J. Math. Kyoto Univ. (JMKYAZ)
45-2 (2005), 257–263

The ideal boundary of the Sol group

By

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Abstract

We obtain equations of geodesic lines in the Lie group **Sol** and prove that the ideal boundary of the **Sol** is a set $\mathcal{R} = \{(x, y, z)|\ xy = 0,\ and\ x^2 + y^2 + z^2 = 1\}$ with a degenerate Tits metric, i.e., the distance between different points equals ∞.

1. Introduction

It is well known that there are 8 three dimensional model geometries [Th]. Each of the 8 three-dimensional model geometries is isometric to a Lie group with a left invariant metric. The **Sol**, one of the eight model geometries, is a Lie group of dimension 3 whose underlying space is $\mathbb{R}^3$. Let (x, y, z) denote a coordinate of $\mathbb{R}^3$. Then, the multiplication rule of the Lie group, **Sol**, is given by

$$(x, y, z) \cdot (x', y', z') = (x + e^{-z}x', y + e^{z}y', z + z'). \tag{1.1}$$

The ideal boundary was introduced to compactify complete Riemannian manifolds or more generally complete locally compact metric spaces (refer to [G1]). Since then, the ideal boundary has become an important part in studying the intrinsic geometry of complete Riemannian manifolds. It is particularly useful for a Hadamard manifold, which is a connected, simply connected complete Riemannian manifold of nonpositive curvature [EO]. The characterization of the ideal boundary of a manifold is a critical issue in the field of the Riemannian geometry. Recently, Valery Marenich [V] showed that the ideal boundary of **Nil** is (S^1, ω) with a natural CR-structure and corresponding Carnot-Caratheodory metric ω [G2], where **Nil** is one of the 8 three dimensional model geometries. Now, the **Sol** group is the only model geometry whose ideal boundary is unknown to us; therefore, in this paper, we study the ideal boundary of the **Sol**. The xz-plane and the yz-plane contained in the **Sol** are isometric to $\mathbb{H}^2$. Moreover, we show that there are not geodesic rays which are not contained in the xz-plane or the yz-plane. Then the ideal boundary of the **Sol** can be determined and characterized completely as in the main theorem.

Received December 19, 2003
Revised February 7, 2005

Theorem 1.1. *The ideal boundary of the* **Sol** *is a* $(\mathcal{R}, d_\infty)$ *with a degenerate Tits metric, i.e., the distance between different points equals* ∞*, where* $\mathcal{R} = \{(a,b,c)|\ ab = 0$ *and* $a^2 + b^2 + c^2 = 1\}$.

2. Left invariant metric, Levi-Civita connection and curvature tensor of the Sol

The element zero, $\mathbf{0} = (0,0,0)$, is the unit of the **Sol** group structure and the vector fields

$$X_1 = (e^{-z}, 0, 0),\ X_2 = (0, e^z, 0),\ X_3 = (0,0,1). \tag{2.1}$$

are then left-invariant fields. We define a left-invariant metric of the **Sol** by taking X_1, X_2, X_3 as the orthonormal frame. The left invariant metric on the **Sol** is given by the formula $ds^2 = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2$. By direct computation, we derive the following lemmas.

Lemma 2.1. *For the covariant derivatives of the Levi-Civita connection of the left-invariant metric, defined above, the following holds*:

$$\nabla = \begin{pmatrix} -X_3 & 0 & X_1 \\ 0 & X_3 & -X_2 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.2}$$

where the (i,j)*-element in the table above equals* $\nabla_{X_i} X_j$.

Lemma 2.2. *The curvature tensor of the* **Sol** *satisfies the following*:

$$R(X_1,X_2)X_1 = X_2,\ R(X_1,X_2)X_2 = -X_1,\ R(X_1,X_2)X_3 = 0, \tag{2.3}$$

$$R(X_2,X_3)X_1 = 0,\quad R(X_2,X_3)X_2 = -X_3,\ R(X_2,X_3)X_3 = X_2, \tag{2.4}$$

$$R(X_3,X_1)X_1 = X_3,\ R(X_3,X_1)X_2 = 0,\quad R(X_3,X_1)X_3 = -X_1. \tag{2.5}$$

From lemma 2.2, we obtain the sectional curvatures of the **Sol** as follows.

$$K(X_1,X_2) = 1,\ K(X_2,X_3) = -1,\ K(X_3,X_1) = -1. \tag{2.6}$$

This lemma immediately tells us that the **Sol** is not a Hadamard manifold.

