac{1}{2}}(\mathbb{T}^2)} \\
&+ \left| \partial_\ell \left(\frac{\partial_k\left((1+x_3)\,\tilde{\eta}\right)}{J} \right) \partial_t v_m \right|_{H^0(\mathbb{T}^2)} |\partial_t v_n|_{H^0(\mathbb{T}^2)} \\
&+ \left| \partial_t \left(\frac{\partial_k\left((1+x_3)\,\tilde{\eta}\right)}{J} \right) \partial_\ell v_m \right|_{H^0(\mathbb{T}^2)} |\partial_t v_n|_{H^0(\mathbb{T}^2)} .
\end{aligned}
$$

Since η satisfies (1.2), $\partial_t \eta$ in $\partial_t \left(\frac{\partial_k\left((1+x_3)\,\tilde{\eta}\right)}{J} \right)$ can be replaced by $v_3(x', 0)$, $x' \in \mathbb{T}^2$. Using the fact (6.1), we can derive from the above

$$
|I_3(t)| \le C\delta_0 \left(|\partial_t v(t)|^2_{H^{\frac{1}{2}}(\mathbb{T}^2)} + |\nabla_F\, v(t)|_{H^0(\mathbb{T}^2)} |\partial_t v(t)|_{H^0(\mathbb{T}^2)} \right) .
$$

As stated in the case for $I_1(t)$, if $\ell = 3$, we can replace $\partial_3 v_m$ in $I_3(t)$ by a linear combination of tangential derivatives $\partial_a v$ with coefficients written in terms of η and its derivatives. Thus we can get the same estimate as for the case $\ell = 1, 2$. One of the terms containing the highest order derivative in $(\partial_t F_3, \partial_t v_3)_{\mathbb{T}^2}$ is the following:

$$
\begin{aligned}
(6.35)\qquad &\left(\partial_t \left(\sigma \left\{ \frac{1}{\sqrt{1 + |\nabla_F\, \eta|^2}} - 1 \right\} \Delta_F\, \eta \right), \partial_t v_3 \right)_{\mathbb{T}^2} \\
&= \left(\sigma \left\{ \frac{1}{\sqrt{1 + |\nabla_F\, \eta|^2}} - 1 \right\} \Delta_F\, \partial_t \eta, \partial_t v_3 \right)_{\mathbb{T}^2} \\
&\quad + \left(-\sigma \frac{(\partial_1 \eta)\,(\partial_1 \partial_t \eta) + (\partial_2 \eta)\,(\partial_2 \partial_t \eta)}{\left(1 + |\nabla_F\, \eta|^2\right)^{\frac{3}{2}}} \Delta_F\, \eta, \partial_t v_3 \right)_{\mathbb{T}^2} \equiv I_4(t).
\end{aligned}
$$

As noted in the above, this can be rewritten as

$$I_4(t) = \left(\sigma \left\{ \frac{1}{\sqrt{1+|\nabla_F \eta|^2}} - 1 \right\} \Delta_F v_3(x',0), \partial_t v_3 \right)_{\mathbb{T}^2} + \left(-\sigma \frac{(\partial_1 \eta)(\partial_1 v_3(x',0)) + (\partial_2 \eta)(\partial_2 v_3(x',0))}{\left(1+|\nabla_F \eta|^2\right)^{\frac{3}{2}}} \Delta_F \eta, \partial_t v_3 \right)_{\mathbb{T}^2} .$$

Using (6.1), we can estimate this as follows

$$|I_4(t)| \leq C\delta_0 \left(\left|\nabla_F{}^2 v\right|_{H^0(\mathbb{T}^2)} |\partial_t v|_{H^0(\mathbb{T}^2)} + |\nabla_F \eta|_{H^1(\mathbb{T}^2)} |\partial_t v|_{H^0(\mathbb{T}^2)} \right).$$

The other terms in $\mathcal{N}_B(t)$ can be treated in a similar way as above. Hence by the trace theorem, Lemma 2.5 and by Proposition 6.1, we can derive the desired bound. □

Proposition 6.3. *Let η, v, q be the solution given in Theorem* 1.1. *Then it holds that*

$$|\mathcal{N}_\Omega(t)| \leq C\delta_0 \mathcal{F}(t) \qquad \textit{for} \quad t > T_1 \tag{6.36}$$

with a constant $C > 0$ independent of the solution.

