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Inverse functions of Grötzsch's and Teichmüller's modulus functions

By

Shinji Yamashita

Abstract

Let χ be the inverse of the Grötzsch modulus function and let σ_n be the n-th iteration of the function $\sigma(r) = 2\sqrt{r}/(1+r)$, $r > 0$. For a real constant $\beta \neq 0$ with $\beta \geqslant -2$, the difference $\chi(x)^\beta - \sigma_n(4e^{-2^n x})^\beta$ is estimated. In the particular case where $\beta = -2$ one has an approximation of the inverse S of the Teichmüller modulus function, which is applied to improving the known upper and lower estimates concerning the error term of $\lambda(K) = \chi(\pi K/2)^{-2} - 1$ from $16^{-1}e^{\pi K} - 2^{-1}$ for the variable $K \geqslant 1$. Expressions of χ and S in terms of theta functions are studied. Lipschitz continuity of f or $\log f$ for $f = \chi$, S, as well as other functions are proved.

1. Introduction

The disk $D = \{z;\, |z| < 1\}$ in the complex plane $\mathbb{C} = \{z;\, |z| < +\infty\}$, slit along the closed interval $[0, r] = \{x;\, 0 \leqslant x \leqslant r\}$ for $0 < r < 1$, is conformally mapped onto the ring domain $\{z;\, 1 < |z| < e^{\mu(r)}\}$. H. Grötzsch's modulus function $\mu(r)$ is decreasing from $+\infty$ to 0 as r increases from 0 to 1, and μ admits the inverse function χ defined in $(0, +\infty)$. More explicitly, C. G. J. Jacobi's identity

$$\chi(x) = 4e^{-x} \prod_{n=1}^{+\infty} \left(\frac{1 + e^{-4nx}}{1 + e^{-(4n-2)x}} \right)^4 \tag{1.1}$$

for $x > 0$ is known, where the right-hand side can be regarded as a function of e^{-x}; for the details see Section 7 in the present paper. On the other hand, $\mathbb{C}$ minus the intervals $[-1, 0]$ and $[t, +\infty)$, $t > 0$, is conformally mapped onto $\{z;\, 1 < |z| < e^{T(t)}\}$, where $T(t) = 2\mu(1/\sqrt{1+t})$; see [LV, p. 55]. The inverse S of O. Teichmüller's modulus function T is, therefore, given by $S(x) = \chi(x/2)^{-2} - 1$ for $x > 0$. As will be seen in Section 7,

$$S(x) = 16^{-1} e^{x} \prod_{n=1}^{+\infty} \left(\frac{1 - e^{-(2n-1)x}}{1 + e^{-2nx}} \right)^8; \tag{1.2}$$

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this is not a trivial consequence of (1.1).

Both functions μ and T appear in the celebrated extremal problems in [Gr] and [T], respectively.

Although both χ and S are limits of partial products both of which are rational functions of e^{-x}, there is another point of view. Let σ_n be the n-th iteration, or the n times composed function, of $\sigma_1(r) \equiv \sigma(r) = 2\sqrt{r}/(1+r)$, $r \geqslant 0$; the function σ_n is increasing from 0 to 1 on the closed interval $[0,1]$ and decreasing from 1 to 0 on $[1,+\infty)$. One of the main subjects is the following: For a natural number n and a real constant $\beta \neq 0$ with $\beta \geqslant -2$, the function $\Delta_{n,\beta}(x)$ of $x > 0$ which appears in

$$\chi(x)^\beta = \sigma_n(4e^{-2^n x})^\beta + \Delta_{n,\beta}(x)e^{-(\beta+2^{n+1})x}$$

is estimated. The case where $\beta = 1$ or $\beta = -2$ is of use for approximating χ or S in terms of functions $\sigma_n(4e^{-2^n x})$ or $\sigma_n(4e^{-2^{n-1}x})^{-2} - 1$ of e^{-x}, respectively.

A special emphasis is placed on χ and S because the function $\varphi_K(r) = \chi(\mu(r)/K)$ of r with $0 \leqslant r < 1$ for a fixed $K \geqslant 1$, and the function $\lambda(K,t) = S(KT(t))$ of two variables $K \geqslant 1$ and $t \geqslant 0$, where $\varphi_K(0) = \lambda(K,0) = 0$, are important in Geometric Function Theory; see [LV, p. 64, Theorem 3.1] for $\varphi_K(r)$, and [LV], [LVV] for $\lambda(K) \equiv \lambda(K,1)$. The function $\lambda(K)$ of $K \geqslant 1$ appears in the sharp inequality [LV, p. 81, (6.6)] for the boundary values of a K-quasiconformal self-mapping of the upper half-plane preserving the point at infinity. Both functions $\varphi_K(r)$ and $\lambda(K,t)$ are linked by the equations $\varphi_K(r) = \chi(K^{-2}\mu(1/\sqrt{1+\lambda(K,r^{-2}-1)}))$ for $0 < r < 1$ and $\lambda(K,t) = \chi(K^2\mu(\varphi_K(1/\sqrt{1+t})))^{-2} - 1$ for $t > 0$. Note that the function $\eta_\kappa(t)$ has been studied in [AVV1], [AVV2], [QV] and others is exactly $S(\kappa T(t))$ for $\kappa > 0$ and $t > 0$; see Remark 2 in Section 12. Actually, $\eta_K(t) = \lambda(K,t)$ for $K \geqslant 1$ and $t > 0$. A Schottky-type theorem by G. J. Martin [Ma, Theorem 1.1] claims that, for f holomorphic in D with $f(D) \subset \mathbb{C} \setminus \{0,1\}$, the inequality $|f(z)| \leqslant \lambda(K,t)$ for $z \in D$ holds, where $K = (1+|z|)/(1-|z|)$ and $t = |f(0)|$. The bound $\lambda(K,t)$ is sharp for each pair $K \geqslant 1$ and $t > 0$. See Remark 1 in Section 12.

Concerning $\lambda(K)$ it will be proved in Section 2 that

$$1.2425\ldots < (\lambda(K) - 16^{-1}e^{\pi K} + 2^{-1})e^{\pi K} < 1.25. \tag{1.3}$$

for $K \geqslant 1$; the right constant 1.25 is the best possible in the sense that the central term in (1.3) tends to 1.25 as $K \to +\infty$. Earlier and weaker estimations are in

$$1 < (\lambda(K) - 16^{-1}e^{\pi K} + 2^{-1})e^{\pi K} < 35/24 = 1.458333\ldots \tag{1.4}$$

for $K \geqslant 1$, the details of which may be found in [LVV, pp. 12–13], in [AVV1, p. 7], and, in particular, in [AVV2, p. 406] for the upper bound 35/24.

The functions χ and S, together with their derivatives up to the second order, are expressed in terms of basic theta functions of Jacobi in Theorems 4 and 5 in Section 7. Theta functions are made effective use of in Sections 8, 9, and 10. Estimates of χ and S are obtained in Theorem 6 in Section 8; they are "local" in contrast with (3.1) for $\beta = 1$ and (2.1) in the forthcoming Theorems 2 and 1, respectively. Beginning with Theorem 7 functions relating to μ, T, χ, and S are shown to be Lipschitz continuous in Section 9. Theorem 8 in Section 10 reveals that the Poincaré density of the domain $\mathbb{C}\setminus\{-1,0\}$ on the real axis is important for estimating the difference $|\log\mu(r_1) - \log\mu(r_2)|$ for $r_1, r_2 \in (0,1)$. In Section 11 two series expansions of $\mu(r)$ in r due to Jacobi and C. F. Gauss are reduced to the expressions in terms of σ_n. In the final Section 12 remarks on the preceding results are given.

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2. Theorem 1 on S

The present paper begins with a theorem on S in conjunction with (1.3), a typical one following in reality from the forthcoming Theorem 2 in Section 3.

Theorem 1. *For $n \geqslant 1$ and $x > 0$,*

$$S(x) = \sigma_n(4e^{-2^{n-1}x})^{-2} - 1 + \Delta_{S,n}(x)e^{(1-2^n)x}, \tag{2.1}$$

where the function $\Delta_{S,n}(x)$ satisfies

(2.2)
$$0 < \Delta_{S,n}(x) < 2^{1-n}(1+\sqrt{1-16L^{-4}})^{-1} \qquad \textit{for} \quad x \geqslant 2^{2-n}\log L \quad \textit{with} \quad L \geqslant 2;$$

in particular,

$$0 < \Delta_{S,n}(x) < 2^{1-n} \qquad \textit{for} \quad x \geqslant 2^{2-n}\log 2. \tag{2.3}$$

Furthermore,

(2.4)
$$1 - \sigma_n(4)^{-2} < \Delta_{S,n}(x) < 16^{1-2^{-n}}(\sigma_n(\sqrt{2})^{-2} - 1) \qquad \textit{for} \quad 0 < x \leqslant 2^{2-n}\log 2$$

and

$$0 \leqslant \limsup_{x\to+\infty} \Delta_{S,n}(x) \leqslant 2^{-n}. \tag{2.5}$$

Actually, as x increases from 0 to $2^{2-n}\log 2$, the function $\Delta_{S,n}(x)$ increases from $1 - \sigma_n(4)^{-2} < 0$ to $16^{1-2^{-n}}(\chi(2^{1-n}\log 2)^{-2} - 1) > 0$ which is, as will be proved, less than the upper bound in (2.4).