3. Geodesic lines in the Sol

First we determine equations of geodesics issuing from $\mathbf{0} = (0,0,0)$. The geodesic equations are

$$\frac{d^2x_k}{dt^2} + \sum_{i,j} \Gamma^k_{ij} \frac{dx_i}{dt}\frac{dx_j}{dt} = 0 \quad (k = 1,2,3). \tag{3.1}$$

By direct computation, we find that $\Gamma^3_{11} = -e^{2z}$, $\Gamma^1_{13} = \Gamma^1_{31} = 1$, $\Gamma^3_{22} = e^{-2z}$, $\Gamma^2_{23} = \Gamma^2_{32} = -1$ and the other Christoffel symbols are zeros. Then the geodesic equations are

$$\ddot{x} + 2\dot{x}\dot{z} = 0, \tag{3.2}$$

$$\ddot{y} - 2\dot{y}\dot{z} = 0, \tag{3.3}$$

$$\ddot{z} - e^{2z}(\dot{x})^2 + e^{-2z}(\dot{y})^2 = 0. \tag{3.4}$$

Let $(x(0), y(0), z(0)) = (0,0,0)$, $(\dot{x}(0), \dot{y}(0), \dot{z}(0)) = (a,b,c)$ and $a^2+b^2+c^2 = 1$. From differential equations (3.1) and (3.2), we know that

$$\dot{x} = ae^{-2z}, \ \dot{y} = be^{2z}. \tag{3.5}$$

Since a geodesic is an arc length parameterized curve, the length of the vector $(\dot{x}, \dot{y}, \dot{z})$ at (x, y, z) is 1. By the left invariant metric $ds^2 = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2$, we have

$$a^2e^{-2z} + b^2e^{2z} + \dot{z}^2 = 1. \tag{3.6}$$

If one let $u = e^{2z}$, after some easy computation one could find that

$$\dot{x} = \frac{a}{u}, \tag{3.7}$$

$$\dot{y} = bu, \tag{3.8}$$

$$\dot{u}^2 = 4(u^2 - a^2u - b^2u^3). \tag{3.9}$$

In the end, we know that the geodesic lines are determined by the function u. Notice that u is an elliptic function in some values of a and b. Let's recall the elliptic function.

Let L be a lattice in the complex plane, by which we mean the set of all integral linear combinations of two given complex numbers ω_1 and ω_2, where ω_1 and ω_2 do not lie on the same line through the origin.

Definition 3.1. For a given lattice L, a meromorphic function f on $\mathbb{C}$ is said to be an *elliptic function* relative to L if $f(z + l) = f(z)$ for all $l \in L$.

Let $\wp(z; \omega_1, \omega_2)$ be the Weierstrass $\wp-$function. It is known that

$$\dot{\wp}(z)^2 = f(\wp(z)), \ f(x) = 4x^3 - g_2x - g_3 \in \mathbb{C}[x]. \tag{3.10}$$

and the function f has three distinct roots. If we put $v = -b^2u + \frac{1}{3}$, then we obtain

$$\dot{v}^2 = 4v^3 - h_2v - h_3. \tag{3.11}$$

from (3.9), where $h_2 = \frac{4}{3}(1 - 3a^2b^2)$ and $h_3 = \frac{4}{27}(9a^2b^2 - 2)$. If we assume that a and b are not zeros and that $1 - 4a^2b^2 > 0$, then the cubic polynomial $4x^3 - h_2x - h_3$ has three distinct real roots. Thus, v is a Weierstrass $\wp-$function and ω_2 corresponding to v is real (see p.28 in [KO]). This means that v is a periodic function on the real line, as is u, because the linear transformation preserves the property of periodicity. We can conclude that z is a periodic function and it will be very important property in determining whether a geodesic is a ray or not.

4. Rays in the Sol

We can not calculate a geodesic line explicitly, so we have difficulty in determining whether a geodesic line is a ray or not and, therefore, have to find useful properties of geodesic lines in the **Sol** group to solve this problem.

Lemma 4.1. *Two geodesics issuing from* $\mathbf{0}$ *with initial vectors* (a, b, c), $(a, b, -c)$, *respectively, for* $abc \neq 0$ *and* $1 - 4a^2b^2 > 0$, *meet at some point.*

Proof. Let's assume $\dot{z}(0) = c > 0$ and $(x(t), y(t), z(t))$, $(x_1(t), y_1(t), z_1(t))$ are geodesics issuing from $\mathbf{0}$ with initial vectors (a, b, c), $(a, b, -c)$, respectively.

$$t_0 = \min\{t | z(t) = 0 \text{ for } t \in (0, T]\} \text{ where } T \text{ is the period of the function } z.$$

Then, we claim $\dot{z}(t_0) = -c$. First, note that $z(T) = 0$ guarantees the existence of t_0, and $\dot{z}(t_0)$ has the value either c or $-c$ from the differential equation of geodesics. If the claim does not hold, we may assume $\dot{z}(t_0) = c$. By the choice of t_0, we have $z(t) \geq 0$ for all $t \in [0, t_0]$. Furthermore, both $\dot{z}(t_0) = c > 0$ and $z(t_0) = 0$ indicate that the function z has a local minimum at t_0. This implies $\dot{z}(t_0) = 0$, contradicting that $\dot{z}(t_0)$ has the value either c or $-c$. Thus, the above claim holds.