The outline of the proof. The terms of the highest order derivatives of v in $F_{0,\alpha}$ are $C_{cd}\partial_c\partial_d v_\ell$, where

$$\begin{aligned} C_{cd} \equiv C_{cd}\left(\tilde{\eta}, \nabla\tilde{\theta}\right) &= \zeta_{ce}\zeta_{de} - \delta_{ce}\delta_{de} \\ &= -2\frac{\delta_{3d}}{J}\partial_c\left((1+x_3)\tilde{\eta}\right) + \frac{\delta_{3c}\delta_{3d}}{J^2}\partial_e\left((1+x_3)\tilde{\eta}\right)\partial_e\left((1+x_3)\tilde{\eta}\right). \end{aligned}$$

It is clear that this coefficient vanishes if $c, d = 1, 2$. The terms corresponding to $c = 3$ and $d = 1, 2$ in $\mathcal{N}_\Omega(t)$ are written as, by integrating by parts

$$\begin{aligned} &\left(\partial^2_{\alpha\beta}\left(C_{3d}\partial_3\partial_d v_\ell\right), \partial^2_{\alpha\beta} v_\ell\right)_\Omega \\ &\quad = -\left(\partial_\beta\left(C_{3d}\partial_3\partial_d v_\ell\right), \partial^2_\alpha\partial_\beta v_\ell\right)_\Omega \equiv I_5(t), \quad \alpha, \beta = 1, 2. \end{aligned}$$

This term can be estimated by

$$|I_5(t)| \leq C\,|\tilde{\eta}(t)|_{H^2(\Omega)} \left|\nabla_F{}^2 v(t)\right|^2_{H^1(\Omega)}.$$

Changing the order of differentiation and integrating by parts, we get

$$\begin{aligned}
&\left(\partial^2_{\alpha\beta}\left(C_{33}\left(\tilde\eta,\nabla\tilde\eta\right)\partial_3^2 v_\ell\right),\partial^2_{\alpha\beta}v_\ell\right)_\Omega\\
&\quad=\left(\partial^2_{\alpha\beta}\left(\partial_3\left(C_{33}\left(\tilde\eta,\nabla\tilde\eta\right)\partial_3 v_\ell\right)-\partial_3 C_{33}\left(\tilde\eta,\nabla\tilde\eta\right)\partial_3 v_\ell\right),\partial^2_{\alpha\beta}v_\ell\right)_\Omega\\
&\quad=\left(\partial_3\left(\partial^2_{\alpha\beta}\left(C_{33}\left(\tilde\eta,\nabla\tilde\eta\right)\partial_3 v_\ell\right)\right),\partial^2_{\alpha\beta}v_\ell\right)_\Omega\\
&\qquad-\left(\partial^2_{\alpha\beta}\left(\partial_3 C_{33}\left(\tilde\eta,\nabla\tilde\eta\right)\partial_3 v_\ell\right),\partial^2_{\alpha\beta}v_\ell\right)_\Omega\\
&\quad=\left(\partial^2_{\alpha\beta}\left(C_{33}\left(\tilde\eta,\nabla\tilde\eta\right)\partial_3 v_\ell\right),\partial^2_{\alpha\beta}v_\ell\right)_{\mathbb{T}^2}\\
&\qquad-\left(\partial^2_{\alpha\beta}\left(C_{33}\left(\tilde\eta,\nabla\tilde\eta\right)\partial_3 v_\ell\right),\partial_3\partial^2_{\alpha\beta}v_\ell\right)_\Omega\\
&\qquad-\left(\partial^2_{\alpha\beta}\left(\partial_3 C_{33}\left(\tilde\eta,\nabla\tilde\eta\right)\partial_3 v_\ell\right),\partial^2_{\alpha\beta}v_\ell\right)_\Omega\equiv I_6(t).
\end{aligned}$$

Note that the boundary condition (1.6) holds. By Lemmas 6.1, 6.4 and (6.1) we get

$$\begin{aligned}
&\left|\left(\partial^2_{\alpha\beta}\left(C_{33}\left(\tilde\eta,\nabla\tilde\eta\right)\partial_3 v_\ell\right),\partial^2_{\alpha\beta}v_\ell\right)_{\mathbb{T}^2}\right|\\
&\quad\le C\delta\left(\left|\nabla_F{}^2 v(t)\right|^2_{H^{\frac12}(\mathbb{T}^2)}+\left|\nabla_F{}^2 v(t)\right|_{H^{\frac12}(\mathbb{T}^2)}\left|\nabla_F{}^2\eta(t)\right|_{H^{\frac12}(\mathbb{T}^2)}\right).
\end{aligned}$$