It follows on setting $x = \pi K$ and $n = 1$ in Theorem 1 that

$$\lambda(K) = 16^{-1}e^{\pi K} - 2^{-1} + \delta_{LVV}(K), \tag{2.6}$$

where the function $\delta_{LVV}(K) \equiv (1 + \Delta_{S,1}(\pi K))e^{-\pi K}$ of $K \geqslant 1$ is studied in [LVV, Theorem 3] and $0 < \Delta_{S,1}(\pi K) < (1 + \sqrt{1-16e^{-2\pi}})^{-1} < 1$ for

$\pi K \geqslant \pi = 2\log L_1$ with $L_1 = e^{\pi/2} > 2$ by (2.2). Also the case $n = 2$ yields that

$$\lambda(K) = \sigma_2(4e^{-2\pi K})^{-2} - 1 + \Delta_{S,2}(\pi K)e^{-3\pi K}, \tag{2.7}$$

where $0 < \Delta_{S,2}(\pi K) < 2^{-1}(1+\sqrt{1-16e^{-4\pi}})^{-1} < 2^{-1}$ for $\pi K \geqslant \pi = \log L_2$ with $L_2 = e^{\pi}$ by (2.2). Equating (2.6) and (2.7) one has that

$$\delta_{LVV}(K)e^{\pi K} = 4^{-1} + (1+4y)^{-1} + \Delta_{S,2}(\pi K)y, \tag{2.8}$$

where $y = e^{-2\pi K} \leqslant e^{-2\pi}$. Consequently

$$\delta_{LVV}(K)e^{\pi K} > 4^{-1} + (1+4e^{-2\pi})^{-1}. \tag{2.9}$$

On the other hand, since the function $4^{-1} + (1+4y)^{-1} + 2^{-1}y$ of $y \leqslant e^{-2\pi}$ is strictly decreasing by $e^{-2\pi} < (\sqrt{8}-1)/4$, it follows that $\delta_{LVV}(K)e^{\pi K} < 5/4$. This, combined with (2.9), establishes that

$$1.2425\ldots = 4^{-1} + (1+4e^{-2\pi})^{-1} < \delta_{LVV}(K)e^{\pi K} < 5/4 \tag{2.10}$$

which is promised in (1.3). It follows from (2.8) that

$$\lim_{K\to+\infty} \delta_{LVV}(K)e^{\pi K} = 5/4,$$

so that, the constant $5/4$ in (2.10) can not be replaced with any smaller one.

Since $\delta_{LVV}(1)e^{\pi} = 16^{-1}(24 - e^{\pi})e^{\pi} = 1.2428\ldots$ by $\lambda(1) = 1$, the lower bound of $\delta_{LVV}(K)e^{\pi K}$ does not exceed $1.2428\ldots$. Further conjecture might be, therefore, that $\delta_{LVV}(K)e^{\pi K}$ were an increasing function of $K \geqslant 1$.

A generalization of δ_{LVV} will be discussed later in Section 6.

3. Theorem 2 and outline of proof

As was stated, Theorem 1 follows from

Theorem 2. *Let $\beta \neq 0$ be real, $\beta \geqslant -2$, $L \geqslant 2$, and n be natural. Then*

$$\chi(x)^{\beta} = \sigma_n(4e^{-2^n x})^{\beta} + \Delta_{n,\beta}(x)e^{-(\beta+2^{n+1})x} \tag{3.1}$$

for $x > 0$, where the function $\Delta_{n,\beta}(x)$ satisfies

(3.2)
$$-2^{2\beta-n+4}(1+\sqrt{1-16L^{-4}})^{-1} < \beta^{-1}\Delta_{n,\beta}(x) < 0 \qquad \text{for} \quad x \geqslant 2^{1-n}\log L;$$

in particular,

$$-2^{2\beta-n+4} < \beta^{-1}\Delta_{n,\beta}(x) < 0 \qquad \text{for} \quad x \geqslant 2^{1-n}\log 2. \tag{3.3}$$

Suppose that $0 < x \leqslant 2^{1-n}\log 2$. If $\beta > 0$, then

$$2^{2^{1-n}\beta+4}(\sigma_n(\sqrt{2})^{\beta} - 1) < \Delta_{n,\beta}(x) < 2^{2^{1-n}\beta+4}(1 - \sigma_n(4)^{\beta}). \tag{3.4}$$

For $-2 \leqslant \beta < 0$ *the function* $\Delta_{n,\beta}(x)$ *increases from* $1 - \sigma_n(4)^\beta < 0$ *to*

$$2^{2^{1-n}\beta+4}(\chi(2^{1-n}\log 2)^\beta - 1) > 0, \tag{3.5}$$

which is strictly less than

$$2^{2^{1-n}\beta+4}(\sigma_n(\sqrt{2})^\beta - 1), \tag{3.6}$$

as x *increases from* 0 *to* $2^{1-n}\log 2$*. Finally, for all* $\beta \neq 0$ *with* $\beta \geqslant -2$*,*

$$-2^{2\beta-n+3} \leqslant \liminf_{x\to+\infty} \beta^{-1}\Delta_{n,\beta}(x) \leqslant 0. \tag{3.7}$$

A reason why $\sigma_n(\sqrt{2})$ is chosen on the left-hand side in (3.4) and in (3.6) is that this is an algebraic number.

Theorem 1 follows from Theorem 2 by setting $\beta = -2$ and by replacing x with $x/2$. More explicitly, $\Delta_{S,n}(x) = \Delta_{n,-2}(x/2)$.

Before the detailed proof of Theorem 2 its principal idea is here outlined. Set

$$\Phi(y) \equiv \Phi_{n,\beta}(y) \equiv \sigma_n(4y^{-2})^\beta \qquad \text{for} \quad y > 0, \tag{3.8}$$

and set

$$\alpha_n = 2^{-n} \qquad \text{for} \quad n = 0, 1, 2, \cdots. \tag{3.9}$$

Then $\sigma_n(4e^{-2^n x})^\beta$ in (3.1) for $n \geqslant 1$ is exactly $\Phi(e^{x/\alpha_{n-1}})$ for $x > 0$. Set

$$r = \chi(x) \qquad \text{for} \quad x > 0 \qquad \text{or} \quad x = \mu(r) \qquad \text{for} \quad 0 < r < 1. \tag{3.10}$$

Then the function $\delta(r) \equiv \delta_n(r) > 0$ of r, $0 < r < 1$, with $n \geqslant 1$ will be found, where $\delta(r)$ appears in

$$\chi(x)^\beta = \Phi(e^{x/\alpha_{n-1}} + \delta(r)) \qquad \text{for} \quad r = \chi(x); \tag{3.11}$$

see the forthcoming (4.4). The Mean-Value Theorem applied to Φ then yields that

$$\chi(x)^\beta - \sigma_n(4e^{-2^n x})^\beta = \Phi'(\overline{Y}(r))\delta(r), \tag{3.12}$$

where $\overline{Y}(r) \equiv \overline{Y}_{n,\beta}(r) \equiv e^{x/\alpha_{n-1}} + \vartheta\delta(r)$ for a ϑ with $0 < \vartheta < 1$.

The main part in the proof is, therefore, upward estimation of $\Phi'(\overline{Y}(r))$ and $\delta(r)$ in (3.12).

For $n \geqslant 1$, and for x, r in (3.10), set

$$Y(r) \equiv Y_n(r) \equiv e^{\mu(r)/\alpha_{n-1}} + \delta(r) = e^{x/\alpha_{n-1}} + \delta(r);$$

this appears on the right-hand side of (3.11). It will be seen that $Y(r) > 2$ for all r, $0 < r < 1$. Obviously,

$$e^{x/\alpha_{n-1}} < \overline{Y}(r) < Y(r) \qquad \text{for} \quad 0 < r < 1. \tag{3.13}$$

In Section 4 the inequality

$$0 < \delta(r) < 2^3 Y(r)^{-3} A(r) < 1 \quad \text{for} \quad 0 < r < 1 \tag{3.14}$$

is proved, where

$$A(r) \equiv A_n(r) = (1 + \sqrt{1 - 16Y(r)^{-4}})^{-1} < 1. \tag{3.15}$$

In Section 5, first the inequality for Φ',

$$0 > \beta^{-1}\Phi'(\overline{Y}(r)) > 2^{-3} C_{n,\beta} \overline{Y}(r)^{\gamma} \overline{Y}(r)^3 \tag{3.16}$$

is established under the restriction that $n \geqslant 1$ and $x \geqslant \alpha_{n-1} \log L \geqslant \alpha_{n-1} \log 2$ which assures the inequality $\overline{Y}(r) > 2$. Here $C_{n,\beta} \equiv -2^{2\beta - n + 4} < 0$ and

$$\gamma \equiv -\beta\alpha_{n-1} - 4 < 0 \tag{3.17}$$

for which $\beta + 2^{n+1} = -\gamma/\alpha_{n-1}$ appears in the second term in the right of (3.1). It then follows from (3.1), (3.12), (3.16), (3.13), and (3.14) that

$$\begin{aligned} 0 > \beta^{-1}\Delta_{n,\beta}(x) &= \beta^{-1}\{\chi(x)^{\beta} - \sigma_n(4e^{-2^n x})^{\beta}\} e^{(\beta + 2^{n+1})x} \\ &= \beta^{-1}\Phi'(\overline{Y}(r))\delta(r) e^{-(\gamma/\alpha_{n-1})x} \\ &> C_{n,\beta} A(r) (e^{-x/\alpha_{n-1}} \overline{Y}(r))^{\gamma} > C_{n,\beta} A(r). \end{aligned} \tag{3.18}$$

Since $A(r) < (1 + \sqrt{1 - 16L^{-4}})^{-1}$ for $x \geqslant \alpha_{n-1} \log L$ by the forthcoming formula (4.7), estimation (3.2) in Theorem 2 follows from (3.18). In the remaining case where $0 < x \leqslant \alpha_{n-1} \log 2$, bounds are determined by fairly direct method. The proof of Theorem 2 is completed in Section 5.