Now, we will prove that two geodesics meet at $t = T$. Two functions $z(t + t_0)$ and $z_1(t)$ satisfy the same first-order differential equation and have the same initial values. Therefore,

$$z_1(t) = z(t + t_0). \tag{4.1}$$

Clearly $z(T) = z(T + t_0) = z_1(T) = 0$.

$$x_1(T) = \int_0^T a e^{-2z_1(t)} dt = \int_0^T a e^{-2z(t+t_0)} dt \tag{4.2}$$

$$= \int_{t_0}^{t_0+T} a e^{-2z(s)} ds = \int_0^T a e^{-2z(s)} ds = x(T). \tag{4.3}$$

Similarly, one can obtain $y(T) = y_1(T)$. □

Corollary 4.1. *The geodesic issuing from* $\mathbf{0}$, *with an initial vector for* $abc \neq 0$ *and* $1 - 4a^2b^2 > 0$, *is not a ray.*

Proof. Let $\gamma(t)$ be a geodesic satisfying conditions in the statement. Then, a geodesic different from $\gamma(t)$ exists which connects $\mathbf{0}$ and $\gamma(T)$ with a length equal to $\gamma([0, T])$ by the lemma 4.1. Then, $\gamma(t)$ is not a ray (see corollary 2.111 in [GHL]). □

Lemma 4.2. *The geodesic issuing from* $\mathbf{0}$ *with an initial vector* (a, b, c) *for* $ab \neq 0$, $c = 0$ *and* $1 - 4a^2b^2 > 0$, *is not a ray.*

Proof. Let $\gamma(t) = (x(t), y(t), z(t))$ be a geodesic issuing from $\mathbf{0}$ with an initial vector $(a, b, 0)$. Choose some $t_0 > 0$, at which the value of $\dot{z}$ is nonzero. Since the length of $\dot{\gamma}(t_0)$ is 1 in the **Sol**, we have

$$a^2 e^{-2z(t_0)} + b^2 e^{2z(t_0)} + \dot{z}(t_0)^2 = 1. \tag{4.4}$$

Then, we regard $(ae^{-z(t_0)}, be^{z(t_0)}, \dot{z}(t_0))$ as an unit vector at origin. Let $\gamma_1(t) = (x_1(t), y_1(t), z_1(t))$ be the geodesic issuing from $\mathbf{0}$ with this velocity vector. One can easily check that the left multiplication $L_{\gamma(t_0)}$ in the Lie group transforms $\gamma_1(0), \dot{\gamma_1}(0)$ to $\gamma(t_0), \dot{\gamma}(t_0)$, respectively. These two curves $\gamma(t + t_0)$ and $L_{\gamma(t_0)}(\gamma_1(t))$ are geodesics sharing a common starting point and velocity vector; thus we conclude that

$$\gamma(t + t_0) = L_{\gamma(t_0)}(\gamma_1(t)). \tag{4.5}$$

We know that the geodesic $\gamma_1(t)$ is not a ray according to the previous lemma. Therefore, $t_1 > 0$ exists such that $\gamma_1(t)$ is not a length-minimizing curve connecting $\mathbf{0}$ and $\gamma_1(t_1)$. Let $\alpha(t)$ be a length-minimizing curve connecting $\mathbf{0}$ and $\gamma_1(t_1)$. Since the left multiplication is an isometry, $L_{\gamma(t_0)}(\alpha(t))$ is a length-minimizing curve connecting $\gamma(t_0)$ and $\gamma(t_1 + t_0)$ different from γ. Therefore, γ is not a ray. □

One can easily notice that the xz-plane and yz-plane are isometric to $\mathbb{H}^2$, and thus geodesics for $ab = 0$ are rays.