For the other two terms in $I_6(t)$, we see that

$$\begin{aligned}
&\left|\left(\partial^2_{\alpha\beta}\left(C_{33}\left(\tilde\eta,\nabla\tilde\eta\right)\partial_3 v_\ell\right),\partial_3\partial^2_{\alpha\beta}v_\ell\right)_\Omega\right|+\left|\left(\partial^2_{\alpha\beta}\left(\partial_3 C_{33}\left(\tilde\eta,\nabla\tilde\eta\right)\partial_3 v_\ell\right),\partial^2_{\alpha\beta}v_\ell\right)_\Omega\right|\\
&\quad\le C\delta\left(\left|\nabla_F{}^2 v\right|^2_{H^1(\Omega)}+\left|\nabla_F v\right|_{H^1(\Omega)}\left|\nabla_F{}^2 v\right|_{H^1(\Omega)}+|v|_{H^1(\Omega)}\left|\nabla_F{}^2 v\right|_{H^1(\Omega)}\right).
\end{aligned}$$

The term containing the highest order derivative of $\tilde\eta$ in $\mathcal{N}_\Omega(t)$ is

$$\left(\partial^2_{\alpha\beta}\left(\zeta_{ce}\zeta_{de}\zeta_{\ell 3}\partial_c\partial_d\partial_k\left((1+x_3)\tilde\eta\right)v_k\right),\partial^2_{\alpha\beta}v_\ell\right)_\Omega\equiv I_7(t),\ \ \alpha,\beta=1,2.$$

Integrating by parts implies

$$I_7(t)=-\left(\partial_\beta\left(\zeta_{ce}\zeta_{de}\zeta_{\ell 3}\partial_c\partial_d\partial_k\left((1+x_3)\tilde\eta\right)v_k\right),\partial^2_\alpha\partial_\beta v_\ell\right)_\Omega.$$

By virtue of (6.1) and estimate (6.32) we can derive from this

$$\begin{aligned}
|I_7(t)|&\le C\left|\tilde\eta(t)\right|_{H^4(\Omega)}|v(t)|_{H^1(\Omega)}\left|\nabla_F{}^2 v(t)\right|_{H^1(\Omega)}\\
&\le C\delta_0\left(\left|\partial_t v(t)\right|_{H^1(\Omega)}+\left|\nabla_F{}^2 v\right|_{H^1(\Omega)}\right)\left|\nabla_F{}^2 v(t)\right|_{H^1(\Omega)}.
\end{aligned}$$

In the same way as above we can deal with the term

$$\left(\partial_t\left(C_{33}\left(\tilde\eta,\nabla\tilde\eta\right)\partial_3^2 v_\ell\right),\partial_t v_\ell\right)_\Omega\equiv I_8(t)$$

in $(\partial_t F_0,\partial_t v)_\Omega$.

$$\begin{aligned}
I_8(t)&=\left(\partial_t\partial_3\left(C_{33}\left(\tilde\eta,\nabla\tilde\eta\right)\partial_3 v_\ell\right),\partial_t v_\ell\right)_\Omega-\left(\partial_t\left(\partial_3 C_{33}\left(\tilde\eta,\nabla\tilde\eta\right)\partial_3 v_\ell\right),\partial_t v_\ell\right)_\Omega\\
&=\left(\partial_t\left(C_{33}\left(\tilde\eta,\nabla\tilde\eta\right)\partial_3 v_\ell\right),\partial_t v_\ell\right)_{\mathbb{T}^2}-\left(\partial_t\left(C_{33}\left(\tilde\eta,\nabla\tilde\eta\right)\partial_3 v_\ell\right),\partial_t\partial_3 v_\ell\right)_\Omega\\
&\quad-\left(\partial_t\left(\partial_3 C_{33}\left(\tilde\eta,\nabla\tilde\eta\right)\partial_3 v_\ell\right),\partial_t v_\ell\right)_\Omega.
\end{aligned}$$

