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Dihedral covers and an elementary arithmetic on elliptic surfaces

By

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1. Introduction

Let X and Y be normal projective varieties defined over $\mathbb{C}$, the complex number field. We call X a cover of Y if there exists a finite surjective morphism $\pi : X \to Y$. The rational function field, $\mathbb{C}(X)$, is regarded as an algebraic extension of that of Y, $\mathbb{C}(Y)$, with $\deg \pi = [\mathbb{C}(X) : \mathbb{C}(Y)]$. The branch locus of a cover $\pi : X \to Y$, denoted by $\Delta(X/Y)$ or Δ_π, is the subset of Y given by

$$\Delta_\pi = \{y \in Y \mid \pi \text{ is not locally isomorphic over } y\}.$$

It is well-known that Δ_π is an algebraic subset of codimension 1 if Y is smooth ([19]). We call X a D_{2n}-cover if (i) $\mathbb{C}(X)/\mathbb{C}(Y)$ is Galois and (ii) $\mathrm{Gal}(\mathbb{C}(X)/\mathbb{C}(Y)) \cong D_{2n}$, the dihedral group of order $2n$. To present D_{2n}, we use the notation

$$D_{2n} = \langle \sigma, \tau \mid \sigma^2 = \tau^n = (\sigma\tau)^2 = 1 \rangle,$$

and fix it throughout this article. Given a D_{2n}-cover $\pi : X \to Y$, we canonically obtain the double cover, $D(X/Y)$, of Y by taking the $\mathbb{C}(X)^\tau$-normalization of Y, where $\mathbb{C}(X)^\tau$ is the fixed field of $\langle \tau \rangle$. X is an n-cyclic cover of $D(X/Y)$ by its definition. We denote these covering morphisms by $\beta_1(\pi) : D(X/Y) \to Y$ and $\beta_2(\pi) : X \to D(X/Y)$, respectively. In [13], the author gave a method to deal with D_{2n}-covers. He exploited it in order to study D_{2n}-covers of $\mathbb{P}^2$ ([14], [15], and [16]) in the following setting:

(i) Y is a surface obtained by a succession of blowing-ups from $\mathbb{P}^2$.

(ii) $D(X/Y)$ has an elliptic fibration $\varphi : D(X/Y) \to \mathbb{P}^1$ with section O and $\beta_1(\pi) : D(X/Y) \to Y$ coincides with the quotient map induced by the inversion homomorphism $z \mapsto -z$ with respect to the group law.

(iii) X also has an elliptic fibration and $\beta_2(\pi)$ is the quotient map by the translation-by-n-torsion element in the Mordell-Weil group.

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The results obtained under the above setting are, for example, existence theorems of D_{2p}-covers (p: odd prime) of $\mathbb{P}^2$ branched along reduced plane curves of degrees 4, 5 and 6 ([13], [15], [17], [18]), and several examples of Zariski pairs ([1], [14], [16]). Many of them gave interesting examples in the study of the complement to a plane algebraic curve. We here introduce terminologies to describe D_{2n}-covers with the above setting.

Definition 1.1. A D_{2n}-cover of a surface satisfying the condition (ii) is called an elliptic D_{2n}-cover. An elliptic D_{2n}-cover is called *torsion type* if X is also an elliptic surface and $\beta_2(\pi)$ is the quotient map by the translation-by-n-torsion element in the Mordell-Weil group. We call X *non-torsion type* if X is not torsion type.

Remark 1.1. For an elliptic D_{2n}-cover $\pi : X \to Y$, it is of torsion type if all the irreducible components of $\Delta_{\beta_2(\pi)}$ are those of fibers of φ, while it is of non-torsion type if there exists a horizontal component in $\Delta_{\beta_2(\pi)}$.

All the previous results are by-products from the investigation of elliptic D_{2n}-covers of torsion type. In this article, we go on to study elliptic D_{2p}-cover (p: odd prime) of non-torsion type.

In the first half of this article, we reduce our problem on elliptic D_{2p}-covers to the problem in solving the equation $px = s$ in the Mordell-Weil group (Propositions 3.1 and 3.2). Hence the solvability of the equation $px = s$ plays an important role. In this article, as the first case, we study it in the case when $D(X/Y)$ is a rational elliptic surface and s has the height ≤ 2. This will be done in Section 4. As an application, we consider D_{2p}-covers of $\mathbb{P}^2$ branched along quintic curves (Theorem 6.2). It gives another proof for the main result in [18].

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2. Preliminaries on elliptic surfaces

The references for this section are [6], [7], [8], [9] and [12].

A smooth surface $\mathcal{E}$ is said to be an elliptic surface over a smooth curve C if there exists a morphism $\varphi : \mathcal{E} \to C$ such that $\varphi^{-1}(v)$ is a smooth curve of genus 1 for $v \in C$ except finite points. We denote the subset of such exceptional points by $\mathrm{Sing}(\varphi)$. We call $\varphi^{-1}(v)$, $v \in R$, a singular fiber. We also define the subset, R, of $\mathrm{Sing}(\varphi)$ as follows:

$$R := \{v \in \mathrm{Sing}(\varphi) \mid \varphi^{-1}(v) \text{ is reducible.}\}.$$

The classification of singular fibers was done by Kodaira in [7]. We use his notation in [7] to describe the type of a singular fiber. Throughout this paper, we always assume that $\varphi : \mathcal{E} \to C$ satisfies the following conditions:

• No exceptional curve of the first kind is contained in any fiber. Namely, φ is relatively minimal.

• $\varphi : \mathcal{E} \to C$ has a section O; we identify O with its image.

• φ has at least one singular fiber.

For a singular fiber F_v, $v \in R$, we write it as follows:

$$F_v = \Theta_{v,0} + \sum_{i=1}^{r_v - 1} m_{v,i}\Theta_{v,i},$$

where r_v is the number of irreducible components, and $\Theta_{v,0}$ is the component with $\Theta_{v,0}O = 1$. For a singular fiber of type I_n, we label its irreducible components, $\Theta_{v,i}$, in such a way that $\Theta_{v,i}\Theta_{v,i+1} = 1\,(0 \le i \le n-2)$ and $\Theta_{v,0}\Theta_{v,n-1} = 1$.

