J. Math. Kyoto Univ. (JMKYAZ)
44-3 (2004), 679–684

On the Stiefel-Whitney classes of the adjoint representation of E_8

By

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Introduction

Exceptional Lie groups G_2, F_4 and $E_l\,(l = 6, 7, 8)$ have been studied by many topologists, where the subscript refers to the rank and we agree to consider 1-connected and compact ones tacitly. The cohomology of the classifying space of them is determined to a large extent. The mod 2 cohomology of BE_8, however, is left unknown. The ring structure of that of BE_7 is not determined yet.

It is known classically that an elementary abelian 2-subgroup, a 2-torus in other words, of the maximal rank is useful. This rank is called the 2-rank of the Lie group. Note that a maximal 2-torus does not necessarily give the 2-rank (see [1], [11]). On the other hand, the 3-connected covering $\widetilde{E}_l$ of E_l has been also utilized. In this paper we determine the image of the Stiefel-Whitney classes of the adjoint representaion of E_8 in $H^*(B\widetilde{E}_8; \boldsymbol{F}_2)$. In particular, we give some results on the image of $H^*(BE_8; \boldsymbol{F}_2)$ in it. We denote the mod 2 cohomology of X simply by $H^*(X)$ and by A^* the mod 2 Steenrod algebra. If S is a non-empty subset of an algebra, $\langle S \rangle$ denotes the subalgebra generated by S.

The author is very grateful to Professor Akira Kono for his helpful advices during the preparation of this paper.

1. Cohomology of the classifying spaces of 3-connected cover

First we recall here facts related to BE_l for later use. Let T^l be a maximal torus of E_l. Denote by q' a generator of $H^4(BE_l; \boldsymbol{Z})$ and by q'' the induced map defined on BT^l. Let $B\widetilde{E}_l$ and $B\widetilde{T}^l$ be the homotopy fibres of these maps, respectively. We have the natural maps $\lambda_l : BT^l \to BE_l$, $\widetilde{\lambda}_l : B\widetilde{T}^l \to B\widetilde{E}_l$, $\pi_l : B\widetilde{E}_l \to BE_l$, and $\widehat{\pi}_l : B\widetilde{T}^l \to BT^l$. Let us denote by φ_l and $\widetilde{\varphi}_l$ the natural maps $BE_{l-1} \to BE_l$ and $B\widetilde{E}_{l-1} \to B\widetilde{E}_l$, respectively. The following diagrams are commutative.

Received September 24, 2004

$$
\begin{array}{ccccc}
B\widetilde{T}^l & \xrightarrow{\widehat{\pi}_l} & BT^l & \xrightarrow{q''} & K(\boldsymbol{Z},4) \\
\downarrow{\scriptstyle \widetilde{\lambda}_l} & & \downarrow{\scriptstyle \lambda_l} & & \| \\
B\widetilde{E}_l & \xrightarrow{\pi_l} & BE_l & \xrightarrow{q'} & K(\boldsymbol{Z},4)
\end{array}
\qquad
\begin{array}{ccc}
B\widetilde{E}_{l-1} & \xrightarrow{\pi_{l-1}} & BE_{l-1} \\
\downarrow{\scriptstyle \widetilde{\varphi}_l} & & \downarrow{\scriptstyle \varphi_l} \\
B\widetilde{E}_l & \xrightarrow{\pi_l} & BE_l
\end{array}
$$

The mod 2 cohomology of these coverings is completely determined in [10] and [9]. For details, also refer to [8] or [18]. As is well known, $H^*(BT^l) \cong \boldsymbol{F}_2[t_1, \ldots, t_l]$, where $\deg t_i = 1$. Let c_i be the i-th elementary symmetric polynomial in t_i's, and also its image in $H^*(B\widetilde{T}^l)$. Define elements c_5', c_7', c_9' by $c_5 + c_4c_1$, $c_7 + c_6c_1$, $c_8c_1 + c_7{c_1}^2 + c_6{c_1}^3$, respectively. Furthermore, we define some elements of $H^*(B\widetilde{T}^l)$ as follows, where for generators γ_i we refer to the next theorem.