By Lemma 6.1 $\partial_3 v_\ell$ in the boundary term is replaced by a linear combination of tangential derivatives $\partial_a v$ with coefficients written in terms of η and its derivatives. Since (1.2) holds $\partial_t \tilde{\eta}$ in the coefficients is replaced by the extension of $v_3(x',0)$ (see (1.1)). Taking (6.1) into account we can obtain

$$|I_8(t)| \le C\delta_0 \Big(|\partial_t v|^2_{H^{\frac{1}{2}}(\mathbb{T}^2)} + |\nabla_F v|_{H^0(\mathbb{T}^2)} |\partial_t v|_{H^0(\mathbb{T}^2)} + |\partial_t v|^2_{H^1(\Omega)} + |v|_{H^1(\Omega)} |\partial_t v|_{H^0(\Omega)} \Big).$$

The term which contains the highest order derivative of η in $(\partial_t F_0, \partial_t v)_\Omega$ is

$$(6.37) \qquad (\partial_t (\zeta_{ce}\zeta_{de}\zeta_{\ell 3}\partial_c\partial_d\partial_k ((1+x_3)\tilde{\eta}) v_k), \partial_t v_\ell)_\Omega .$$

As noticed above, $\partial_t \eta$ is replaced by Rv by means of (1.2)(see (1.1), (2.10)). According to the definition of extension (1.1), we see

$$\left|\widetilde{Rv}\right|_{H^3(\Omega)} \le C |Rv|_{H^{\frac{5}{2}}(\mathbb{T}^2)} .$$

Hence the term

$$((\zeta_{ce}\zeta_{de}\zeta_{\ell 3}\partial_c\partial_d\partial_k ((1+x_3)\partial_t\tilde{\eta}) v_k), \partial_t v_\ell)_\Omega = \left(\zeta_{ce}\zeta_{de}\zeta_{\ell 3}\partial_c\partial_d\partial_k \left((1+x_3)\widetilde{Rv}\right) v_k, \partial_t v_\ell\right)_\Omega \equiv I_9(t)$$

appearing in (6.37) is estimated as follows

$$|I_9(t)| \le C\delta_0 |Rv|_{H^{\frac{5}{2}}(\mathbb{T}^2)} |\partial_t v|_{H^0(\Omega)} \le C\delta_0 \left|\nabla_F{}^2 Rv\right|_{H^{\frac{1}{2}}(\mathbb{T}^2)} |\partial_t v|_{H^0(\Omega)} .$$

The other terms in (6.37)

$$(\partial_t (\zeta_{ce}\zeta_{de}\zeta_{\ell 3}) \partial_c\partial_d\partial_k ((1+x_3)\tilde{\eta}) v_k, \partial_t v_\ell)_\Omega + (\zeta_{ce}\zeta_{de}\zeta_{\ell 3}\partial_c\partial_d\partial_k ((1+x_3)\tilde{\eta}) \partial_t v_k, \partial_t v_\ell)_\Omega$$

can be estimated in a similar manner. Here we again replace $\partial_t \eta$ in $\partial_t (\zeta_{ce}\zeta_{de}\zeta_{\ell 3})$ by $v_3(x',0)$. Thus, by the trace theorem and (6.1), we have

$$|(\partial_t (\zeta_{ce}\zeta_{de}\zeta_{\ell 3}\partial_c\partial_d\partial_k ((1+x_3)\tilde{\eta}) v_k), \partial_t v_\ell)_\Omega| \le C\delta \left(\left|\nabla_F{}^2 v\right|_{H^1(\Omega)} + |\partial_t v|_{H^0(\Omega)} + |v|_{H^0(\Omega)} \right) |\partial_t v|_{H^0(\Omega)} .$$

We can treat the other terms in $\mathcal{N}_\Omega(t)$ similarly. Collecting these estimates and using Lemma 2.5, we obtain the desired estimate. □

The next proposition is shown in a same manner as above, therefore we only state the result.

Proposition 6.4. *Let η, v, q be the solution given in Theorem* 1.1. *Then it holds that*

$$(6.38) \qquad \mathcal{Q}(t) \le C\mathcal{F}(t) \quad \text{for} \quad t > T_1$$

with a constant $C > 0$ independent of the solution.