1. Double cover construction of elliptic surfaces

Let $\varphi : \mathcal{E} \to C$ be an elliptic surfaces as above. Since φ has a section, the generic fiber $\mathcal{E}_\eta$ is regarded as an elliptic curve over $\mathbb{C}(C)$, and we can define a group law, O being the zero, in a usual manner. The inversion morphism with respect to the group law induces a fiber preserving automorphism of $\mathcal{E}$ of order 2. We denote it by σ_φ. Let $\overline{\mathcal{E}}$ be the normal surface obtained by contracting all irreducible components of singular fibers not meeting O. σ_φ also induces an involution on $\overline{\mathcal{E}}$, which we denote by $\overline{\sigma}_\varphi$. Now let $\Sigma := \mathcal{E}/\langle\sigma_\varphi\rangle$ and $\overline{\Sigma} := \overline{\mathcal{E}}/\langle\overline{\sigma}_\varphi\rangle$ be the quotient surfaces by these involutions, respectively. Then $\mathcal{E}$, $\overline{\mathcal{E}}$, Σ and $\overline{\Sigma}$ satisfies the following properties:

1. Both Σ and $\overline{\Sigma}$ are smooth. $\overline{\Sigma}$ is a ruled surface over C and Σ is obtained from $\overline{\Sigma}$ by a succession of blowing-ups, $q : \Sigma \to \overline{\Sigma}$.

2. The branch locus of $\overline{\mathcal{E}} \to \overline{\Sigma}$ is of the form $\Delta(\overline{\mathcal{E}}/\overline{\Sigma}) = \Delta_0 + B$, where Δ_0 is a section and B is a triple section with only simple singularities (see [2] for simple singularities).

3. The contracting morphism $\mu : \mathcal{E} \to \overline{\mathcal{E}}$ gives the *canonical resolution* of the double cover $\overline{\mathcal{E}} \to \overline{\Sigma}$ (see [5] for the canonical resolution). Hence the following diagram commutes:

$$\begin{array}{ccc} \overline{\mathcal{E}} & \leftarrow & \mathcal{E} \\ \downarrow & & \downarrow \\ \overline{\Sigma} & \leftarrow & \Sigma. \end{array}$$

As for the above facts, see [6], [8] and [9] for details.

2. The Mordell-Weil group and Shioda's height pairing

We denote the group of sections, the Mordell-Weil group, by $\mathrm{MW}(\mathcal{E}/\varphi)$. If there is no ambiguity for the fibration, we denote it by $\mathrm{MW}(\mathcal{E})$ for simplicity.

Note that we can also consider $\mathrm{MW}(\mathcal{E})$ as a set of $\mathbb{C}(C)$ rational points on the generic fiber. In our circumstances, these two groups are canonically identified. We denote the Néron-Severi group of $\mathcal{E}$ by $\mathrm{NS}(\mathcal{E})$. Under our assumption, $\mathrm{NS}(\mathcal{E})$ is torsion-free, and it has a lattice structure with respect to the intersection pairing. Let T be the subgroup of $\mathrm{NS}(\mathcal{E})$ generated by O and all the irreducible components of fibers. T is a sublattice of $\mathrm{NS}(\mathcal{E})$ and has a natural basis, O, a fiber F, and $\Theta_{v,i}$ ($v \in R$, $1 \le i \le r_v - 1$). Let T_v be the subgroup generated by $\Theta_{v,i}$ ($1 \le i \le r_v - 1$). Then T has a decomposition

$$T \cong \mathbb{Z}O \oplus \mathbb{Z}F \oplus \bigoplus_{v \in R} T_v.$$

Theorem 2.1 (Shioda [12]). *There exists a natural map* $\tilde{\psi} : \mathrm{NS}(\mathcal{E}) \to \mathrm{MW}(\mathcal{E})$ *such that it induces an isomorphism of groups,*

$$\psi : \mathrm{NS}(\mathcal{E})/T \cong \mathrm{MW}(\mathcal{E}).$$

For a proof, see [12].

Let $\mathrm{NS}_{\mathbb{Q}} := \mathrm{NS}(\mathcal{E}) \otimes \mathbb{Q}$, $T_{\mathbb{Q}} := T \otimes \mathbb{Q}$. Then we have the orthogonal decomposition $\mathrm{NS}_{\mathbb{Q}} = T_{\mathbb{Q}} \oplus (T_{\mathbb{Q}})^{\perp}$. Note that there will be no harm in considering $\mathrm{NS}_{\mathbb{Q}}$ since $\mathrm{NS}(\mathcal{E})$ is torsion-free. Following to [12], we define $\phi : \mathrm{MW}(\mathcal{E}) \to \mathrm{NS}_{\mathbb{Q}}$ as follows:

$$\begin{aligned} \phi : s \in \mathrm{MW}(\mathcal{E}) \mapsto\ & s - O - (sO + \chi(\mathcal{O}_{\mathcal{E}}))F \\ & + \sum_{v \in R} (\Theta_{v,1}, \dots, \Theta_{v,r_v-1})(-A_v^{-1}) \begin{pmatrix} s\Theta_{v,1} \\ \cdot \\ s\Theta_{v,r_v-1} \end{pmatrix} \in \mathrm{NS}_{\mathbb{Q}}, \end{aligned} \tag{1}$$

where A_v is the intersection matrix of T_v with respect to the basis $\Theta_{v,1}, \dots, \Theta_{v,r_v-1}$. ϕ satisfies that (*i*) $\phi(s) \equiv s \bmod T_{\mathbb{Q}}$ and (*ii*) $\phi(s) \perp T_{\mathbb{Q}}$. Moreover, ϕ gives a group homomorphism from $\mathrm{MW}(\mathcal{E})$ to $\mathrm{NS}_{\mathbb{Q}}$ such that $\mathrm{Ker}(\phi) = \mathrm{MW}(\mathcal{E})_{tor}$. See Lemmas 8.1 and 8.4 in [12] for details.