$$
\begin{aligned}
&I_8 = c_8 + c_6{c_1}^2 + {c_4}^2 + c_4{c_1}^4 + {c_1}^8, \\
&I_{12} = Sq^8 I_8 = c_8c_4 + {c_6}^2 + c_6c_4{c_1}^2 + {c_4}^2{c_1}^4 + c_4{c_1}^8, \\
&I_{14} = Sq^4 I_{12} = c_8c_6 + c_7'^2 + {c_6}^2{c_1}^2 + c_6c_4{c_1}^4 + c_6{c_1}^8, \\
&I_{15} = Sq^2 I_{14} = c_8c_7' + c_7'c_6{c_1}^2 + c_7'c_4{c_1}^4 + c_7'{c_1}^8, \\
&I_{17} = \gamma_{17} + \gamma_9 I_8 + \gamma_5 I_{12} + \gamma_3 I_{14} + c_7'c_6c_4, \\
&I_{18} = Sq^2 I_{17} = {\gamma_9}^2 + {\gamma_5}^2 I_8 + {\gamma_3}^2 I_{12} + \gamma_3 I_{15} + c_7'^2 c_4, \\
&I_{20} = Sq^4 I_{18} = {\gamma_5}^4 + \gamma_5 I_{15} + {\gamma_3}^4 I_8 + {\gamma_3}^2 I_{14} + I_{14}c_6 + I_{12}{c_4}^2 + c_7'^2 c_6, \\
&I_{24} = Sq^2 I_{20} = \gamma_9 I_{15} + {\gamma_5}^4 I_{14} + {\gamma_3}^4 I_{12} + {\gamma_3}^8 + I_{14}c_6c_4 + I_{12}{c_6}^2 + I_8{c_4}^4 + c_7'^2 c_6c_4.
\end{aligned}
$$

Ishitoya and Kono show the following result.

Theorem 1.1 ([9]). *The following facts about the* mod 2 *cohomology of $B\widetilde{T}^l$ and $B\widetilde{E}_l$ $(l = 6, 7, 8)$ hold.*

(i) $H^*(B\widetilde{T}^l) = \boldsymbol{F}_2[t_1, t_2, \ldots, t_l, \gamma_3, \gamma_5, \gamma_9, \gamma_{17}, v_{2^j+1}\ (j \geq 5)]/(c_2, c_3, c_5', c_9')$, *where* $\deg \gamma_i = 2i$ *and* $\deg v_i = i$.

(ii) $H^*(B\widetilde{E}_6) =$
$\boldsymbol{F}_2[y_{10}, y_{12}, y_{16}, y_{18}, y_{24}, y_{33}, y_{34}, y_{2^i+1}\ (i \geq 6)]$,
$H^*(B\widetilde{E}_7) =$
$\boldsymbol{F}_2[y_{12}, y_{16}, y_{20}, y_{24}, y_{28}, y_{33}, y_{34}, y_{36}, y_{2^i+1}\ (i \geq 6)]$,
$H^*(B\widetilde{E}_8) =$
$\boldsymbol{F}_2[y_{16}, y_{24}, y_{28}, y_{30}, y_{31}, y_{33}, y_{34}, y_{36}, y_{40}, y_{48}, y_{2^i+1}\ (i \geq 6)]$,
where $\deg y_i = i$.

(iii) *If both $H^*(B\widetilde{E}_l)$ and $H^*(B\widetilde{E}_{l-1})$ have the corresponding generator y_i, ${\widetilde{\varphi}_l}^*(y_i) = y_i$. Otherwise ${\widetilde{\varphi}_l}^*(y_i) = 0$ unless it is mentioned below.*

$$
\begin{aligned}
&{\widetilde{\varphi}_8}^*(y_{40}) = y_{28}y_{12} + y_{24}y_{16} + {y_{20}}^2 + y_{16}{y_{12}}^2, \\
&{\widetilde{\varphi}_8}^*(y_{48}) = y_{28}y_{20} + {y_{24}}^2 + y_{24}{y_{12}}^2 + {y_{16}}^3 + {y_{12}}^4, \\
&{\widetilde{\varphi}_7}^*(y_{20}) = {y_{10}}^2, \quad {\widetilde{\varphi}_7}^*(y_{36}) = y_{24}y_{12} + {y_{18}}^2 + y_{16}{y_{10}}^2.
\end{aligned}
$$