Proof of Theorem 1.3. We add $\gamma\left(\mathcal{E}(t)+\mathcal{H}(t)+\mathcal{Q}(t)\right)$ to the both side of (6.13) to get

$$\frac{1}{2}\frac{d}{dt}\left(\mathcal{E}(t)+\mathcal{H}(t)+\mathcal{Q}(t)\right)+\gamma\left(\mathcal{E}(t)+\mathcal{H}(t)+\mathcal{Q}(t)\right) = \mathcal{N}_B(t)+\mathcal{N}_\Omega(t)+\gamma\left(\mathcal{E}(t)+\mathcal{H}(t)+\mathcal{Q}(t)\right)-\mathcal{F}(t), \tag{6.39}$$

where $\gamma>0$ is to be chosen small later. By virtue of the Poincaré inequality and Lemma 2.5 we have $\mathcal{E}(t)\le C\mathcal{F}(t)$. Using the results in Propositions 6.1, 6.4, we have

$$\mathcal{E}(t)+\mathcal{H}(t)+\mathcal{Q}(t)\le C_1\mathcal{F}(t).$$

By the estimates in Propositions 6.36, 6.34, we can derive from (6.39)

$$\frac{1}{2}\frac{d}{dt}\left(\mathcal{E}(t)+\mathcal{H}(t)+\mathcal{Q}(t)\right)+\gamma\left(\mathcal{E}(t)+\mathcal{H}(t)+\mathcal{Q}(t)\right) \le C_2\delta_0\mathcal{F}(t)+C_1\gamma\mathcal{F}(t)-\mathcal{F}(t). \tag{6.40}$$

In Theorems 1.2, 1.3 we start with the assumption that $\delta>0$ is so small that $C_2\delta_0-1<0$. Further we choose $\gamma>0$ so small that $C_2\delta_0+C_1\gamma-1<0$. Then we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\mathcal{E}(t)+\mathcal{H}(t)+\mathcal{Q}(t)\right)+\gamma\left(\mathcal{E}(t)+\mathcal{H}(t)+\mathcal{Q}(t)\right)\le 0.$$

From this we have

$$\mathcal{E}(t)+\mathcal{H}(t)+\mathcal{Q}(t)\le Ke^{-2\gamma t} \qquad \text{for} \quad t>T_1,$$

where $K=e^{2\gamma T_1}\left(\mathcal{E}(T_1)+\mathcal{H}(T_1)+\mathcal{Q}(T_1)\right)$. Hence we get

$$\mathcal{E}(t)+\mathcal{H}(t)\le Ke^{-2\gamma t} \qquad \text{for} \quad t>T_1. \tag{6.41}$$

Since $\mathcal{H}(t)$ includes

$$\sum_{\alpha,\beta=1}^{2}\left|\nabla_F\,\partial^2_{\alpha\beta}\eta\right|^2_{H^0(\mathbb{T}^2)},$$

by the Poincaré inequality it follows immediately that $|\eta|_{H^3(\mathbb{T}^2)}\le Ce^{-\gamma t}$. Let us show the exponential decay of $|v|_{H^2(\Omega)}$. Using Lemma 6.2 leads to

$$\begin{aligned}&|v(t)|_{H^2(\Omega)}+|\nabla q(t)|_{H^0(\Omega)}\\ &\quad\le C\Big(|\partial_t v(t)\,|_{H^0(\Omega)}+|F_0(t)|_{H^0(\Omega)}+|Q\nabla q|_{H^0(\Omega)}\\ &\qquad+|F_1(t)|_{H^{\frac12}(\mathbb{T}^2)}+|F_2(t)|_{H^{\frac12}(\mathbb{T}^2)}+|v_3(t)|_{H^{\frac32}(\mathbb{T}^2)}\Big).\end{aligned}$$

By (6.1) we see that $|\,Q\nabla q\,|_{H^0(\Omega)}\le C\delta_0|\,\nabla q\,|_{H^0(\Omega)}$. Hence, taking $\delta>0$ small enough if necessary, we have

$$\begin{aligned}|v(t)|_{H^2(\Omega)}\le C\Big(&|\partial_t v(t)|_{H^0(\Omega)}+|F_0(t)|_{H^0(\Omega)}\\ &+|F_1(t)|_{H^{\frac12}(\mathbb{T}^2)}+|F_2(t)|_{H^{\frac12}(\mathbb{T}^2)}+|v_3(t)|_{H^{\frac32}(\mathbb{T}^2)}\Big).\end{aligned} \tag{6.42}$$