4. Upper bound of $\delta(r)$

The function $\sigma(r)$ of $r \geqslant 0$ has the inverse function $\omega(r) = r^2(1 + \sqrt{1 - r^2})^{-2}$ in $[0, 1]$. The n-th iteration ω_n of ω is therefore the inverse of σ_n in $[0, 1]$. Note that $\sigma_n(1/r) = \sigma_n(r)$ for all $r > 0$. Set $\sigma_0(r) \equiv \omega_0(r) = r$ in $[0, 1]$.

Before proceeding further a brief review of the function μ will be given. J. Hersch [H, p. 316, (1)] proved that $\mu(r) = (\pi/2)\ \mathcal{K}(\sqrt{1 - r^2})\ /\mathcal{K}(r)$ for $0 < r < 1$, where

$$\mathcal{K}(r) = \int_0^{\pi/2} \frac{d\vartheta}{\sqrt{1 - r^2 \sin^2 \vartheta}} = \frac{\pi}{2} + \frac{\pi}{2} \sum_{n=1}^{\infty} \left(\frac{(2n-1)!!}{n! 2^n} \right)^2 r^{2n}, \qquad 0 < r < 1,$$

is A. M. Legendre's complete elliptic integral of the first kind; see [BB, pp. 7–8], [WW, p. 499] for $\mathcal{K}$ and also [LV, p. 60, (2.2)] for the expression of μ. The function $\mathcal{K}(r)$ increases from $\pi/2$ to $+\infty$ as r increases from 0 to 1. The function μ is real-analytic and μ becomes continuous in $(0, 1]$ on setting $\mu(1) = 0$; see [LV, p. 62]. Furthermore, $\mu(1/\sqrt{2}) = \pi/2$ is immediately obtained. Among others two series expansions of $\mu(r)$ in r due to Gauss and Jacobi are known; see (11.4) and (11.9). Since $\mu(\sigma(r)) = 2^{-1}\mu(r)$, $0 < r < 1$ ([H, p. 316, (3′)], [BB, p. 16, 1. e)]), it immediately follows that $\mu(\sigma_n(r)) = \alpha_n\mu(r)$ for $n \geqslant 0$ and $0 < r < 1$. Hence $\mu(r) = \alpha_n\mu(\omega_n(r))$ for $n \geqslant 0$ and $0 < r < 1$.

Since $2^{1-n}\log 2 < 2^{1-n}(\pi/4) = 2^{-n}\mu(1/\sqrt{2}) = \mu(\sigma_n(1/\sqrt{2})) = \mu(\sigma_n(\sqrt{2}))$ by $\log 2 = 0.69314\ldots < 0.78539\ldots = \pi/4$, it follows that $\sigma_n(\sqrt{2}) < \chi(2^{1-n}\log 2)$. Hence the constant in (3.6) is greater than that in (3.5) because $\beta < 0$.

Replacing r with $\omega_n(r)$ in the inequalities

$$\log\frac{(1+\sqrt{1-r^2})^2}{r} < \mu(r) < \log\frac{4}{r}, \qquad 0 < r < 1, \tag{4.1}$$

(see [H, p. 318, (9′)] and [LV, p. 61, (2.10)]; see also (11.10) and (11.12)) one obtains the estimates

$$\alpha_n\log\frac{(1+\sqrt{1-\omega_n(r)^2})^2}{\omega_n(r)} < \mu(r) < \alpha_n\log\frac{4}{\omega_n(r)}, \qquad 0 < r < 1. \tag{4.2}$$

It then follows from (4.2), together with $\omega_n = \omega\circ\omega_{n-1}$, that

$$\alpha_{n-1}\log\frac{(1+\sqrt[4]{1-\omega_{n-1}(r)^2})^2}{\omega_{n-1}(r)} < \mu(r) < \alpha_{n-1}\log\frac{2(1+\sqrt{1-\omega_{n-1}(r)^2})}{\omega_{n-1}(r)}, \tag{4.3}$$

for $0 < r < 1$ and for $n \geqslant 1$. The function $\delta(r)$ of $r \in (0,1)$ is then defined by

$$\delta(r) \equiv \delta_n(r) \equiv \frac{2(1+\sqrt{1-\omega_{n-1}(r)^2})}{\omega_{n-1}(r)} - e^{\mu(r)/\alpha_{n-1}}$$

for $n \geqslant 1$, so that, $\delta(r) > 0$ by (4.3) and, for $Y(r) \equiv e^{\mu(r)/\alpha_{n-1}} + \delta(r)$, one has

$$(2Y(r)^{-1})^2 = \omega\circ\omega_{n-1}(r) = \omega_n(r) < 1.$$

Automatically, $Y(r) > 2$ for all r, $0 < r < 1$. Consequently,

$$r = \sigma_n(4Y(r)^{-2}) = \Phi(Y(r))^{1/\beta}, \tag{4.4}$$

$$\omega_{n-1}(r) = \sigma(4Y(r)^{-2}) \tag{4.5}$$

for $0 < r < 1$ and for $n \geqslant 1$. On the other hand, it follows from (4.3) and (4.5) that

$$0 < \delta(r) < \Lambda(\omega_{n-1}(r)) = \Lambda\circ\sigma(4Y(r)^{-2})(< 1) \tag{4.6}$$

for $0 < r < 1$ and for $n \geqslant 1$, where the function of ρ, $0 < \rho \leqslant 1$,

$$\begin{aligned}\Lambda(\rho) &= \{2(1+\sqrt{1-\rho^2}) - (1+\sqrt[4]{1-\rho^2})^2\}/\rho \\ &= \rho^3(1+\sqrt[4]{1-\rho^2})^{-2}(1+\sqrt{1-\rho^2})^{-2}\ (\leqslant 1)\end{aligned}$$

increases from 0 to 1 as ρ increases from 0 to 1. Hence the identity

$$\Lambda \circ \sigma(\rho) = \rho^{3/2}(1+\sqrt{1-\rho^2})^{-1}, \qquad 0<\rho \leqslant 1,$$

together with (4.6), yields (3.14). Furthermore, if $\mu(r)(=x) \geqslant \alpha_{n-1}\log L$, then $Y(r) \geqslant L + \delta(r) > L$, so that

$$A(r) < (1+\sqrt{1-16L^{-4}})^{-1} \leqslant 1. \tag{4.7}$$

5. Derivative Φ'

To establish (3.16) one begins with estimation of $(\sigma_n^\beta)'(r) = (d/dr)\{\sigma_n(r)^\beta\}$ for $n \geqslant 1$ and $0<r<1$. Set $Q_{n,\beta} \equiv 2^{(2-\alpha_{n-1})\beta - n}$ and recall that $\beta \neq 0$ and $\beta \geqslant -2$. To verify inductively that

$$0 < \beta^{-1}(\sigma_n^\beta)'(r) < Q_{n,\beta}\cdot r^{\beta\alpha_n - 1} \tag{5.1}$$

for $n \geqslant 1$ and $0<r<1$, one begins with the identity $F_{n+1} = F_n \circ \sigma$ for $F_n(r) \equiv \sigma_n(r)^\beta$ with $0<r<1$. Because

$$\beta^{-1}F_1'(r) = 2^{\beta-1}r^{\beta/2-1}(1+r)^{-\beta-2}(1-r^2) < 2^{\beta-1}r^{\beta/2-1}$$

by $-\beta-2 \leqslant 0$, the case $n=1$ in (5.1) follows. Next suppose (5.1) for $n \geqslant 1$. Then

$$\beta^{-1}F_{n+1}'(r) = \beta^{-1}F_n'(\sigma(r))\sigma'(r)$$

is positive and is strictly less than $Q_{n,\beta}\sigma(r)^{\beta\alpha_n-1}\sigma'(r)$. Since

$$\begin{aligned}\sigma(r)^{\beta\alpha_n-1}\sigma'(r) &= 2^{\beta\alpha_n-1}r^{\beta\alpha_{n+1}-1}(1+r)^{-\beta\alpha_n-2}(1-r^2)\\ &< 2^{\beta\alpha_n-1}r^{\beta\alpha_{n+1}-1}\end{aligned}$$

because $-\beta\alpha_n - 2 < 0$ by $\beta \geqslant -2 > -2/\alpha_n$, it follows that (5.1) is valid for $n+1$ instead of n.

Precisely, $\Phi'(y) = -2^3y^{-3}(\sigma_n^\beta)'(2^2y^{-2})$ for the function Φ of (3.8), from which, together with (5.1), results the estimate

$$0 > \beta^{-1}\Phi'(y) > -R_{n,\beta}\cdot y^{-\beta\alpha_{n-1}-1} = -R_{n,\beta}\cdot y^{\gamma+3}$$

for $y>2$, γ of (3.17), and $n \geqslant 1$. Here, $R_{n,\beta} = 2^{\beta\alpha_{n-1}+1}Q_{n,\beta} = 2^{2\beta-n+1} > 0$.

Setting $C_{n,\beta} = -2^3R_{n,\beta}$ one immediately obtains (3.16) for $x \geqslant \alpha_{n-1}\log 2$ because $\overline{Y}(r) > 2$ by (3.13).

It follows from (3.18) that $\beta^{-1}\Delta_{n,\beta}(x) > C_{n,\beta}A(r)$ for $x \geqslant \alpha_{n-1}\log 2$. Since $e^{x/\alpha_{n-1}} < Y(r) \to +\infty$ as $x \to +\infty$, it follows that $\lim_{x\to+\infty} A(r) = 2^{-1}$. Hence (3.7) is established.