(iv) *For the case $l = 8$,*

$$\widetilde{\lambda}_8{}^*(y_i) = \begin{cases} I_{i/2}, & (i = 16, 24, 28, 30, 34, 36, 40, 48), \\ v_i, & (i = 2^j + 1,\ j \geq 5), \\ 0, & (i = 31). \end{cases}$$

(v) *For the case $l = 7$,*

$$\widetilde{\lambda}_7{}^*(y_i) = \begin{cases} I_{i/2}, & (i = 12, 16, 20, 24, 28, 34, 36), \\ v_i, & (i = 2^j + 1,\ j \geq 5), \end{cases}$$

where $I_6 = {\gamma_3}^2 + c_4{c_1}^2 + {c_1}^6$, $I_{10} = Sq^8 I_6 = {\gamma_5}^2 + c_6{c_1}^4 + {c_4}^2{c_1}^2 + {c_1}^{10}$.

(vi) *For the case $l = 6$,*

$$\widetilde{\lambda}_6{}^*(y_i) = \begin{cases} I_{i/2}, & (i = 10, 12, 16, 18, 24, 34), \\ v_i, & (i = 2^j + 1,\ j \geq 5), \end{cases}$$

where $I_5 = \gamma_5 + c_4 c_1 + {c_1}^5$, $I_9 = Sq^8 I_5 = \gamma_9 + {c_4}^2 c_1 + {c_1}^9$, *and I_6 denotes the image of the corresponding elements of $H^*(B\widetilde{T}^7)$.*

(vii) *The action of A^* on $H^*(B\widetilde{E}_l)$ satisfies the table below and $Sq^{2^j} y_{2^i+1} = 0$ $(j < i)$. These suffices to determine the action completely.*

	Sq^1	Sq^2	Sq^4	Sq^8	Sq^{16}	Sq^{32}	Sq^{2^i}
y_{16}	0	0	0	y_{24}	${y_{16}}^2$	0	
y_{24}	0	0	y_{28}	0	$y_{24}y_{16}$	0	
y_{28}	0	y_{30}	0	0	$y_{28}y_{16}$	0	
y_{30}	y_{31}	0	0	0	$y_{30}y_{16}$	0	
y_{31}	0	0	0	0	$y_{31}y_{16}$	0	
y_{33}	y_{34}	0	0	0	$y_{33}y_{16}$	y_{65}	
y_{34}	0	y_{36}	0	0	$y_{34}y_{16}$	$y_{36}y_{30} + {y_{33}}^2$	
y_{36}	0	0	y_{40}	0	$y_{36}y_{16}$	$y_{40}y_{28} + {y_{34}}^2$	
y_{40}	0	0	0	y_{48}	$y_{40}y_{16}$	$y_{48}y_{24} + {y_{36}}^2$	
y_{48}	0	0	0	0	$y_{40}y_{24} + y_{36}y_{28}$ $+ y_{34}y_{30} + y_{33}y_{31}$	$y_{48}{y_{16}}^2 + {y_{40}}^2 + y_{40}y_{24}y_{16}$ $+ y_{36}y_{28}y_{16} + y_{34}y_{30}y_{16} + y_{33}y_{31}y_{16}$	
y_{12}	0	0	y_{16}	y_{20}	0	0	
y_{20}	0	0	${y_{12}}^2$	y_{28}	$y_{36} + y_{24}y_{12} + y_{20}y_{16}$	0	
y_{10}	0	y_{12}	0	y_{18}	0	0	
y_{18}	0	${y_{10}}^2$	0	0	$y_{34} + y_{24}y_{10} + y_{18}y_{16}$	0	
y_{2^i+1}	0	0	0	0	0	0 $(i \geq 6)$	$y_{2^{i+1}+1}$