Theorem 2.2 ([12, Theorem 8.4]). *Let*

$$\langle s_1, s_2 \rangle = -\phi(s_1)\phi(s_2), \quad s_1, s_2 \in \mathrm{MW}(\mathcal{E}).$$

Then it defines a symmetric bilinear form on $\mathrm{MW}(\mathcal{E})$ *which induces the structure of a positive definite lattice on* $\mathrm{MW}(\mathcal{E})/\mathrm{MW}(\mathcal{E})_{tor}$.

For a proof, see [12].

The pairing in Theorem 2.2 is called the *height pairing.* For $s_1\, s_2 \in \mathrm{MW}(\mathcal{E})$,

$$\langle s_1, s_2 \rangle = \chi(\mathcal{O}_{\mathcal{E}}) + s_1O + s_2O - s_1s_2 - \sum_{v \in R} \mathrm{Contr}_v(s_1, s_2),$$

$$\langle s_1, s_1 \rangle = 2\chi(\mathcal{O}_{\mathcal{E}}) + 2s_1O - \sum_{v \in R} \mathrm{Contr}_v(s_1, s_1),$$

where the contribution terms $\mathrm{Contr}_v(s_1, s_2)$ and $\mathrm{Contr}_v(s_1, s_1)$ are given as in [12, p. 229].

Corollary 2.1. *If p^e (p : prime, $e \geq 1$) divides the denominator of $\langle s,\, s\rangle$ for some s, then $\varphi : \mathcal{E} \to C$ has at least one singular fiber F_v as follows*:

p^e	2	2^2	2^e ($e \geq 3$)
Type of F_v	III, III^*, I_n (n: *even*), I_n^*	I_n ($2^2 \mid n$), I_n^*	I_n ($2^e \mid n$)

3	3^e ($e \geq 2$)	p^e ($p \geq 5$)
IV, IV^*, I_n ($3 \mid n$)	I_n ($3^e \mid n$)	I_n ($p^e \mid n$)

Proposition 2.1. *Let p^e as in Corollary* 2.1. *Then*

$$p^e \leq \min(10\chi(\mathcal{O}_{\mathcal{E}}) + 2q - 1, 12\chi(\mathcal{O}_{\mathcal{E}})),$$

where q is the irregularity of $\mathcal{E}$.

Proof. Suppose that p^e divides the denominator of $\langle s,\, s\rangle$ for some $s \in \mathrm{MW}(\mathcal{E})$. Then φ has a singular fiber described in Corollary 2.1, and $p^e \leq$ the topological Euler number of the corresponding F_v. By §12 in [7], $12\chi(\mathcal{O}_{\mathcal{E}})$ is equal to the sum of the topological Euler numbers of all singular fibers. Hence $p^e \leq 12\chi(\mathcal{O}_{\mathcal{E}})$. On the other hand, a reducible singular fiber F_v gives $r_v - 1$ independent elements in $\mathrm{NS}(\mathcal{E})$. Since

$$\operatorname{rank} \mathrm{NS}(\mathcal{E}) \leq 10\chi(\mathcal{O}_{\mathcal{E}}) + 2q,$$

we have

$$2 + p^e - 1 \leq \operatorname{rank} T \leq 10\chi(\mathcal{O}_{\mathcal{E}}) + 2q,$$

where T is the lattice introduced at the beginning of this section. □

Corollary 2.2. *If $\mathcal{E}$ is a rational elliptic surface, then $\langle s,\, s\rangle \in 1/(2^3 \cdot 3 \cdot 5 \cdot 7)\mathbb{Z}$ for any $s \in \mathrm{MW}(\mathcal{E})$.*

Proof. By Proposition 2.1, possible pairs (p, e) are $(2, 1)$, $(2, 2)$, $(2, 3)$, $(3, 1)$, $(3, 2)$, $(5, 1)$ and $(7, 1)$. We show that $(3, 2)$ does not occur. Suppose that it occurs. Then, by Corollary 2.1, φ has a singular fiber of type I_n ($3^2 \mid n$). Since $\mathcal{E}$ is a rational elliptic surface, the configuration of singular fibers is $\{I_9,\, 3I_1\}$ by [11]. In this case, $\mathrm{MW}(\mathcal{E}) \cong \mathbb{Z}/3\mathbb{Z}$. Hence $\langle s,\, s\rangle = 0$ for any $s \in \mathrm{MW}(\mathcal{E})$. □

3. Dihedral covers

In this section, we summarize some results on D_{2n}-covers. We here consider the case when n is an odd integer. We keep the notations introduced in Introduction.

Proposition 3.1. *Let n be an odd integer ≥ 3. Let Z be a smooth double cover of a smooth projective variety Y and we denote its covering morphism by $f : Z \to Y$. Let σ_f be the covering transformation. Let D be an effective divisor on Z such that*

(i) *D and σ_f^*D have no common component,*

(ii) *if we let $D = \sum_i a_i D_i$ be the irreducible decomposition, then $a_i > 0$ for all i, and the greatest common divisor of a_i's and n is 1, and*

(iii) *there exists a line bundle L such that $D - \sigma_f^*D \approx nL$.*

Then there exists a D_{2n}-cover, $\pi : X \to Y$ such that (a) $D(X/Y) = Z$, $f = \beta_1(\pi)$ and (b) the branch locus of $\beta_2(\pi)$ is contained in $\mathrm{Supp}(D + \sigma_f^*D)$, *i.e., $\Delta_\pi \subset \Delta_f \cup f(\mathrm{Supp}(D))$.*

Proposition 3.2. *Let n be an odd integer ≥ 3. Let $\pi : X \to Y$ be a D_{2n}-cover such that both Y and $D(X/Y)$ are smooth. Let σ be the covering transformation of $\beta_1(\pi)$. Then there exist an effective divisor D and a line bundle L on $D(X/Y)$ satisfying the following four conditions*:

(i) *D and σ^*D have no common component.*

(ii) *If we let $D = \sum_i a_i D_i$ be the irreducible decomposition, then $0 \leq a_i \leq (n-1)/2$ for every i.*

(iii) *$D - \sigma D \sim nL$.*

(iv) $\mathrm{Supp}(D + \sigma^*D) = \Delta_{\beta_2(\pi)}$.