For $0 < x \leqslant \alpha_{n-1}\log 2$ it is convenient to introduce the functions $\mathcal{G}(x) = \chi(x)^\beta - \sigma_n(4e^{-2^nx})^\beta$ and $\mathcal{H}(x) = e^{(\beta+2^{n+1})x}$, so that $\Delta_{n,\beta}(x) = \mathcal{G}(x)\mathcal{H}(x)$. Then $\mathcal{H}$ increases from 1 to $2^{\beta\alpha_{n-1}+4}$ because $\beta \geqslant -2 > -2^{n+1}$. Notice that $4e^{-2^nx} \geqslant 1$. If $\beta > 0$ then $\mathcal{G}$ decreases from $1 - \sigma_n(4)^\beta > 0$ to $\chi(\alpha_{n-1}\log 2)^\beta -$

$1 < 0$. Consequently, (3.4) follows from $\sigma_n(\sqrt{2}) < \chi(\alpha_{n-1} \log 2)$. In the case where $(-2^{n+1} <) - 2 \leqslant \beta < 0$, the function $\mathcal{G}$ increases from $1 - \sigma_n(4)^\beta < 0$ to $\chi(\alpha_{n-1} \log 2)^\beta - 1 > 0$. Hence $\Delta_{n,\beta}(x)$ increases from $1 - \sigma_n(4)^\beta$ to the quantity in (3.5).

6. The function δ_{LVV} revisited

Although the choice $x = KT(t) = 2K\mu(1/\sqrt{1+t})$ in Theorem 1 leads to the expansion of the function $\lambda(K,t)$ of K and t, there is another approach with t limited. Set $L(n,t) = \exp\{2^{n-1}\mu(1/\sqrt{1+t})\}$, so that $L(n+1,t) > L(n,t)$ for $n \geqslant 1$ and $t > 0$. For example, $L(2,1) = e^\pi > L(1,1) = e^{\pi/2} > 2$. Under the condition that

$$L(n,t) \geqslant 2 \tag{6.1}$$

for $n \geqslant 1$ and $t > 0$, Theorem 1 for $x = 2K\mu(1/\sqrt{1+t})$, together with $L = L(n,t)$ in (2.2), immediately yields

Theorem 3. *Suppose that $n \geqslant 1$ and $t > 0$ satisfy* (6.1). *Then for every $K \geqslant 1$,*

$$\begin{aligned} \lambda(K,t) = {} & \sigma_n(4\exp\{-2^n K\mu(1/\sqrt{1+t})\})^{-2} - 1 \\ & + \Delta_{\lambda,n}(K,t)\exp\{(2-2^{n+1})K\mu(1/\sqrt{1+t})\}, \end{aligned} \tag{6.2}$$

where $\Delta_{\lambda,n}(K,t) = \Delta_{S,n}(2K\mu(1/\sqrt{1+t}))$ and

$$0 < \Delta_{\lambda,n}(K,t) < 2^{1-n}(1 + \sqrt{1 - 16L(n,t)^{-4}})^{-1}. \tag{6.3}$$

Furthermore, for all fixed $n \geqslant 1$ and $t > 0$, possibly $L(n,t) < 2$,

$$0 \leqslant \limsup_{K\to+\infty} \Delta_{\lambda,n}(K,t) \leqslant 2^{-n}, \tag{6.4}$$

where $\Delta_{\lambda,n}(K,t)$ is, this time, defined directly by (6.2).

Set

$$\delta_{LVV}(K,t) = \lambda(K,t) - \frac{1}{16}\exp\{2K\mu(1/\sqrt{1+t})\} + \frac{1}{2} \tag{6.5}$$

for $K \geqslant 1$ and $t > 0$, so that $\delta_{LVV}(K) = \delta_{LVV}(K,1)$ by (2.6). Furthermore, set

$$\zeta_K(t) \equiv \exp\{-2K\mu(1/\sqrt{1+t})\}(\leqslant \exp\{-2\mu(1/\sqrt{1+t})\}) \tag{6.6}$$

and $\Psi_n(K,t) \equiv \sigma_n(4\zeta_K(t)^{2^{n-1}})^{-2}$ for $n \geqslant 1$ and $t > 0$. The latter is exactly the first term in the right of (6.2) even in the case $L(n,t) < 2$. Then

$$\Psi_1(K,t) = \zeta_K(t) + \frac{1}{2} + \frac{1}{16\zeta_K(t)}. \tag{6.7}$$

Suppose that for $t > 0$ the function $\Delta_{\lambda,n}(K,t)$ is defined directly by (6.2). Then

$$\lambda(K,t) = \Psi_n(K,t) - 1 + \Delta_{\lambda,n}(K,t)\zeta_K(t)^{2^n-1} \tag{6.8}$$

for $n \geqslant 1$ and $t > 0$. Set

$$W_n(K,t) \equiv \Psi_n(K,t) - \Psi_1(K,t) + \zeta_K(t). \tag{6.9}$$

Then it follows from (6.5), (6.8), (6.9), and (6.7) that

$$\delta_{LVV}(K,t)\zeta_K(t)^{-1} = W_n(K,t)\zeta_K(t)^{-1} + \Delta_{\lambda,n}(K,t)\zeta_K(t)^{2^n-2} \tag{6.10}$$

for $n \geqslant 1$ and $t > 0$.

On the other hand, for each fixed $t > 0$ the function

$$W_2(K,t)\zeta_K(t)^{-1} = \frac{5}{4} - \frac{4}{\zeta_K(t)^{-2} + 4} \tag{6.11}$$

of $K \geqslant 1$ increases from $5/4 - 4/(e^{4\mu(1/\sqrt{1+t})} + 4)$ to $5/4$ as K increases from 1 to $+\infty$.

Fix $t > 0$ and consider (6.10) for $n = 2$. Since $W_2(K,t)\zeta_K(t)^{-1} \to 5/4$ as $K \to +\infty$ by (6.11), it follows from (6.3) that $\delta_{LVV}(K,t)\zeta_K(t)^{-1} \to 5/4$ as $K \to +\infty$.

In the present and next paragraphs the condition that $L(1,t) \geqslant 2$ is supposed, so that $\mu(1/\sqrt{1+t}) \geqslant \log 2$. Since $L(n,t) \geqslant L(1,t) \geqslant 2$, estimates (6.3) in Theorem 3 are valid for n, t, with $L = L(n,t)$. It then follows from (6.3) and (6.10) that

$$\begin{aligned} &W_n(K,t)\zeta_K(t)^{-1} < \delta_{LVV}(K,t)\zeta_K(t)^{-1} \\ &\quad < W_n(K,t)\zeta_K(t)^{-1} \\ &\qquad + 2^{1-n}(1 + \sqrt{1 - 16L(n,t)^{-4}})^{-1} \exp\{(2^2 - 2^{n+1})\mu(1/\sqrt{1+t})\}. \end{aligned} \tag{6.12}$$

It further follows from (6.12) for $n = 2$, together with the monotone property of the function $W_2(K,t)\zeta_K(t)^{-1}$ of $K \geqslant 1$, that

$$\begin{aligned} \frac{5}{4} - \frac{4}{e^{4\mu(1/\sqrt{1+t})} + 4} &< \delta_{LVV}(K,t)\zeta_K(t)^{-1} \\ &< \frac{5}{4} + (2 + 2 \cdot \sqrt{1 - 16L(2,t)^{-4}})^{-1} \exp\{-4\mu(1/\sqrt{1+t})\}. \end{aligned} \tag{6.13}$$

On setting $t = 1$ in (6.13) one immediately has

$$\begin{aligned} 1.2425\ldots = 5/4 - 4(e^{2\pi} + 4)^{-1} &< \delta_{LVV}(K)e^{\pi K} \\ &< 5/4 + e^{-2\pi}(2 + 2 \cdot \sqrt{1 - 16e^{-4\pi}})^{-1} = 1.2504\ldots; \end{aligned}$$

the right most is worse than $5/4$ in (2.10).

Since $\mu(1/\sqrt{1+t}) \geqslant \log 2$ by $L(1,t) \geqslant 2$, it follows that $L(2,t) \geqslant 4$. Hence (6.13) can be reduced to a weaker form with the bounds independent of t,

$$\begin{aligned} 1.05 = \frac{5}{4} - \frac{4}{16+4} &< \delta_{LVV}(K,t)\exp\{2K\mu(1/\sqrt{1+t})\} \\ &< \frac{5}{4} + (2 + 2\cdot\sqrt{1-16\cdot 4^{-4}})^{-1}\cdot\frac{1}{16} = \frac{14-\sqrt{15}}{8} = 1.2658\ldots \end{aligned} \tag{6.14}$$

for t with $L(1,t) \geqslant 2$.

More precisely, if

$$t \geqslant \sigma(\sqrt{2})^{-2} - 1 = (3\sqrt{2}-4)/8 = 0.03033\ldots, \tag{6.15}$$

then

$$\mu(1/\sqrt{1+t}) \geqslant \mu(\sigma(1/\sqrt{2})) = 2^{-1}\mu(1/\sqrt{2}) = \pi/4 > \log 2.$$

Hence $L(1,t) \geqslant e^{\pi/4} > 2$ and moreover, $L(2,t) \geqslant e^{\pi/2}$. Consequently, (6.13) is reduced to

$$\begin{aligned} 1.1026\ldots = \frac{5}{4} - \frac{4}{e^{\pi}+4} &< \delta_{LVV}(K,t)\exp\{2K\mu(1/\sqrt{1+t})\} \\ &< \frac{5}{4} + \frac{e^{-\pi}}{2(1+\sqrt{1-16e^{-2\pi}})} = 1.2608\ldots \end{aligned} \tag{6.16}$$

for t satisfying (6.15).