Note that $(\,\widetilde{\varphi}_l{}^*(y_i)\,)_i$ forms a regular sequence for each l if we exclude $\widetilde{\varphi}_l{}^*(y_i)$ which is null. Thus $\operatorname{Ker} \widetilde{\varphi}_7{}^* = (y_{28})$ and $\operatorname{Ker} \widetilde{\varphi}_8{}^* = (y_{30}, y_{31})$. Also note that $(\,\widetilde{\lambda}_l{}^*(y_i)\,)_i$ does, and if $\widetilde{\lambda}_l{}^*(y_i)$ is non-zero and contained in $\langle t_1, \ldots, t_l \rangle$, then $i = 16, 24, 28, 30$ for $l = 8$, $i = 16, 24, 28$ for $l = 7$, and $i = 16, 24$ for $l = 6$.

Corollary 1.1. (i) $\operatorname{Ker} \widetilde{\varphi}_7{}^* = (y_{28})$, *and* $\operatorname{Ker} \widetilde{\varphi}_8{}^* = (y_{30}, y_{31})$.
(ii) $\operatorname{Ker} \widetilde{\lambda}_6{}^* = 0$, $\operatorname{Ker} \widetilde{\lambda}_7{}^* = 0$, *and* $\operatorname{Ker} \widetilde{\lambda}_8{}^* = (y_{31})$.
(iii) $\operatorname{Im} \pi_6{}^* \subset \boldsymbol{F}_2[y_{16}, y_{24}]$, $\operatorname{Im} \pi_7{}^* \subset \boldsymbol{F}_2[y_{16}, y_{24}, y_{28}]$, *and* $\operatorname{Im} \pi_8{}^* \subset \boldsymbol{F}_2[y_{16}, y_{24}, y_{28}, y_{30}] \oplus (y_{31})$.
(iv) *In particular,* $\operatorname{Ker} \widetilde{\varphi}_8{}^* \cap \operatorname{Im} \pi_8{}^* \subset y_{30} \cdot \boldsymbol{F}_2[y_{16}, y_{24}, y_{28}] \oplus (y_{31})$.

Proof. The equalities are immediate. For the third inclusion notice that $\widetilde{\lambda}_8{}^*(\operatorname{Im}\pi_8{}^*) \subset \operatorname{Im}\widehat{\pi}_8{}^* \cap \operatorname{Im}\widetilde{\lambda}_8{}^* = \langle t_1, \ldots, t_8\rangle \cap \operatorname{Im}\widetilde{\lambda}_8{}^*$. Thus $\operatorname{Im}\pi_8{}^*$ is contained in $\langle y_{16}, y_{24}, y_{28}, y_{30}\rangle \oplus \operatorname{Ker}\widetilde{\lambda}_8{}^*$. Other inclusions are proved similarly. $\square$

2. Stiefel-Whitney class of the adjoint representation of E_8

Let Ad_{E_l} be the adjoint representation of E_l ($l = 6, 7, 8$). It is known that Ad_{E_8} satisfies $Ad_{E_8}|_{E_7} = Ad_{E_7} \oplus \mu \oplus$ (3-dimensional trivial representation), where $\mu : E_7 \to U(56) \to O(112)$ is the realization of the 56-dimensional complex representation. We refer, for example, to [1, Case 2 in page 52].

As for the Stiefel-Whitney class of Ad_{E_l}, the following facts are known. Firstly, $H^*(BE_6)$ is generated by x_4 and $w_{32}(\lambda)$ as an A^*-algebra, where x_4 is a generater of $H^4(BE_6)$ and λ is a representation of E_6 whose degree is 54. This fact is shown in [12, Theorem 6.21 and Remark following it].

Secondly, $H^*(BE_7)$ is generated by x_4 and $w_{64}(Ad_{E_7})$ as an A^*-algebra, and also by x_4 and $w_{64}(\mu)$. For these we refer to [14, Corollary 4.6, Proposition 6.1 and Corollary 6.9], and to [13, Proposition 2.11, Theorem 2.12 and Corollary 3.7].