For a proof, see [13, §2].

Corollary 3.1. *Under the condition of Proposition* 3.2, *if n is an odd prime p and $\Delta_{\beta_2(\pi)} \neq \emptyset$, then we can choose a divisor D in such a way that $a_1 = 1$.*

For a proof, see [13, Corollary 2.3].

Corollary 3.2. *Let D be an irreducible component of $\beta_1(\pi)(\Delta_{\beta_2(\pi)})$. Then $\beta_1(\pi)^*D$ is of the form $D' + \sigma^*D'$ for some irreducible divisor on $D(X/Y)$. In other words, $\beta_2(\pi)$ is not branched along any irreducible divisor D with $D = \sigma^*D$.*

4. Elliptic D_{2p}-covers of non-torsion type

Let $\varphi : \mathcal{E} \to C$ be an elliptic surface over C. Note that we always assume that φ satisfies the three conditions in Section 2.

Lemma 4.1. *Let $f : \mathcal{E} \to \Sigma$ be the double cover introduced in Section* 2. *Let σ_f be the covering transformation of f. Let D be an irreducible horizontal divisor on $\mathcal{E}$ such that* (i) *the intersection number of D and a fiber F is an odd number d and* (ii) *$D \not\subset \Delta_f$. Then $D \neq \sigma_f^*D$.*

Proof. Suppose that $D = \sigma_f^*D$. Then there exists an irreducible divisor $\overline{D}$ on Σ such that $f^*\overline{D} = D$ by the second assumption. Then

$$d = DF = f^*\overline{D}f^*(f(F)) = 2\overline{D}f(F).$$

This is impossible. □

Let D be a divisor as in Lemma 4.1. By [12, §5], there exists a unique section, s, on $\mathcal{E}$ such that

$$D \approx s + (d-1)O + nF + \sum_{v \in R} \sum_{i=1}^{r_v - 1} b_{v,i}\Theta_{v,i}, \tag{2}$$

where d, n and $b_{v,i}$ are integers defined as follows:

$$d = DF, \qquad n = (d-1)\chi(\mathcal{O}_{\mathcal{E}}) + OD - sD,$$

and

$$\begin{pmatrix} b_{v,1} \\ \cdot \\ b_{v,r_v-1} \end{pmatrix} = A_v^{-1} \begin{pmatrix} D\Theta_{v,1} - s\Theta_{v,1} \\ \cdot \\ D\Theta_{v,r_v-1} - s\Theta_{v,r_v-1} \end{pmatrix},$$

where A_v is the matrix introduced in Section 2, (1).

Proposition 4.1. *Let p be an odd prime. Let D_o be an irreducible divisor satisfying the conditions in Lemma* 4.1 *and let s be the unique section satisfying* (2) *for D_o. Suppose that there exists an elliptic D_{2p}-cover $\pi : S \to \Sigma$ such that $D(S/\Sigma) = \mathcal{E}$, $\beta_1(\pi) = f$ and the horizontal component of $\Delta_{\beta_2(\pi)}$ is $D_o + \sigma^* D_o$. Then s is p-divisible in* $\mathrm{MW}(\mathcal{E})$, *i.e., there exists a section s_1 such that $ps_1 = s$ in* $\mathrm{MW}(\mathcal{E})$.

Proof. Let D be the divisor on $\mathcal{E}$ as in Proposition 3.2. By Corollary 3.1, we may assume that D is of the form $D = D_o + \Xi$, where Ξ consists of only vertical components. By Proposition 3.2, there exists a divisor D' such that

$$D - \sigma_f^* D \sim pD'.$$

Let s_1 be the unique section corresponding to D' as in (2). Then, by Abel's theorem on the generic fiber of $\mathcal{E}$, we have $ps_1 = s - \sigma_f^* s$ on $\mathrm{MW}(\mathcal{E})$. Since σ_f is induced by the inverse morphism with respect to the group law, we have

$$ps_1 = 2s \qquad \text{on} \qquad \mathrm{MW}(\mathcal{E}).$$

Let k and l be integers such that $2k + pl = 1$. Then, on $\mathrm{MW}(\mathcal{E})$,

$$s = (2k + lp)s = p(ks_1 + ls).$$

Therefore s is p-divisible. □

Corollary 4.1. *Suppose that D is a section s. If $\langle s,\, s\rangle > 0$, elliptic D_{2p}-covers as in Proposition* 4.1 *exist for only finitely many p.*

Proof. Let s_1 be a section such that $ps_1 = s$. Then we have

$$\langle s_1,\, s_1\rangle = \frac{1}{p^2}\langle s,\, s\rangle.$$

By Proposition 2.1, the denominator of $\langle s_1, s_1\rangle$ is bounded by $\chi(\mathcal{O}_{\mathcal{E}})$ and $q(\mathcal{E})$. □

From now on, we assume that $C = \mathbb{P}^1$. By the assumption in Section 2, $\mathcal{E}$ is simply connected. In particular, $\mathrm{NS}(\mathcal{E}) = \mathrm{Pic}(\mathcal{E})$. Hence we may replace algebraic equivalence by linear equivalence.

Proposition 4.2. *Let D be an irreducible divisor as in Lemma* 4.1 *and let s be the section corresponding to D as in* (2)*. If s is p-divisible in* $\mathrm{MW}(\mathcal{E})$*, then there exists an elliptic D_{2p}-cover, S, of Σ such that $\Delta_{\beta_2(\pi)} = \mathrm{Supp}(D_1 + \sigma^* D_1)$, where D_1 is an effective divisor of the form*

$$D_1 = D + \Xi,$$

where Ξ is an effective divisor whose irreducible components are vertical divisor not meeting O.