Setting $t = 1$ in (6.14) or in (6.16) one still has improvement of (1.4).

7. Basic theta functions

Topics on the functions χ and S are picked up in conjunction with theta functions. The main reference is the book [BB].

The basic theta functions ([BB, pp. 52 and 33], [WW, p. 464])

$$\theta_2(q)(=\theta_2(0,q)) = 2\sum_{n=0}^{+\infty} q^{(n+2^{-1})^2} = \sum_{n=-\infty}^{+\infty} q^{(n+2^{-1})^2} = 2q^{1/4}\sum_{n=0}^{+\infty} q^{n(n+1)},$$

$$\theta_3(q)(=\theta_3(0,q)) = 1 + 2\sum_{n=1}^{+\infty} q^{n^2} = \sum_{n=-\infty}^{+\infty} q^{n^2}, \qquad \text{and}$$

$$\theta_4(q)(=\theta_4(0,q)) = 1 + 2\sum_{n=1}^{+\infty} (-q)^{n^2} = \sum_{n=-\infty}^{+\infty} (-q)^{n^2}$$

for $0 < q < 1$ admit respectively infinite-product expressions ([BB, p. 64, Corollary 3.1], [WW, pp. 472–473]),

$$\theta_2(q) = 2q^{1/4}\prod_{n=1}^{+\infty}(1-q^{2n})(1+q^{2n})^2,$$

$$\theta_3(q) = \prod_{n=1}^{+\infty}(1-q^{2n})(1+q^{2n-1})^2, \qquad \text{and}$$

$$\theta_4(q) = \prod_{n=1}^{+\infty}(1-q^{2n})(1-q^{2n-1})^2.$$

In the present Section the dash $'$ means the derivative. Set $\Xi_k(q) = \theta_k(q)'\theta_k(q)^{-1}$ for $k = 2, 3, 4$ and for $0 < q < 1$. Then $\Xi_2(q) = 4^{-1}q^{-1} + Q'(q)Q(q)^{-1} > 0$ where $Q(q) = 2\sum_{n=0}^{+\infty} q^{n(n+1)}$. Obviously $\Xi_3(q) > 0$. It will be soon observed that $\Xi_4(q) < 0$.

Two theorems involving theta functions will be proved.

Theorem 4. *For $x > 0$*

(7.1) $\chi(x) = \theta_2(e^{-2x})^2\theta_3(e^{-2x})^{-2}$,

(7.2) $\chi'(x) = -\theta_2(e^{-2x})^2\theta_3(e^{-2x})^{-2}\theta_4(e^{-2x})^4$,

(7.3) $\chi''(x) = \theta_2(e^{-2x})^2\theta_3(e^{-2x})^{-2}\theta_4(e^{-2x})^4[\theta_4(e^{-2x})^4 + 8e^{-2x}\Xi_4(e^{-2x})]$,

(7.4) $(d^2/dx^2)\log\chi(x) = 8e^{-2x}\theta_4(e^{-2x})^4\Xi_4(e^{-2x})$.

A real function f defined in an open interval (a, b) with $-\infty \leqslant a < b \leqslant +\infty$ is called d-increasing if $f'(x) > 0$ for all $x \in (a, b)$ and f is called d-convex if $f''(x) > 0$ for all $x \in (a, b)$. If $-f$ is d-increasing, then f is called d-decreasing, whereas if $-f$ is d-convex, then f is called d-concave.

Proof of Theorem 4. The quotient $\Omega(q) = \theta_2(q)/\theta_3(q)$ is d-increasing in $(0, 1)$ and it increases from 0 to 1 as the variable q increases from 0 to 1. In reality,

$$\Omega(q) = 2q^{1/4}\prod_{n=1}^{+\infty}\left(\frac{1+q^{2n}}{1+q^{2n-1}}\right)^2 \tag{7.5}$$

and

$$(d/dq)\log\Omega(q) = \Xi_2(q) - \Xi_3(q) = 4^{-1}q^{-1}\theta_4(q)^4; \tag{7.6}$$

see [BB, p. 42, (2.3.11)] which, together with $ds = -\pi^{-1}q^{-1}dq$ for $s = -\pi^{-1}\log q$ there, reads (7.6). Since $2\mu(r) = -\log q$ for $r = \Omega(q)^2$ by [BB, pp. 40–41, Theorem 2.3], the identity (7.1) follows on setting $q = e^{-2x}$.

Taking the square roots of both sides in Jacobi's formula [J, p. 146, (7.)] one actually has (7.5); accordingly the identity (1.1) is Jacobi's. Jacobi's formula can be rewritten as

$$\exp(\mu(r) + \log r) = 4\prod_{n=1}^{+\infty}\left(\frac{1+q^{2n}}{1+q^{2n-1}}\right)^4.$$

Here the variable $q = e^{-2\mu(r)} \in (0,1)$ is called the nome associated with the variable $r \in (0,1)$.

Since $dq/dx = -2q$ for $q = e^{-2x}$, it follows from (7.6), together with $\chi(x) = \Omega(q)^2$, that

$$\chi'(x)/\chi(x) = -\theta_4(q)^4; \tag{7.7}$$

furthermore,

$$\chi''(x)/\chi'(x) - \chi'(x)/\chi(x) = -8q\Xi_4(q). \tag{7.8}$$

Obviously, (7.2) follows from (7.7). One is now able to prove (7.3). The identity

$$(d^2/dx^2)\log\chi(x) = (\chi'(x)/\chi(x))(\chi''(x)/\chi'(x) - \chi'(x)/\chi(x)),$$

together with (7.7) and (7.8), shows (7.4). □

It is known that $\mu''(r_\iota) = 0$ for only one point $r_\iota \in (0,1)$; see [AVV2, p. 84, Theorem 5.13, (1)]. Hence $\chi''(x_\iota) = 0$ for only one point $x_\iota = \mu(r_\iota)$; the derivative of $\theta_4(e^{-2x})^{-4}$ with respect to x at this point x_ι is just -1 by (7.3). Let us introduce Legendre's complete elliptic integral of the second kind

$$\mathcal{E}(r) = \int_0^{\pi/2} \sqrt{1 - r^2\sin^2\vartheta}\, d\vartheta = \frac{\pi}{2} - \frac{\pi}{2}\sum_{n=1}^{\infty}\left(\frac{(2n-1)!!}{n!2^n}\right)^2 \frac{r^{2n}}{2n-1}, \quad 0 < r < 1;$$

see [BB, p. 8] and [WW, p. 518]. The function $\mathcal{E}(r)$ is d-decreasing and it decreases from $\pi/2$ to 1 as r increases from 0 to 1. It then follows from [BB, p. 43, (2.3.17)] that

$$\Xi_4(q) = \pi^{-2}q^{-1}\mathcal{K}(r)(\mathcal{E}(r) - \mathcal{K}(r)) \tag{7.9}$$

for $r = \Omega(q)^2$. Since $\mathcal{E}(r) < \mathcal{K}(r)$, it follows that $\Xi_4(q) < 0$ for $0 < q < 1$.

The d-decreasing function $\log\chi(x) < 0$ of $x > 0$ is d-concave by (7.4). See also [AVV2, p. 96, Theorem 5.46]. A consequence is that the inverse function $x = \mu(e^s)$ of $s = \log\chi(x)$ is a d-decreasing and d-concave function of $s < 0$. Consequently, for a constant $\beta < 0$, the function $\mu(s^\beta) = \mu(\exp(\beta\log s))$ is a d-increasing and d-concave function of $s > 1$ because $\beta\log s$ is d-decreasing and d-convex. Furthermore, the d-increasing function $\chi(x)^\beta = \exp(\beta\log\chi(x))$ of $x > 0$ for a constant $\beta < 0$ is d-convex. In particular, S is seen to be a d-increasing and d-convex function without appealing to the direct calculation of $S''(x)$. Consequently the inverse T of S is a d-increasing and d-concave function. Furthermore, S^β for a constant $\beta > 1$ is d-increasing and d-convex.

The inverse function of $y = \tanh x,\ x > 0$, is $x = \tanh^{-1} y$, where $\tanh^{-1} y \equiv 2^{-1}\log\{(1+y)/(1-y)\},\ 0 < y < 1$. To prove that $\tanh^{-1}\chi$ is d-decreasing and d-convex, the identity [BB, p. 35, (2.1.10)]

$$\theta_3(q)^4 - \theta_2(q)^4 = \theta_4(q)^4 \tag{7.10}$$

for $0 < q < 1$ should be recalled. Then for $q = e^{-2x}$ it follows from (7.1) that $1 - \chi(x)^2 = \theta_3(q)^{-4}\theta_4(q)^4$ for $x > 0$. On the other hand, the identity

$$\Xi_3(q) = \Xi_4(q) + 4^{-1}q^{-1}\theta_2(q)^4 \tag{7.11}$$

follows from [BB, p. 42, (2.3.15)]. Consequently, in view of (7.2) one has

$$(\tanh^{-1}\chi(x))' = -\theta_2(q)^2\theta_3(q)^2 < 0$$

and hence

$$(\tanh^{-1}\chi(x))''/(\tanh^{-1}\chi(x))' = -4q(\Xi_2(q) + \Xi_3(q)) < 0.$$

Let $\mathcal{Q}$ be the first quadrant in the plane. The set $\{(\kappa, t) \in \mathcal{Q};\ S(\kappa^{-1}T(t)) = c\}$ for a constant $c > 0$ is the curve $\{(\kappa, S(\kappa T(c)));\ \kappa > 0\}$. On the other hand, for a fixed $t > 0$ the d-increasing function $S(\kappa T(t))$ of $\kappa > 0$ is d-convex; see also [AVV2, p. 217, Theorem 10.31]. Accordingly the shape of the level set defined above should be clarified. Furthermore, the set $\{(\kappa, t) \in \mathcal{Q};\ S(\kappa T(t)) = c\}$ for a constant $c > 0$ is the curve $\{(\kappa, S(\kappa^{-1}T(c)));\ \kappa > 0\}$. The function $S(\kappa^{-1}T(c))$ of $\kappa > 0$ is d-decreasing and d-convex.