Let A and B be the A^*-subalgebras of $H^*(BE_7)$ generated by x_4 and $w_{64}(\mu)$, respectively. The image of A in $H^*(B\widetilde{E}_7)$ is trivial, and also in $H^*(B\widetilde{T}^7)$. Consequently, $\pi_7{}^*$ assigns 0 to the Stiefel-Whitney classes $w_i(Ad_{E_7})$ and $w_i(\mu)$, if $i \le 63$ or $65 \le i \le 95$.

Lemma 2.1. *$\pi_6{}^* w_{32}(\lambda) = y_{16}{}^2$ and $\pi_7{}^* w_{64}(Ad_{E_7}) = \pi_7{}^* w_{64}(\mu) = y_{16}{}^4$. In lower degrees, it holds that $\pi_6{}^* w_i(\lambda) = 0$ for $i < 32$ and $\pi_7{}^* w_i(Ad_{E_7}) = \pi_7{}^* w_i(\mu) = 0$ for $i < 64$.*

Proof. It suffices to prove the first half. Firstly, we can assume that $\pi_6{}^* w_{32}(\lambda) = \alpha y_{16}{}^2$, where α is a scalar, by Corollary 1.1. We notice that $H^*(BT^6)$ is a finite $H^*(BE_6)$-module. In particular, $\widehat{\pi}_6{}^*(H^*(BT^6))$ is also finite. Suppose that $\alpha = 0$. Then the image $\pi_6{}^*(H^*(BE_6))$ is trivial, and so in $H^*(B\widetilde{T}^6)$. This contradicts the fact above.

Secondly, we verify the case of $H^*(B\widetilde{E}_7)$. $\pi_7{}^* w_{64}(\mu)$ is of the form $\alpha y_{16}{}^4 + \beta y_{24}{}^2 y_{16}$, where $\alpha, \beta \in \boldsymbol{F}_2$. As a result, $Sq^8(\pi_7{}^* w_{64}(\mu)) = \beta y_{28}{}^2 y_{16} + \beta y_{24}{}^3$, which is null as we indicated above. Therefore $\beta = 0$. If $\alpha = 0$, we can show a contradiction similarly to the case of $H^*(B\widetilde{E}_6)$. The assertion on $\pi_7{}^* w_{64}(Ad_{E_7})$ is proved in the same manner. $\square$

Proposition 2.1. *It holds that $\operatorname{Im}\pi_6{}^* = \boldsymbol{F}_2[y_{16}{}^2, y_{24}{}^2]$ and $\operatorname{Im}\pi_7{}^* = \boldsymbol{F}_2[y_{16}{}^4, y_{24}{}^4, y_{28}{}^4]$. In particular, $\operatorname{Im}\pi_8{}^* \subset \boldsymbol{F}_2[y_{16}{}^4, y_{24}{}^4, y_{28}{}^4] \oplus y_{30} \cdot \boldsymbol{F}_2[y_{16}, y_{24}, y_{28}] \oplus (y_{31})$.*

Proof. The first two are clear from Corollary 1.1 and Lemma 2.1. Since $\widetilde{\varphi}_8{}^*(\operatorname{Im}\pi_8{}^*) = \pi_7{}^*(\operatorname{Im}\varphi_8{}^*) \subset \operatorname{Im}\pi_7{}^*$, $\operatorname{Im}\pi_8{}^*$ is contained in $\boldsymbol{F}_2[y_{16}{}^4, y_{24}{}^4, y_{28}{}^4] \oplus \operatorname{Ker}\widetilde{\varphi}_8{}^*$. Thus the last assertion follows from Corollary 1.1. $\square$

Let i be a non-negative integer less than 7 for a while. Note that $\widetilde{\varphi}_8{}^*(\pi_8{}^*(w_{2^i}(Ad_{E_8}))) = 0$ because of Proposition 2.1 and the decomposition of $Ad_{E_8}|_{E_7}$.