Proof. Since s is p-divisible in $\mathrm{MW}(\mathcal{E})$, there exists $s_1 \in \mathrm{MW}(\mathcal{E})$ such that $ps_1 = s$ on $\mathrm{MW}(\mathcal{E})$. This implies

$$s \sim ps_1 - (p-1)O + aF + \sum_{v\in R}\sum_{i=1}^{r_v-1} c_{v,i}\Theta_{v,i},$$

and we have

$$D \sim ps_1 + (d-p)O + (a+n)F + \sum_{v\in R}\sum_{i=1}^{r_v-1} (b_{v,i} + c_{v,i})\Theta_{v,i}.$$

Therefore we have

$$\left(D + \sum_{v\in R}\sum_{i=1}^{r_v-1} (b_{v,i} + c_{v,i})\sigma_f^*\Theta_{v,i}\right) - \sigma_f^*\left(D + \sum_{v\in R}\sum_{i=1}^{r_v-1} (b_{v,i} + c_{v,i})\sigma_f^*\Theta_{v,i}\right)$$
$$\sim p(s_1 - \sigma_f^* s_1).$$

The left hand side contains some redundancy in the sum for $(\Theta_{v,i} - \sigma_f^*\Theta_{v,i})$, but we can rewrite it in the form

$$(D + \Xi) - \sigma_f^*(D + \Xi),$$

where Ξ is an effective vertical divisor such that (i) the irreducible components are those in fibers not meeting O by Corollary 3.2, and (ii) Ξ and $\sigma_f^*\Xi$ have no common component.

Now put $D_1 = D + \Xi$. Then by Proposition 3.1, we have the desired elliptic D_{2p}-cover. □

5. Rational elliptic case

We keep the notations as before. In this section, we consider the case as follows:

(i) $\mathcal{E}$ is a rational elliptic surface.
(ii) D is a section s with $sO = 0$.
Note that s can not be a 2-torsion, since we assume $D \not\subset \Delta_{\beta_1(\pi)}$.

Lemma 5.1. *Let m be the smallest natural number such that $\langle s, s\rangle \in (1/m)\mathbb{Z}$ for all $s \in \mathrm{MW}(\mathcal{E})$. If $\mathcal{E}$ is a rational elliptic surface, then m is equal to one of the following*:

$$1, 2, 3, 4, 5, 6, 7, 8, 10, 12, 14, 15, 20, 30.$$

Proof. The statement immediately follows from the Main Theorem in [10]. Yet we here give another more elementary proof. By Corollary 2.2, $m \mid 2^3 \cdot 3 \cdot 5 \cdot 7$. Let $m = 2^a 3^b 5^c 7^d$. If m is not in the list in the statement, then (a, b, c, d) belongs to the list below:

$$\begin{array}{cccccc} (0,1,0,1) & (0,0,1,1) & (0,1,1,1) & (1,1,0,1) & (1,1,0,1) & (1,1,1,1) \\ (2,0,0,1) & (2,1,1,0) & (2,1,0,1) & (2,0,1,1) & (2,1,1,1) & (3,1,0,0) \\ (3,0,1,0) & (3,0,0,1) & (3,1,1,0) & (3,1,0,1) & (3,0,1,1) & (3,1,1,1) \end{array}$$

We show that all the cases as above do not occur. If (a, b, c, d) is of the form $(a, b, 1, 1)$, then by Corollary 2.1, φ has both I_5 and I_7 singular fibers. The irreducible components of these fibers give 10 independent elements in $\oplus_{v\in R} T_v$. Since $\mathcal{E}$ is rational, $\operatorname{rank} \mathrm{NS}(\mathcal{E}) = 10$. Hence $\operatorname{rank} \oplus_{v\in R} T_v = \operatorname{rank} T - 2 \le \operatorname{rank} \mathrm{NS}(\mathcal{E}) - 2 = 8$. This leads us to a contradiction. Hence this case does not occur. Similarly we see that he remaining cases except $(0, 1, 0, 1)$ do not occur. We need to eliminate the case $(0, 1, 0, 1)$. If this happens, then the reducible singular fibers of φ are of types either IV, I_7 or I_3, I_7. It implies that $\operatorname{rank} T = 10$. Let $\det \mathrm{NS}(\mathcal{E})$ and $\det T$ be the determinant of the intersection matrices for $\mathrm{NS}(\mathcal{E})$ and T, respectively. Since $\operatorname{rank} \mathrm{NS}(\mathcal{E}) = \operatorname{rank} T = 10$, $\mathrm{NS}(\mathcal{E})/T$ is a finite group. These three quantities satisfy the equality $|\det \mathrm{NS}(\mathcal{E})| \sharp (\mathrm{NS}(\mathcal{E})/T)^2 = |\det T|$. This implies the case $(0, 1, 0, 1)$ does not occur, since $|\det T| = 21$, $\det \mathrm{NS}(\mathcal{E}) = 1$. □

Theorem 5.1. *Let $\pi : S \to \Sigma$ be an elliptic D_{2p} covering. Suppose that*
(i) *$\mathcal{E}$ is a rational elliptic surface,*
(ii) *$\Delta_{\beta_2(\pi)}$ is of the form $\Delta_{\beta_2(\pi)} = s + \Xi + \sigma^*(s + \Xi)$, where s is a section with $sO = 0$ and the irreducible components of Ξ consists of $\Theta_{v,i}$'s $(i > 0)$, and*
(iii) *$\langle s, s\rangle > 0$.*
Then $p = 3, 5$.

