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A note on the Stiefel-Whitney classses of representations of exceptional Lie groups

Dedicated to the memory of Professor Masahiro Sugawara

By

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Let E_l be the compact, 1-connected simple exceptional Lie group of rank l $(l = 6, 7, 8)$. Consider the following real representations:

$$\begin{aligned}
\rho_6 &: E_6 \hookrightarrow U(27) \to SO(54),\\
\rho_7 &: E_7 \hookrightarrow U(56) \to SO(112),\\
\rho_7' &: \qquad E_7 \to SO(133)\\
&\qquad\quad \text{and}\\
\rho_8 &: \qquad E_8 \to SO(248),
\end{aligned}$$

where ρ_7' and ρ_8 are the adjoint representations (see Adams [1]). The purpose of this note is to show the following without using the structure of $H^*(BE_l)$ where $H^*(\ )$ denotes the mod 2 cohomology ring:

Theorem 1. *$w_{2^{l-1}}(\rho_l)$ and $w_{64}(\rho_7')$ are not decomposable in $H^*(BE_l)$.*

Let $p = p_n : Spin(n) \to SO(n)$ be the universal covering and $C_n = \ker p_n$. The subgroup of $SO(n)$ which consists of diagonal matrices is denoted by $V(n)$ and $\tilde{V}(n) = p_n^{-1}(V(n))$. Put $p' = p_n' = p_n|_{\tilde{V}(n)}$, $d(6) = 10, d(7) = 11$ and $d(8) = 13$. E_l contains $Spin(d(l))$ as a closed subgroup. Denote by $\Delta(l)$ the unique irreducible representation of $\tilde{V}(d(l))$ on which $C_{d(l)}$ acts non-trivially. Note that $\mathbf{dim}\,\Delta(l) = 2^{l-1}$ and $\Delta(l)|_{C_{d(l)}}$ is isomorphic to $2^{l-1}\epsilon$ where ϵ is the one dimensional non-trivial real representation of $C_{d(l)} \cong \mathbb{Z}/2$ (see Quillen [3]). Note that (the center of E_l) $\cap\, C_{d(l)} = \{0\}$. Therefore $\rho_l|_{C_{d(l)}}$ and $\rho_7'|_{C_{d(7)}}$ are non trivial. Since $\mathbf{dim}\,\rho_l < 2^l$, we have the following:

Lemma 1. *$\rho_l|_{\tilde{V}(d(l))} = \Delta(l) + p'^*\mu_l$ where μ_l is a representation of $V(d(l))$.*

On the other hand, since we may assume $C_{d(7)}$ contained in a maximal torus, $\rho_7'|_{C_{d(7)}}$ contains at least 7-dimensional trivial representation. Therefore we have

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Lemma 2. *$\rho'_7|_{\tilde{V}(11)} = \Delta(7) \oplus p'^*_7\mu'_7$ where μ'_7 is a representation of $V(d(7))$.*

Denote the natural maps $BC_{d(l)} \subset B\tilde{V}(d(l))$, $B\tilde{V}(d(l)) \to BSpin(d(l))$ and $BSpin(d(l)) \to BE_l$ by i_l, j_l and k_l. Put $\xi_l = k_l \circ j_l \circ i_l$. Note that in $H^*(BC_{d(l)}) = \mathbb{Z}/2[t]$ where $\deg t = 1$, $\mathbf{Im}\, i^*_l = \mathbb{Z}/2[t^{2^{l-1}}]$ (see Quillen [3]). Using Lemma 1 and Lemma 2, we have

$$w(\xi^*_l \rho_l) = 1 + t^{2^{l-1}} \tag{1.1}$$

and

$$w(\xi^*_l \rho'_7) = 1 + t^{64}, \tag{1.2}$$

where $w(\ )$ denotes the total Stiefel-Whitney class. We have $\xi^*_l w_{2^{l-1}}(\rho_l)$ and $\xi^*_l w_{64}(\rho'_7)$ are not decomposable in $\mathbf{Im}\, i^*_l$ and therefore we have Theorem 1.