The solenoidal condition implies $v_3(x',0) \in H_0^{\frac{3}{2}}(\mathbb{T}^2)$. Thus from the trace theorem and the Poincaré inequality it follows

$$|v_3(t)|_{H^{\frac{3}{2}}(\mathbb{T}^2)} \le C|\nabla_F v_3(t)|_{H^{\frac{1}{2}}(\mathbb{T}^2)} \le C|\nabla\nabla_F v_3(t)|_{H^0(\Omega)} \le C \sum_{\alpha,\beta=1}^{2} |\partial^2_{\alpha\beta} v(t)|_{H^0(\Omega)}.$$

Here we use the solenoidal condition again in the most right inequality. From this and (6.41), taking the explicit forms of F_0, F_1, F_2 into account, we conclude that

$$(6.43)\quad |\partial_t v(t)|_{H^0(\Omega)} + |F_0(t)|_{H^0(\Omega)} + |F_1(t)|_{H^{\frac{1}{2}}(\mathbb{T}^2)} + |F_2(t)|_{H^{\frac{1}{2}}(\mathbb{T}^2)} + |v_3(t)|_{H^{\frac{3}{2}}(\mathbb{T}^2)} \le Ce^{-\gamma t} \quad \text{for} \quad t > T_1.$$

This completes the proof of Theorem 1.3. □

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Added in proof: After we finished writing this paper we learned from Prof. Padula that she with B. J. Jin published "On existence of compressible viscous flows in a horizontal layer with free upper surface" by M. Padula and B. J. Jin in Comm. Pure Appl. Analysis, Vol. 1, No. 3, 2002.

References

[1] S. Agmon, A. Douglis and L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions* II, Comm. Pure Appl. Math. **17** (1964), 35–92.

[2] J. T. Beale, *The initial value problem for the Navier-Stokes equations with a free surface*, Comm. Pure Appl. Math. **34** (1980), 359–392.

[3] ______, *Large-time regularity of viscous surface waves*, Arch. Rat. Mech. Anal. **84** (1984), 307–352.

[4] J. T. Beale and T. Nishida, *Large-time behavior of viscous surface waves*, Lecture Notes in Num. Appl. Anal. **8** (1985), 1–14.

[5] H. Ito, *Extended Korn's inequalities and the associated best possible constants*, J. Elasticity **24** (1990), 43–78.

[6] S. G. Krein, *Linear differential equations in Banach spaces*, Amer. Math. Soc., Providence, 1979.

[7] J. L. Lions and E. Magenes, *Nonhomogeneous boundary value problems and applications*, Springer, New York-Berlin-Heidelberg, 1972.

[8] I. Sh. Mogilevskii, *Estimates of solutions of a general initial-boundary value problem for the linear time-dependent system of Navier-Stokes equations in a half-space*, J. Soviet Math. **21**-3 (1983), 356–375.

[9] T. Nishida, *Equations of fluid dynamics—Free surface problems*, Comm. Pure Appl. Math. **39** (1986), 221–238.

[10] M. Padula and V. A. Solonnikov, *On Rayleigh-Taylor stability*, Ann. Univ. Ferrara, Sez. VIII, sc. Mat. **46** (2000), 307–336.

[11] J. Peetre, *On the differentiability of the solutions of quasilinear partial differential equations*, Trans. Amer. Math. Soc. **104** (1962), 476–482.

[12] V. A. Solonnikov and V. E. Scadilov, *On a boundary value problem for a stationary system of Navier-Stokes equations*, Proc. Steklov Inst. Math. **125** (1973), 186–199.

[13] V. A. Solonnikov, *On an initial-boundary value problem for the Stokes systems arising in the study of a problem with a free boundary*, Trudy. Math Inst. Steklov **188** (1990), 191–239.

[14] N. Tanaka and A. Tani, *Surface waves for a compressible viscous fluid*, Preprint.

[15] A. Tani, *Small-time existence for the three-dimensional Navier-Stokes equations for an incompressible fluid with a free surface*, Arch. Rational Mech. Anal. **133** (1996), 299–331.

[16] R. Temam, *Navier-Stokes equations*, North-Holland, Amsterdam, 1977.

[17] J. V.Wehausen and E. V. Laitone, *Surface waves*, Handbuch der Physik IX, pp. 446–778, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1960.