Proof. By Proposition 4.1, there exists $s_1 \in \mathrm{MW}(\mathcal{E})$ such that $ps_1 = s$ in $\mathrm{MW}(\mathcal{E})$. Let m be as above. Put $\langle s_1, s_1\rangle = n_1/m$ and $\langle s, s\rangle = n_2/m$. Then the equality $p^2\langle s_1, s_1\rangle = \langle s, s\rangle$ implies $p^2 \mid n_2$. Since $0 < \langle s, s\rangle \le 2\chi(\mathcal{O}_{\mathcal{E}}) = 2$, $0 < n_2 < 2m$. By Lemma 5.1, we infer that $p = 3, 5, 7$. We prove that $p = 7$ does not happen. Suppose that a D_{14}-cover satisfying the condition (i), (ii) and (iii) exists. In this case, $2m \ge 49$. Hence $m = 30$ by Lemma 5.1 and $\langle s, s\rangle = 49/30$. By the list in [10] and [11], $m = 30$ occurs only if the configuration of singular fibers of $\mathcal{E}$ are either $\{I_5,\ I_3,\ I_2,\ 2I_1\}$, $\{I_5,\ I_3,\ I_2,\ II\}$ or $\{I_5,\ IV,\ I_2,\ I_1\}$. Since $m = 30$, s meets some $\Theta_{v,i}$ $(i > 0)$ for every $v \in R$. By the explicit formula for the height pairing in Section 2, we have

$$\langle s,\ s\rangle = \frac{5}{6} - \frac{k(5-k)}{5},$$

where $k \in \{1, 2, 3, 4\}$. This leads us to a contradiction. $\square$

We next consider the case when s is a torsion element. Note that the order of s is ≥ 3 and $sO = 0$ always holds.

Theorem 5.2. *Let $\pi : S \to \Sigma$ be an elliptic D_{2p}-cover. Suppose that*
(i) *$\mathcal{E}$ is a rational elliptic surface,*
(ii) *$\Delta_{\beta_2(\pi)}$ is of the form $\Delta_{\beta_2(\pi)} = s + \Xi + \sigma^*(s + \Xi)$, where s is a section as above and the irreducible components of Ξ consists of $\Theta_{v,i}$'s $(i > 0)$, and*
(iii) *s is a torsion element.*
Then we have the following table:

Order of s	3	4	5	6
p	$p \neq 3$	*any odd prime*	$p \neq 5$	$p \neq 3$.

Proof. Let m_s be the order of s. By Proposition 4.1, s is p-divisible. If $(m_s, p) = (3, 3)$ occurs, it implies that $\mathrm{MW}(\mathcal{E})$ has a torsion element of order 9. On the other hand, by [3] and [4], the possible values of m_s are $2, 3, 4, 5, 6$. Hence the case $(m_s, p) = (3, 3)$ does not occur. The cases $(m_s, p) = (5, 5), (6, 3)$ are also ruled out in the same manner.

Conversely, if $\gcd(m_s, p) = 1$, then s is p-divisible. By Proposition 4.2, there exists an elliptic D_{2p}-cover with desired properties. $\square$

In the rest of this section, we consider the solvability of the equation $px = s$ $(p = 3, 5)$. To this purpose, we make use of the results in [10]. The case numbers refers to those in the Main Theorem in [10].

Theorem 5.3. *Suppose that there exists a section s satisfying that* (i) *$sO = 0$,* (ii) *$\langle s,\ s\rangle > 0$ and* (iii) *s is 3-divisible. Then $\mathrm{MW}(\mathcal{E})$ is one of the following cases*:

No. 20, 29, 31, 37, 40, 45, 47, 49, 50, 53, 55, 56, 59, 61.

Conversely, for each case in the above, there exists s satisfying (i), (ii) *and* (iii).

Proof. Let m be the natural number in Lemma 5.1. Let s_1 be a section such that $3s_1 = s$. Put $\langle s_1, s_1\rangle = a/m$, $a \in \mathbb{Z}_{>0}$. Then $9a/m = \langle s, s\rangle \leq 2$ and we have $m \geq 5$. Hence by the Main Theorem in [10], there are 24 cases: No. 6, 8, 12, 15, 19, 20, 23, 25, 29, 33, 37, 40, 41, 44, 45, 47, 49, 50, 51, 53, 55, 56, 59, 61. We prove the statement by the case-by-case checking. We here explain how we show it for three cases, since the remaining cases are similarly checked.

No. 8. In this case, φ has a unique reducible singular fiber, F_v, of type I_5. Suppose that there exists s with (i), (ii) and (iii) and s meets $\Theta_{v,k}$ $(1 \leq k \leq 4)$ at F_v. By (ii) and (iii), $\langle s,\, s\rangle = 9/5$. On the other hand, we have

$$\langle s,\, s\rangle = 2 - \frac{k(5-k)}{5}$$

by the explicit formula. This lead us to a contradiction.

No. 20. Let s_1 be a section such that $\langle s_1,\, s_1\rangle = 1/6$. Let s be the section given by $3s_1$. Then $\langle s,\, s\rangle = 3/2$. On the other hand,

$$\langle s,\, s\rangle = 2 + 2sO - \frac{2}{3}a - \frac{1}{2}b,$$

where $a \in \{0,1,2\}$ and $b \in \{0,1\}$. This implies that $sO = 0$.

No. 56. Let s_0 be a section such that $\langle s_0,\, s_0\rangle = 1/30$. Put $s_1 = s_0$ and $s_1' = 2s_0$. Then section $s := 3s_1$ and $s' := 3s_1'$ satisfy the three conditions. □

Remark 5.1. The calculation for No. 32 in the Main Theorem in [10] seems to be false. In fact,

$$\langle s, s\rangle = 2 + 2sO - \frac{2}{3}a - b,$$

where $a, b \in \{0,1\}$, and no section attains 1/6. Also No. 70 seems to be miscalculation. In this case, $\mathrm{MW}(\mathcal{E}) \cong \mathbb{Z}/4\mathbb{Z}$.

Theorem 5.4. *Suppose that there exists a section s satisfying that* (i) $sO = 0$, (ii) $0 < \langle s,\, s\rangle$ *and* (iii) *s is* 5*-divisible. Then* $\mathrm{MW}(\mathcal{E})$ *is either No.* 55 *or No.* 56*. Conversely, for each case in the above, there exists s with* (i), (ii) *and* (iii).