From the infinite-product formula for $\theta_4(q)$, together with (1.1) and (7.7), it follows that

$$\chi'(x) = -4e^{-x}\prod_{n=1}^{+\infty}(1 - e^{-8nx})^4(1 - e^{-(4n-2)x})^8(1 + e^{-(4n-2)x})^{-4}.$$

Theorem 5. *For $x > 0$*

$$S(x) = \theta_2(e^{-x})^{-4}\theta_4(e^{-x})^4, \tag{7.12}$$

$$S'(x) = \theta_2(e^{-x})^{-4}\theta_3(e^{-x})^4\theta_4(e^{-x})^4, \tag{7.13}$$

$$S''(x) = \theta_2(e^{-x})^{-4}\theta_3(e^{-x})^4\theta_4(e^{-x})^4[\theta_3(e^{-x})^4 - 4e^{-x}\Xi_3(e^{-x})], \tag{7.14}$$

$$(d^2/dx^2)\log S(x) = -4e^{-x}\theta_3(e^{-x})^4\Xi_3(e^{-x}). \tag{7.15}$$

Proof of Theorem 5. It follows from (7.10) that $\Omega(q)^{-4} - 1 = \theta_2(q)^{-4}\theta_4(q)^4$ for all q with $0 < q < 1$, so that, one has $S(x) = \theta_2(p)^{-4}\theta_4(p)^4$ or (7.12) on setting $p = e^{-x}$ for $x > 0$. Hence $S'(x)/S(x) = -4p(\Xi_4(p) - \Xi_2(p))$. On the other hand, it follows from [BB, p. 42, (2.3.16)] that

$$\Xi_4(q) - \Xi_2(q) = -4^{-1}q^{-1}\theta_3(q)^4 \tag{7.16}$$

for $0 < q < 1$, so that one may replace q with p; actually, (7.16) is a consequence of (7.6), (7.11), and (7.10). Consequently,

$$S'(x)/S(x) = \theta_3(p)^4, \tag{7.17}$$

whence,

$$S''(x)/S'(x) - S'(x)/S(x) = -4p\Xi_3(p). \tag{7.18}$$

Both (7.13) and (7.14) follow from (7.17) and (7.18). Multiplying (7.17) and (7.18) one immediately obtains (7.15). □

An immediate consequence of

$$\theta_2(q)^{-1}\theta_4(q) = 2^{-1}q^{-1/4}\prod_{n=1}^{+\infty}\left(\frac{1-q^{2n-1}}{1+q^{2n}}\right)^2,$$

combined with (7.12), accomplishes (1.2). From the infinite-product formula for $\theta_3(p)$, $p = e^{-x}$, together with (1.2) and (7.17), it follows that

$$S'(x) = 16^{-1}e^x\prod_{n=1}^{+\infty}(1-e^{-2nx})^4(1-e^{-(4n-2)x})^8(1+e^{-2nx})^{-8}.$$

One can express the right-hand side in (7.15) in a series form. Let us recall the identity $\theta_3(q)^4 = 1+8\sum^* nq^n(1-q^n)^{-1}$ for $0 < q < 1$, where $\sum^*$ means the summation taken over all integers $n \geqslant 1$ with $n \not\equiv 0 \pmod 4$; see [BB, p. 71, (3.2.23)]. Differentiation then yields that $4\theta_3(q)^4\Xi_3(q) = 8\sum^* n^2q^{n-1}(1-q^n)^{-2}$ for $0 < q < 1$. The derivative $(d^2/dx^2)\log S(x)$ in (7.15) is hereby $-8e^{-x}\sum^* n^2e^{-(n-1)x}(1-e^{-nx})^{-2} < 0$.

Consequently, the d-increasing function $\log S(x)$ of $x > 0$ is d-concave. An additional conclusion is that the inverse function $x = T(e^s)$ of $s = \log S(x)$ is d-increasing and d-convex for $-\infty < s < +\infty$. Furthermore, for a constant $\beta < 0$, the function $T(s^\beta) = T(\exp(\beta\log s))$ of $s > 0$ is d-decreasing and d-convex. For a constant $\beta < 0$, the d-decreasing function $S(x)^\beta = \exp(\beta\log S(x))$ of $x > 0$ is d-convex.

In particular, for each fixed $t > 0$, the d-increasing function $\log S(\kappa T(t))$ of $\kappa > 0$ is d-concave; see [AVV2, p. 217, Theorem 10.31]. The function $\log S(\kappa T(t))$ of $t > 0$ for a fixed $\kappa > 0$ is d-increasing and d-concave; see [AVV2, p. 213, Theorem 10.23]. For $\beta < 0$, the function $S(\kappa T(t))^\beta = \exp(\beta\log S(\kappa T(t)))$ of $t > 0$ is d-decreasing and d-convex.

The function S is seen to be d-convex. This fact, together with (7.14), reveals that $\theta_3(q)^4 > 4q\Xi_3(q)$ for $0 < q < 1$, a direct proof of which is obtained from (7.11), $\theta_3(q) > \theta_2(q)$ and $\Xi_4(q) < 0$.

The following notice on $\Omega(q)$ might be significant. Consider the particular case where $\beta = 1/2$, $n = 1$, and $x = -2^{-1}\log q$ in Theorem 2. Then (3.1), (3.3), and (3.4) yield that $\Omega(q) = 2q^{1/4}(1+4q)^{-1/2} + \Delta(q)q^{9/4}$, for $0 < q < 1$, where $-8 < \Delta(q) < 16\sqrt{2}(1-2/\sqrt{5}) = 2.3883\ldots$.

8. Inequalities for χ and S

Let $f = \chi$ or $f = S$, and let $A_n = 2^{n-1}\pi$ for all integers n. "Good" functions g_n and h_n will be found so that $g_n \leqslant f \leqslant h_n$ in each closed interval $[A_n, A_{n+1}]$.

Hereafter for a negative integer n and for $0 < r < 1$, let us set $\sigma_n(r) = \omega_{-n}(r)$ and $\omega_n(r) = \sigma_{-n}(r)$. Then $\mu(\sigma_n(r)) = 2^{-n}\mu(r)$ for all $r \in (0,1)$ and for all integers n. Set $\psi_n = \omega_n(1/\sqrt{2})$ for all n. Since $\mu(\psi_n) = 2^n\mu(1/\sqrt{2}) = A_n$, it follows that $0 < \psi_{n+1} < \psi_n < 1$ for all n. Moreover, $\psi_n \to 0$ as $n \to +\infty$, whereas, $\psi_n \to 1$ as $n \to -\infty$ because $\mu(\psi_n) \to +\infty$ as $n \to +\infty$ and $\mu(\psi_n) \to 0$ as $n \to -\infty$.

Next, four constants are defined in terms of ψ.

$$B_{n,1} = 2^{1-n}\pi^{-1}\log(\psi_{n+1}/\psi_n), \qquad B_{n,2} = \psi_n^2 - 1,$$
$$B_{n,3} = 2^{1-n}\pi^{-1}\log\frac{\psi_n^{-2}-1}{\psi_{n-1}^{-2}-1}, \qquad B_{n,4} = 2^{1-n}\pi^{-1}\cdot\frac{\psi_n^{-2}-\psi_{n-1}^{-2}}{\psi_{n-1}^{-2}-1}.$$

Obviously $B_{n,1} < 0$ and $B_{n,2} < 0$; furthermore, $B_{n,3} > 0$ and $B_{n,4} > 0$.

An absolute constant $c_0 = 4^{-1}\pi^{-3}\Gamma(1/4)^4 = 1.39320\ldots$ will become important, where $\Gamma(1/4) = 3.62560990822190\ldots$. It follows from

$$\mathcal{K}(1/\sqrt{2}) = 4^{-1}\pi^{-1/2}\Gamma(1/4)^2 = 1.85407\ldots \tag{8.1}$$

(see [BB, p. 25, Theorem 1.7]) that $c_0 = 4\pi^{-2}\mathcal{K}(1/\sqrt{2})^2$.

Set $c_n = c_0\prod_{k=1}^{n}(1+\psi_k)^{-2}$ for $n > 0$ and $c_n = c_0\prod_{k=0}^{n+1}(1+\psi_k)^2$ for $n < 0$. Then $c_{n+1} < c_n$ for $n \geqslant 0$ and $c_{n+1} > c_n$ for $n \leqslant 0$. Since $\psi_n \to 1$ as $n \to -\infty$, it follows that $\sum_{k=0}^{n+1}\psi_k \to +\infty$ as $n \to -\infty$, whence $c_n \to +\infty$ as $n \to -\infty$. At the end of the present Section it will be proved that c_n has a finite limit as $n \to +\infty$.