Lemma 4.3. *The geodesic issuing from* $\mathbf{0}$ *with an initial vector* (a, b, c) *for* $1 - 4a^2b^2 = 0$ *is not a ray.*

Proof. The inequality $a^2+b^2 \leq 1$ means that the solution for $1-4a^2b^2 = 0$ is only $a^2 = b^2 = \frac{1}{2}$. Let's assume $a = b = \frac{1}{\sqrt{2}}$. Then, the geodesic corresponding to the vector $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$ can be easily derived from the geodesic equations as $\gamma(t) = \frac{1}{\sqrt{2}}(t, t, 0)$. Let's calculate the Jacobi field along γ with $J(0) = 0$ and $\dot{J}(0) = (1, -1, 0)$ and set $J(t) = f_1(t)X_1 + f_2(t)X_2 + f_3(t)X_3$. The Jacobi equation is

$$\ddot{J} + R\,(\dot{\gamma}, J)\dot{\gamma} = \ddot{J} + \frac{1}{2}R(X_1 + X_2, f_1X_1 + f_2X_2 + f_3X_3)(X_1 + X_2) \tag{4.6}$$

$$= \ddot{J} + \frac{1}{2}\{f_2(X_2 - X_1) + f_3(-X_3) + f_1(-X_2 + X_1) + f_3(-X_3)\} \tag{4.7}$$

$$= \left(\ddot{f_1} + \frac{f_1 - f_2}{2}\right) X_1 + \left(\ddot{f_2} + \frac{f_2 - f_1}{2}\right) X_2 + (\ddot{f_3} - f_3)X_3 = 0. \tag{4.8}$$

In sum, the components of the Jacobi equation satisfy

$$2\ddot{f_1} + f_1 - f_2 = 0, \tag{4.9}$$

$$2\ddot{f_2} + f_2 - f_1 = 0, \tag{4.10}$$

$$\ddot{f_3} - f_3 = 0. \tag{4.11}$$

Through this simple calculation, we have $\ddot{f}_1(t)+f_1(t)=0$, $\ddot{f}_3(t)-f_3(t)=0$ and $f_1(t)+f_2(t)=0$. One can easily find that $f_1(t)=\sin t$, $f_3(t)=0$ are solutions of each equation with each initial value. Therefore, we have the following Jacobi field

$$J(t)=(\sin t, -\sin t, 0). \tag{4.12}$$

Thus γ is not a ray, because J has a conjugate point at $t=\pi$. Since $f(x,y,z)=(\pm x, \pm y, z)$ is an isometry, the other four geodesics are not rays. $\square$

Theorem 4.1. *The set $\mathcal{R}$ of directions of all rays issuing from* **0** *in the* **Sol** *is*

$$\mathcal{R}=\{(a,b,c)|\ ab=0,\ a^2+b^2+c^2=1\}. \tag{4.13}$$

5. Ideal boundary of the Sol

Recall the definition of the ideal boundary $(M(\infty), d_\infty)$ of an open manifold (M, d_M). For two rays $l_1(t)$ and $l_2(t)$ issuing from some fixed point of M denote

$$\widetilde{d}_\infty(l_1,l_2)=\lim_{t\to\infty}\frac{d_M(l_1(t),l_2(t))}{t}. \tag{5.1}$$

Rays are equivalent if $\widetilde{d}_\infty(l_1,l_2)=0$. The class of equivalence of l we denote by $[l]$ and the set of all classes of equivalent rays by $\mathcal{R}/\sim$. The metric $\widetilde{d}_\infty(l_1,l_2)$ in a standard way defines lengths of continuous curves in $\mathcal{R}/\sim$, which in turn generates the so-called inner metric $d_\infty([l_1],[l_2])$ which is by definition the infimum of lengths of all continuous curves in $\mathcal{R}/\sim$ connecting $[l_1]$ and $[l_2]$. Finally, the metric space $(\mathcal{R}/\sim, d_\infty)$ of classes of equivalent rays issuing from some fixed point of M is the ideal boundary of a manifold (M, d_M). For instance, the ideal boundary of the hyperbolic plane $\mathbb{H}^2$ of constant curvature -1 is a circle with so-called Tits metric, where the distance between different points equals ∞ [BP]. By theorem 4.1, the set of rays in the **Sol** is

$$\mathcal{R}=\{(a,b,c)|\ ab=0 \textit{ and } a^2+b^2+c^2=1\}. \tag{5.2}$$

In other words, it is the collection of unit parameterized geodesics issuing from **0** contained in the xz-plane or yz-plane. The metric d_∞ on $\mathcal{R}$ is a Tits metric on the ideal boundary of $\mathbb{H}^2$.

Theorem 5.1. *The ideal boundary of the* **Sol** *is a $(\mathcal{R}, d_\infty)$ with a Tits metric, where the distance between different points equals ∞, and where $\mathcal{R}=\{(a,b,c)|\ ab=0$ and $a^2+b^2+c^2=1\}$.*

Acknowledgements. I would like to thank the refree for his(her) useful suggestions.

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