Remark 2. The fact that $w_{128}(\rho_8)$ is not decomposable in $H^*(BE_8)$ is also obtained by Mimura and Nishimoto using $\varphi^*(w(\rho_8))$ where $\varphi : BSpin(16) \to BE_8$ (Talk in Naha 2004).

Consider the following commutative diagram

$$\begin{array}{ccc} H^4(BE_l;\mathbb{Z}) & \xrightarrow{H^4(\xi_l;\mathbb{Z})} & H^4(BC_{d(l)};\mathbb{Z}) \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\rho'} \\ H^4(BE_l) & \xrightarrow{\xi^*_l} & H^4(BC_{d(l)}), \end{array}$$

where ρ and ρ' are mod 2 reductions. Note that ρ is epic and ρ' is isomorphic. Since $i^*_l = 0$, $\xi^*_l = i^*_l \circ j^*_l \circ k^*_l = 0$. Therefore we have

$$H^4(\xi_l;\mathbb{Z}) = 0.$$

Therefore there exists $\tilde{\xi}_l : BC_{d(l)} \to \widetilde{BE}_l$ such that $\pi_l \circ \tilde{\xi}_l \simeq \xi_l$ where $\pi_l : \widetilde{BE}_l \to BE_l$ is the 4-connected cover. In Ohsita [2] $\pi^*_l(w(\rho_l))$ and $\pi^*_7(w(\rho'_7))$ are determined. To determine $\pi^*_l(w(\rho_l))$ $l = 6, 7$ and $\pi^*_7(w(\rho'_7))$ the structures of $H^*(BE_6)$ and $H^*(BE_7)$ are used. In this section we determine $\pi^*_6 w(\rho_6)$ and $\pi^*_7 w(\rho_7)$ without using $H^*(BE_6)$ and $H^*(BE_7)$. For symbols and notation see Ohsita [2]. Since $\xi^*_l \neq 0$, $\tilde{\xi}^*_l \neq 0$. Note that $\mathbf{Im}\, \pi^*_6 \subset \mathbb{Z}/2[y_{16}, y_{24}]$ and $\mathbf{Im}\, \pi^*_7 \subset \mathbb{Z}/2[y_{16}, y_{24}, y_{28}]$ where $|y_j| = j, Sq^8 y_{16} = y_{24}$ and $Sq^4 y_{24} = y_{28}$. Therefore $\tilde{\xi}^*_l y_{16} = t^{16}$. By (1.1), $\pi^*_6 w_j(\rho_6) = 0 \quad 1 \leq j \leq 31$ and $\pi^*_6 w_{32}(\rho_6) = y^2_{16}$. By (1.1), $\pi^*_7 w_j(\rho_7) = 0 \quad 1 \leq j \leq 63$ and $\pi^*_7 w_{64}(\rho_7) = y^4_{16} + \beta y_{16} y^2_{24}$ for some $\beta \in \mathbb{Z}/2$. Applying Sq^8 we have $0 = \beta(y^3_{24} + y_{16} y^2_{28})$ and therefore $\beta = 0$. Using (1.2), we can prove $\pi^*_7 w(\rho'_7) = y^4_{16}$ similarly. Note that $\rho'_7|_{T^7}$ contains 7-dimensional trivial representation and therefore $w_{128}(\rho'_7|_{T^7}) = 0$. Using this fact we have $\pi^*_7 w_{128}(\rho'_7) = 0$.

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References

[1] J. F. Adams, Lectures on exceptional Lie groups, Chicago Univ. Press, 1996.

[2] A. Ohsita, *On the Stiefel-Whitney class of the adjoint representation of* E_8, J. Math. Kyoto Univ. **44** (2004), 679–684.

[3] D. Quillen, *The mod* 2 *cohomology ring of extra-special* 2*-groups and the spinor groups*, Math. Ann. **194** (1971), 197–212.