Theorem 6. *Let an integer n be arbitrary. Then for all $x \in [A_n, A_{n+1}]$,*

$$\psi_n\exp\{B_{n,1}(x-A_n)\} \leqslant \chi(x) \leqslant \psi_n\exp\{c_n B_{n,2}(x-A_n)\}; \tag{8.2}$$

$$\begin{aligned}(\psi_{n-1}^{-2}-1)\max\Big[\exp\{B_{n,3}(x-A_n)\},\ 1+c_{n-1}(x-A_n)\Big] &\leqslant S(x)\\ \leqslant (\psi_{n-1}^{-2}-1)\min\Big[\exp\{c_{n-1}(x-A_n)\},\ 1+B_{n,4}(x-A_n)\Big]&.\end{aligned} \tag{8.3}$$

Equality holds in the left in (8.2) *if and only if $x \in \{A_n,\ A_{n+1}\}$, whereas, in the right if and only if $x = A_n$. All the equalities hold in* (8.3) *if and only if $x \in \{A_n,\ A_{n+1}\}$.*

Proof. The proof depends on fairly elementary treatment. For a d-convex function f in an open interval (a,b) with $-\infty \leqslant a < b \leqslant +\infty$, and for $A \in (a,b)$, the quotient $F(x) = (f(x)-f(A))/(x-A)$ becomes a continuous function in (a,b) on setting $F(A) = f'(A)$. The derivative $f''(x)(x-A)$ of the function

$$g(x) = (x-A)^2F'(x) = f'(x)(x-A) - (f(x)-f(A))$$

of $x \in (a,b)\setminus\{A\}$ is positive for $x > A$ and negative for $x < A$, and furthermore, $g(x) \to 0$ as $x \to A$. Hence $g(x) > 0$ for all $x \in (a,b)\setminus\{A\}$. This implies that $F'(x) > 0$ for $x \in (a,b)\setminus\{A\}$, whence $F(x) < F(y)$ for $a < x < y < b$. Thus, for $x \in [A,B] \subset (a,b)$ with $A < B$,

$$f'(A) \leqslant \frac{f(x)-f(A)}{x-A} \leqslant \frac{f(B)-f(A)}{B-A}. \tag{8.4}$$

Equality holds in the first if and only if $x = A$, whereas it holds in the second if and only if $x = B$. The right-most is strictly less than $f'(B)$ by the Mean-Value Theorem with the monotone property of f'. Furthermore,

$$-\infty \leqslant \lim_{x\to a} \frac{f(x)-f(A)}{x-A} < \frac{f(x)-f(A)}{x-A} < \lim_{x\to b} \frac{f(x)-f(A)}{x-A} \leqslant +\infty \tag{8.5}$$

for all $x \in (a,b)$. All the inequalities in (8.4) and in (8.5) should be reversed if f is d-concave in (a,b).

Since $\log\chi$ and $\log S$ both are d-concave in $(0,+\infty)$, one immediately obtains for $x \in [A_n, A_{n+1}]$ that

$$\begin{aligned} h(A_n)\exp\left\{\left(\frac{1}{A_n}\log\frac{h(A_{n+1})}{h(A_n)}\right)(x-A_n)\right\} &\leqslant h(x) \\ &\leqslant h(A_n)\exp\left\{\frac{h'(A_n)}{h(A_n)}(x-A_n)\right\} \end{aligned}$$

for $h = \chi,\ S$. Equality holds in the left if and only if $x \in \{A_n,\ A_{n+1}\}$ and in the right if and only if $x = A_n$. Furthermore, since S is d-convex in $(0,+\infty)$, one also has for $x \in [A_n, A_{n+1}]$ that

$$\begin{aligned} S(A_n)\left(1+\frac{S'(A_n)}{S(A_n)}(x-A_n)\right) &\leqslant S(x) \\ &\leqslant S(A_n)\left(1+\frac{1}{S(A_n)}\left(\frac{S(A_{n+1})-S(A_n)}{A_n}\right)(x-A_n)\right); \end{aligned}$$

again equality holds in the left if and only if $x = A_n$ and in the right if and only if $x \in \{A_n, A_{n+1}\}$.

One thus observes that (8.2) and (8.3) depend finally on proofs of a string of identities

$$\chi(A_n) = \psi_n, \tag{8.6}$$

$$S(A_n) = \psi_{n-1}^{-2} - 1, \tag{8.7}$$

$$\chi'(A_n)/\chi(A_n) = (\psi_n^2 - 1)c_n, \qquad \text{and} \tag{8.8}$$

$$S'(A_n)/S(A_n) = c_{n-1} \tag{8.9}$$

for all integers n.

Identities (8.6) and (8.7) are obvious from $\mu(\psi_n) = A_n$ and $\chi(2^{-1}A_n) = \psi_{n-1}$.

Proofs of (8.8) and (8.9) begin with establishing that $4\pi^{-2}\mathcal{K}(\psi_n)^2 = c_n$ for all integers n. This is obvious for $n = 0$ by (8.1). First, the identity $\mathcal{K}(r) = (1+r)^{-1}\mathcal{K}(\sigma(r))$ for $0 < r < 1$ ([BB, p. 12, Theorem 1.2, (a)]) should be changed into $\mathcal{K}(\omega(r)) = (1+\omega(r))^{-1}\mathcal{K}(r)$. Then induction to both identities shows that $\mathcal{K}(\sigma_n(r)) = \mathcal{K}(r)\prod_{k=1}^{n}(1+\sigma_{k-1}(r))$ and $\mathcal{K}(\omega_n(r)) = \mathcal{K}(r)\prod_{k=1}^{n}(1+\omega_k(r))^{-1}$ for $n \geqslant 1$ and $0 < r < 1$. Setting $r = 1/\sqrt{2}$ in these formulae, one obtains the requested $c_n = 4\pi^{-2}\mathcal{K}(\psi_n)^2$.

Since

$$\theta_3(q)^4 = 4\pi^{-2}\mathcal{K}(r)^2 \tag{8.10}$$

for $r = \theta_2(q)^2\theta_3(q)^{-2}$ with $0 < q < 1$ by [BB, p. 35, (2.1.13)], it follows that

$$\theta_2(q)^4 = 4\pi^{-2}r^2\mathcal{K}(r)^2, \tag{8.11}$$

so that the identity $\theta_4(q)^4 = \theta_3(q)^4 - \theta_2(q)^4$ reveals further that

$$\theta_4(q)^4 = 4\pi^{-2}(1-r^2)\mathcal{K}(r)^2. \tag{8.12}$$

Here $\mu(r) = -2^{-1}\log q$.

One thus has (8.8) with the aid of (7.7) and (8.12) for $r = \chi(A_n) = \psi_n$ by (8.6), whereas one has (8.9) with the aid of (7.17) and (8.10) for $r = \chi(2^{-1}A_n) = \psi_{n-1}$. □

In addition to (8.4) one has

$$f'(A) < \frac{f(B)-f(A)}{B-A} = \frac{f(A)-f(B)}{A-B} \leqslant \frac{f(x)-f(B)}{x-B} \leqslant f'(B)$$

for $x \in [A, B]$ on considering the function $(f(x) - f(B))/(x - B)$ instead of F there. Again the inequalities are reversed if f is *d*-concave. One can then obtain obvious counterparts of (8.2) and (8.3) the details of which are left as exercises.

Return to (8.5) and set $f = -\log\chi$. Since $\chi(x) \to 1$ as $x \to 0$, it follows that

$$\lim_{x\to 0}\frac{\log\chi(A) - \log\chi(x)}{x-A} = -A^{-1}\log\chi(A).$$

On the other hand, since $\theta_3(q) \to 1$ and $\theta_2(q)^2q^{-1/2} = Q(q)^2 \to 4$ as $q \to 0$, it follows from (7.1) with $q = e^{-2x}$ that

$$\lim_{x\to+\infty}\frac{\log\chi(A) - \log\chi(x)}{x-A} = 1.$$

One now obtains that

$$(-A^{-1}\log\chi(A))|x-A| \leqslant |\log\chi(x) - \log\chi(A)| \leqslant |x-A| \tag{8.13}$$

for all $x > 0$; both equalities hold if and only if $x = A$.

Next, set $f = -\log S$ and also $f = S$ in (8.5). Since $S(x) \to 0$ as $x \to 0$ and $A > 0$, it immediately follows that

$$\lim_{x\to 0}\frac{\log S(A) - \log S(x)}{x-A} = -\infty,$$

whereas, since $\theta_4(p) \to 1$ and $\theta_2(p)^{-4}p = Q(p)^{-4} \to 16^{-1}$ as $p \to 0$, it follows from (7.12) with $p = e^{-x}$ that

$$\lim_{x\to+\infty}\frac{\log S(A) - \log S(x)}{x-A} = -1.$$

Consequently,

$$|x - A| \leqslant |\log S(x) - \log S(A)| \tag{8.14}$$

for all $x > 0$; equality holds if and only if $x = A$.

On the other hand,

$$\lim_{x\to 0} \frac{S(x) - S(A)}{x - A} = A^{-1}S(A)$$

because $S(x) \to 0$ as $x \to 0$. Furthermore, one is now able to prove that

$$\lim_{x\to +\infty} \frac{S(x) - S(A)}{x - A} = \lim_{p\to 0} \frac{p^{-1}}{\log(p^{-1})} = +\infty$$

with the aid of (7.12) again. Hence

$$A^{-1}S(A)|x - A| \leqslant |S(x) - S(A)|$$

for all $x > 0$; equality holds if and only if $x = A$.

Explicit estimations will be obtained on setting $A = A_n$.

In Section 7 functions other than $\log\chi$, $\log S$, and S are shown to be d-concave or d-convex. For example, χ^β for $\beta < 0$ is d-convex in $(0, +\infty)$, so that one has estimations of χ^β in intervals $[A_n, A_{n+1}]$ on following the described argument. In case $-2 \leqslant \beta < 0$, these estimations are different, in spirit, from (3.1) in Theorem 2.