Thus ${\pi_8}^*(w_{2^i}(Ad_{E_8}))$ is lying in $\operatorname{Ker}{\widetilde{\varphi}_8}^*$. Corollary 1.1 implies ${\pi_8}^*(w_{2^i}(Ad_{E_8})) = 0$ for $i \leq 5$ and ${\pi_8}^*(w_{64}(Ad_{E_8}))$ is expressed in the form $\alpha y_{31}y_{33}$. Therefore, applying Sq^1, we deduce that $\alpha = 0$ since ${\pi_8}^*(w_{2^i}(Ad_{E_8})) = 0$ for $i \leq 5$.

Lemma 2.2. ${\pi_8}^* w_{2^i}(Ad_{E_8}) = 0$ *for* $i < 7$. *Therefore* ${\pi_8}^* w_i(Ad_{E_8}) = 0$ *for* $i < 128$.

Now we begin to show ${\widetilde{\varphi}_8}^*({\pi_8}^*(w_{128}(Ad_{E_8}))) = {y_{16}}^8$. In this time we need an additional fact. The root space decomposition of E_7 shows $Ad_{E_7}|_{T^7} = \xi \oplus$ (7-dimensional trivial representation), where ξ is a representation of T^7 of degree 126. Thus ${\lambda_7}^*(w_i(Ad_{E_7})) = 0$ for $i \geq 127$. In particular, ${\widetilde{\lambda}_7}^*{\pi_7}^*(w_{128}(Ad_{E_7})) = 0$. Corollary 1.1 then implies ${\pi_7}^*(w_{128}(Ad_{E_7})) = 0$. Since ${\widetilde{\varphi}_8}^*({\pi_8}^*(w_{128}(Ad_{E_8}))) = {\pi_7}^*(w_{128}(Ad_{E_7} \oplus \mu))$, we obtain ${\widetilde{\varphi}_8}^*({\pi_8}^*(w_{128}(Ad_{E_8}))) = {y_{16}}^8$.

Theorem 2.1. ${\pi_8}^* w_{128}(Ad_{E_8}) = {y_{16}}^8$.

Proof. We can assume that ${\pi_8}^* w_{128}(Ad_{E_8}) = {y_{16}}^8 + \alpha {y_{30}}^2 y_{28} y_{24} y_{16} + {y_{31}}^2(\beta {y_{33}}^2 + \gamma y_{30} y_{36} + \delta y_{34} {y_{16}}^2) + y_{31} y_{33}(\varepsilon y_{30} y_{34} + p) + \zeta y_{31} y_{65} {y_{16}}^2$, where $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in \boldsymbol{F}_2$ and $p \in \langle y_{16}, y_{24}, y_{28}, y_{36}, y_{40}, y_{48} \rangle$. Since $Sq^1 {\pi_8}^* w_{128}(Ad_{E_8}) = 0$ and $Sq^1 {\pi_8}^* w_{128}(Ad_{E_8}) = \gamma {y_{31}}^3 y_{36} + \varepsilon y_{30} y_{31} {y_{34}}^2 + \varepsilon {y_{31}}^2 y_{33} y_{34} + y_{31} y_{34} p$, $\gamma = \varepsilon = 0$ and $p = 0$. We apply Sq^2 and then we conclude $\alpha = \beta = \delta = 0$. Lastly, applying Sq^{16}, we obtain $\zeta = 0$. $\square$

The following is an easy consequence of Wu formulae.

Corollary 2.1. $\boldsymbol{F}_2[{y_{16}}^8, {y_{24}}^8, {y_{28}}^8, {y_{30}}^8, {y_{31}}^8] \subset \operatorname{Im}{\pi_8}^* \subset \boldsymbol{F}_2[{y_{16}}^8, {y_{24}}^8, {y_{28}}^8, {y_{30}}^8, {y_{31}}^8] + Q$, *where* $Q \subset y_{30} \cdot \boldsymbol{F}_2[y_{16}, y_{24}, y_{28}] \oplus (y_{31})$.

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