Proof. Let m be the natural number in Lemma 5.1. Then, likewise our proof of Theorem 5.3, $m \geq 13$. Hence possible cases are No. 31, 47, 55, 56. We can check the statement by the case-by-case checking in the same manner as in Theorem 5.3. □

6. Application: D_{2p}-covers of $\mathbb{P}^2$ branched along quintic curves

In this section, we apply the results in Section 5 to investigate D_{2p}-covers branched along reduced quintic curves. It gives a geometric interpretation of p-divisibility of a section s.

Let $\varpi : S \to \mathbb{P}^2$ be a D_{2p}-cover branched along a reduced quintic curve. Since $\mathbb{P}^2$ is simply connected, $\Delta_{\beta_1(\pi)} \neq \emptyset$ and it is either a conic or a quartic. We call the former type I and the latter type II. If ϖ is of type II, Δ_ϖ is of the form $Q + l$, where l is a line and $Q = \Delta_{\beta_1(\varpi)}$. By Corollary 3.2, l does not meet Q transversely. As for their relative positions, it falls into one of the following:

The case $Q \cap l = \{y_1, y_2\}$.

(i) l is bitangent to Q at y_1 and y_2.

(ii) l is tangent to one point and the other is a singular point of Q.
(iii) Both of y_1 and y_2 are singular points of Q.
The case $Q \cap l = \{y\}$.
(iv) l is a tangent line at y with multiplicity 4.
(v) y is a singular point of Q.
(vi) Q consists of four lines meeting at one point y and l is another line through y.

In the following except Theorem 6.2, we always assume:

Assumption: $Q \neq$ four lines meeting at one point.

We apply the results in §4 to study D_{2p}-covers of type II. To this purpose, we introduce a rational elliptic surface, $\mathcal{E}(\varpi)$, related with $\varpi : S \to \mathbb{P}^2$ in the following manner:

Choose a general point x on Q. We call x the distinguished point. Let l_x be the tangent line at x (in the case when Q is four lines, l_x is the component containing x). Let $q_1 : (\mathbb{P}^2)_x \to \mathbb{P}^2$ be the blowing-up at x. The strict transform, $\bar{l}_x$, of l_x meets the exceptional curve E_x of q_1 at a point, y. Let $q_2 : \Sigma_0 \to (\mathbb{P}^2)_x$ be the blowing-up at y. Let $\mathcal{E}_1$ be the $\mathbb{C}(S)^\tau$-normalization of $D(S/\mathbb{P}^2)$. $\mathcal{E}_1$ is a double cover of Σ_0 branched at $\overline{E}_x$ and $\overline{Q}$, where $\overline{E}_x$ and $\overline{Q}$ are the strict transforms of E_x and Q in Σ_0, respectively.

The canonical resolution

$$\begin{array}{ccc} \mathcal{E}_0 & \leftarrow & \mathcal{E}(\varpi) \\ \downarrow & & \downarrow \\ \Sigma_0 & \leftarrow & \Sigma \end{array}$$

gives rise to a rational elliptic surface, $\mathcal{E}(\varpi)$. Note that $\mathcal{E}(\varpi)$ satisfies the three conditions in Section 2, since (a) there is a section O coming from $\overline{E}_x$, (b) it has a singular fiber arising from l_x and (c) Q has only simple singularities. Moreover the covering transformation of $\mathcal{E}(\varpi) \to \Sigma$ coincides with the one induced by the inversion morphism with respect to the group law with O as the zero. Let $\pi : \tilde{S} \to \Sigma$ be the $\mathbb{C}(S)$-normalization of Σ. Then $\tilde{S}$ is an elliptic D_{2p}-cover with $D(\tilde{S}/\Sigma) = \mathcal{E}(\varpi)$.

Lemma 6.1. *Let $q_2^{-1}(\bar{l}_x)$ be the strict transform of $\bar{l}_x$. Let F_x be the fiber containing an irreducible component coming from $q_2^{-1}(\bar{l}_x)$. Then F_x is either I_0^*, I_1^* or I_2.*

Proof. By carrying out the canonical resolution, we can check

(i) F_x is I_0^*, if Q consists of 4 lines, l_x and l_i $(i = 1, 2, 3)$ such that the three intersection points $l_x \cap l_i$ $(i = 1, 2, 3)$ are all distinct.

(ii) F_x is I_1^*, if Q consists of 4 lines, l_x and l_i $(i = 1, 2, 3)$, such that two of the three intersection points $l_x \cap l_i$ $(i = 1, 2, 3)$ coincide, i.e., Q has an ordinary triple point and l_x is a component through it.

(iii) F_x is I_2, if Q contains a component of degree ≥ 2 and x is a general point on this component.

□

Theorem 6.1. *Let $\varpi : S \to \mathbb{P}^2$ be a D_{2p}-cover of $\mathbb{P}^2$ branched along a reduced quintic curve. Suppose that ϖ is of type II. Then p and the structure of* $\mathrm{MW}(\mathcal{E}(\varpi))$ *falls into one of the following*:

$p = 3$: No. 20, 29, 27, 37, 40, 47, 49, 56, 58, 59, 61, 70, 74.

$p = 5$: No. 55, 56, 58, 61, 66, 70, 74.

$p \geq 7$: No. 58, 61, 66, 70, 74.

Proof. By our construction of $\mathcal{E}(\varpi)$, l gives rise to two sections, s^+ and s^-. Since $\Delta_\varpi = Q + l$, any irreducible component of $\Delta_{\beta_2(\varpi)}$ other than s^+ and s^- is that of exceptional set of $\mathcal{E}(\varpi) \to D(S/\Sigma)$. Since the strict transform of E_x is contained in $\Delta_{\beta_1(\varpi)}$, $\Delta_{\beta_2(\varpi)}$ is of the form $s^+ + s^- +$ vertical divisors. Hence $s^\pm$ are torsions or $\mathrm{MW}(\mathcal{E}(\varpi))$ belongs to the lists in Theorems 5.3 and 5.4. Moreover, $\mathcal{E}(\varpi)$ has a singular fiber described in Lemma 6.1. Therefore we have the statement. □

Theorem 6.2. *Let $\varpi : S \to \mathbb{P}^2$ be a D_{2p}-cover of $\mathbb{P}^2$ branched along a quintic curve. If π is of type* II, *then the branch locus, $Q + l$, of ϖ falls into one of the following*:

	p	Q	$Q \cap l$	*Type No. of* $\mathrm{MW}(\mathcal{E}(\varpi))$
1	3	Q_1	(i)	20
2	3	Q_1	(iv)	20
3	3	Q_2	(i)	40
4	3	Q_2	(iv)	40
5	3	Q_3	(i)	61
6	3	Q_4	(ii), a_3	37
7	3	Q_5	(i)	29
8	3	Q_5	(iv)	29
9	3	Q_6	(ii), a_6	47
10	3	Q_7	(v), a_4	56
11	3	Q_8	(i)	49
12	3	Q_8	(iv)	49
13	3	Q_9	(ii), a_3	59
14	3	Q_{12}	(i)	53
15	5	Q_7	(ii), a_2	56
16	$p \neq 3$	Q_{10}	(iii), $a_2 + a_5$	66
17	*arbitrary*	Q_{11}	(iii), $a_3 + a_3$	58
18	*arbitrary*	Q_{13}	(v), a_7	70
19	*arbitrary*	Q_{14}	(iii), $a_3 + a_3$	74
20	*arbitrary*	Q_{15}	*ordinary* 4*-ple point*	*See Remark* 6.1 (ii) *below.*

The second column refers to its type (*see the table below*), *the third refers to the relative position between Q and l, the number being the one introduced in this section, and the singularities of Q in $Q \cap l$.*

	Irreducible components	*Singularities*
Q_1	*irreducible*	$2a_2$
Q_2	*irreducible*	$2a_2 + a_1$
Q_3	*irreducible*	$3a_2$
Q_4	*irreducible*	$a_3 + a_2$
Q_5	*irreducible*	a_5
Q_6	*irreducible*	a_6
Q_7	*irreducible*	$a_4 + a_2$
Q_8	*irreducible*	e_6
Q_9	*a cuspidal cubic and a line*	$a_3 + a_2 + a_1$
Q_{10}	*a cuspidal cubic and a line*	$a_5 + a_2$
Q_{11}	*two conics*	$2a_3$
Q_{12}	*two conics*	$a_3 + a_1$
Q_{13}	*two conics*	a_7
Q_{14}	*a conic and two lines*	$2a_3 + a_1$
Q_{15}	*four lines meeting at one point*	*an ordinary* 4*-ple point*

Remark 6.1. (i) The above table for the possible branch loci was obtained in [17]. Since it has never been published anywhere, we put it here.

(ii) If $Q = Q_{15}$, the minimal resolution of $D(S/\mathbb{P}^2)$ is not a rational elliptic surface. In fact, the minimal resolution of $D(S/\mathbb{P}^2)$ is a ruled surface over a curve of genus 1.

Proof. If $Q \neq Q_{15}$, one can make use of Theorem 6.1. We only show that the branch loci of D_{2p}-covers of type II corresponding to No. 20 and No. 70 belong to the above table, since the remaining cases are proved in the same manner.

$p = 3$, No. 20. Let $\varpi : S \to \mathbb{P}^2$ be a D_6-cover of type II such that $\mathrm{MW}(\mathcal{E}(\varpi))$ is No. 20. Since we choose a general point as x, the reduced fibers of $\mathcal{E}(\varpi)$ are of types $2I_3$, I_2 and the singular fiber F_x is of type I_2. Since $\mathrm{MW}(\mathcal{E}(\varpi))$ has no 2-torsion, Q is irreducible. Q has $2a_2$ singularities by Table 6.2 in [9]. Let l be the line component of Δ_ϖ. Let s^+ and s^- be the sections arising from l. Since both s^+ and s^- are 3-divisible and $s^{\pm}O = 0$, $\langle s^+, s^+\rangle = \langle s^-, s^-\rangle = 3/2$. On the other hand, for any $s \in \mathrm{MW}(\mathcal{E}(\varpi))$,

$$\langle s, s\rangle = 2 + 2sO - \frac{2}{3}a - \frac{1}{2}b,$$

where $a \in \{0, 1, 2\}$ and $b \in \{0, 1\}$. Hence we have $a = 0$, $b = 1$ for $s^{\pm}$. This implies l does not pass through the singularity of Q, i.e., l is either a bitangent or a hyperflex tangent. These are the cases 1 and 2, respectively.

$p =$ arbitrary, No. 70. Let $\varpi : S \to \mathbb{P}^2$ be a D_{2p}-cover of type II such that $\mathrm{MW}(\mathcal{E}(\varpi))$ is No. 70. In this case, Q is not 4 lines by Lemma 6.1. By choosing a general point on a non-linear component as x, we can assume that the configuration of fibers of $\mathcal{E}(\varpi)$ is I_8 I_2, $2I_1$ and the singular fiber F_x is of type I_2. As $\mathrm{MW}(\mathcal{E}(\varpi)) \cong \mathbb{Z}/4\mathbb{Z}$, it has only one 2-torsion. Hence Q consists of two components. By Table 6.2 in [9], Q has an a_7-singularity. Hence we infer that Q consists of two conics intersecting at one point y. Let l be the line component of Δ_{ϖ}. Let s^+ and s^- be the sections arising from l. Since $\mathrm{MW}(\mathcal{E}(\varpi)) \cong \mathbb{Z}/4\mathbb{Z}$ and $s^{\pm}$ are not 2-torsions, they must be 4-torsions. Hence $\langle s^+, s^+\rangle = \langle s^-, s^-\rangle = 0$. By the similar argument to the previous case, s^+ (resp. s^-) meets $\Theta_{v,2}$ (resp. $\Theta_{v,6}$) at the I_8 fiber. This implies that l is the tangent line at y. □

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