In view of (7.9) for $\mu(r) = -2^{-1}\log q$, one can prove with the aid of (7.11) and (8.11) that

$$\Xi_3(q) = \pi^{-2}q^{-1}\mathcal{K}(r)(\mathcal{E}(r) - (1 - r^2)\mathcal{K}(r)) \tag{8.15}$$

and with the aid of (7.16) and (8.10) that

$$\Xi_2(q) = \pi^{-2}q^{-1}\mathcal{E}(r)\mathcal{K}(r). \tag{8.16}$$

It follows from (7.7), together with (8.12), that $\chi'(x)/\chi(x) = 4\pi^{-2}(r^2 - 1)\mathcal{K}(r)^2$, whence

$$\chi'(x) = 4\pi^{-2}r(r^2 - 1)\mathcal{K}(r)^2 \tag{8.17}$$

for $r = \chi(x)$. It further follows from (7.8), together with (7.9) and (8.17), that

$$\chi''(x) = 16\pi^{-4}r(1 - r^2)\mathcal{K}(r)^3(2\mathcal{E}(r) - (1 + r^2)\mathcal{K}(r))$$

for $r = \chi(x)$. Since $2\mathcal{E}(r)/((1 + r^2)\mathcal{K}(r))$ decreases from 2 to 0 as r increases from 0 to 1, the function χ is d-convex in $(0, x_\iota)$ and d-concave in $(x_\iota, 1)$. It is now an exercise to obtain the following for S, where, this time, $r = \chi(2^{-1}x)$.

$$\begin{aligned} S(x) &= r^{-2} - 1, \\ S'(x) &= 4\pi^{-2}(r^{-2} - 1)\mathcal{K}(r)^2, \\ S''(x) &= 16\pi^{-4}(r^{-2} - 1)\mathcal{K}(r)^3((2 - r^2)\mathcal{K}(r) - \mathcal{E}(r)). \end{aligned}$$

As for T, one obtains on setting $r = (1+t)^{-1/2}$ that $T'(t) = -r^3\mu'(r)$ and

$$T''(t) = 2^{-1}r^5\mu'(r)(3 + r\mu''(r)/\mu'(r))$$

for $t > 0$. Since $T'' < 0$ and since $\mu' < 0$, one has the inequality $\mu''(r)/\mu'(r) > -3r^{-1}$ for $r \in (0,1)$.

To prove that $c_n \to c_0 \prod_{n=1}^{+\infty}(1+\psi_n)^{-2} \neq 0, \neq +\infty$, as $n \to +\infty$, it suffices to set $r = 1/\sqrt{2}$ in

$$0 < \sum_{n=0}^{+\infty} \log(1 + \omega_n(r)) < +\infty \tag{8.18}$$

which is valid for all $r \in (0,1)$. For the proof, first, $\omega_n(r) < r^{2^n}$ or $\sigma_n(r) > r^{2^{-n}}$ for $n > 1$ and $r \in (0,1)$, is obtained by induction; in particular, $\omega_n(r) \to 0$ as $n \to +\infty$. On the other hand, there exists a unique $r_o \in (\sqrt{\sqrt{17}-3}/2,\ 1)$ such that $(1+r)^2 < 1+\sigma(r)$ if and only if $r < r_o$. Consequently, $(1+r)^{2^n} < 1+\sigma_n(r)$ for $n > 1$ by induction. Hence $1 + \omega_n(r) < (1+r)^{2^{-n}}$, from which

$$\log(1 + \omega_n(r)) < 2^{-n}\log(1 + r) \tag{8.19}$$

for $n > 1$ and $r \in (0, r_o)$. Given $r \in (0,1)$, choose N such that $\omega_N(r) < r_o$. Replace then r with $\omega_N(r)$ in (8.19) to have $\log(1+\omega_{n+N}(r)) < 2^{-n}\log(1+r_o)$ for all $n > 1$. The proof of (8.18) is herewith complete.

9. Lipschitz continuity

In the present Section, Lipschitz continuity and "inverse" Lipschitz continuity of f or $\log f$ for $f = \mu, \chi, T$, or S are mainly investigated.

Theorem 7. *For $r_k \in (0,1)$ with $k = 1, 2$,*

$$|\log r_1 - \log r_2| \leqslant |\mu(r_1) - \mu(r_2)|; \tag{9.1}$$

inequality is strict if and only if $r_1 \neq r_2$. For each constant $a \in (0,1)$, and for $r_k \in (0, a]$ with $k = 1, 2$,

$$|\mu(r_1) - \mu(r_2)| \leqslant -a\mu'(a)|\log r_1 - \log r_2|; \tag{9.2}$$

inequality is strict if and only if $r_1 \neq r_2$. The constant

$$-a\mu'(a) = 4^{-1}\pi^2(1-a^2)^{-1}\mathcal{K}(a)^{-2}$$

depending on a, increases from 1 to $+\infty$ as a increases from 0 to 1. For each constant $A > 0$, and for $x_k \in [A, +\infty)$ with $k = 1, 2$,

$$|\log S(x_1) - \log S(x_2)| \leqslant (S'(A)/S(A))|x_1 - x_2|; \tag{9.3}$$

inequality is strict if and only if $x_1 \neq x_2$. The constant

$$S'(A)/S(A) = 4\pi^{-2}\mathcal{K}(\chi(2^{-1}A))^2$$

depending on A, decreases from $+\infty$ to 1 as A increases from 0 to $+\infty$.

For $x_k > 0$ with $k = 1, 2$,

$$(9.4) \qquad |\log \chi(x_1) - \log \chi(x_2)| \leqslant |\log S(x_1) - \log S(x_2)|;$$

inequality is strict if and only if $x_1 \neq x_2$.

For (9.1) see also [AVV2, p. 84, Theorem 5.13, (2)].

Consequences are listed. Inequality (9.1) is equivalent to

$$(9.5) \qquad |\log \chi(x_1) - \log \chi(x_2)| < |x_1 - x_2|,$$

so that

$$(9.6) \qquad |\log(S(x_1) + 1) - \log(S(x_2) + 1)| < |x_1 - x_2|;$$

both for $x_1 > 0$ and $x_2 > 0$ with $x_1 \neq x_2$. The right inequality in (8.13) is, in reality, equivalent to (9.5). A direct proof of (9.1) in connection with that of (9.2) will be given. Incidentally, the inequality

$$(9.7) \qquad |x_1 - x_2| \leqslant |\log S(x_1) - \log S(x_2)| \qquad (x_1 > 0,\ x_2 > 0)$$

is equivalent to (8.14).

Furthermore, it follows from (9.2), (9.3), (9.7), and (9.6), respectively, that

$$(9.8) \qquad |x_1 - x_2| < -(\chi(b)/\chi'(b))|\log \chi(x_1) - \log \chi(x_2)|$$

for $x_k \in [b, +\infty)$, $k = 1, 2$, with $x_1 \neq x_2$ and $b > 0$;

$$(9.9) \qquad |\log t_1 - \log t_2| < B^{-1}T'(B)^{-1}|T(t_1) - T(t_2)|,$$

for $t_k \in [B, +\infty)$, $k = 1, 2$, with $t_1 \neq t_2$, where $B^{-1}T'(B)^{-1} = 4\pi^{-2}\mathcal{K}(\chi(2^{-1}T(B)))^2$ for $B > 0$;

$$|T(t_1) - T(t_2)| < |\log t_1 - \log t_2| \qquad \text{and}$$
$$|\log(t_1 + 1) - \log(t_2 + 1)| < |T(t_1) - T(t_2)|$$

both for $t_1 > 0$, $t_2 > 0$ with $t_1 \neq t_2$.

It follows from (8.8) that the Lipschitz constant $-a\mu'(a)$ in (9.2), for $a = \chi(A_n)$ with integer n, is $-\chi(A_n)/\chi'(A_n) = (1 - \psi_n^2)^{-1}c_n^{-1}$. Furthermore, the constant $S'(A)/S(A)$ in (9.3) for $A = A_{n+1}$ is c_n by (8.9). It is now obvious that the constant in (9.8) for $b = A_n$ is $(1 - \psi_n^2)^{-1}c_n^{-1}$, while the constant in (9.9) for $B = S(A_{n+1})$ is c_n.

Combinations of the above inequalities yield, for example, the following two, where $\kappa > 0$ is fixed. For $0 < r_k \leqslant a < 1$, $k = 1, 2$,

$$|\log \chi(\kappa\mu(r_1)) - \log \chi(\kappa\mu(r_2))| \leqslant -\kappa a\mu'(a)|\log r_1 - \log r_2|,$$

whereas, for $t_k \geqslant C > 0$, $k = 1, 2$,

$$|\log S(\kappa T(t_1)) - \log S(\kappa T(t_2))| \leqslant 4\pi^{-2}\kappa\mathcal{K}(\chi(2^{-1}\kappa T(C)))^2|\log t_1 - \log t_2|.$$

Remaining cases are left as exercises.

Before the proof of Theorem 7 expressions of $\mu'(r)$ and $\mu''(r)$ in terms of $\theta_k(q)$, $k = 2, 3, 4$, and $\Xi_4(q)$ are proposed, where $\mu(r) = -2^{-1}\log q$. It follows from (7.2) with $\mu'(r) = 1/\chi'(x)$, $q = e^{-2x}$, and $r = \Omega(q)^2$, that

$$\mu'(r) = -\theta_2(q)^{-2}\theta_3(q)^2\theta_4(q)^{-4} = -r^{-1}\theta_4(q)^{-4}. \tag{9.10}$$

Since $\mu''(r) = -\chi''(x)\mu'(r)^3$, it further follows from (7.3) and (9.10) that

$$\begin{aligned}\mu''(r) &= \theta_2(q)^{-4}\theta_3(q)^4\theta_4(q)^{-8}(\theta_4(q)^4 + 8q\Xi_4(q)) \\ &= r^{-2}\theta_4(q)^{-8}(\theta_4(q)^4 + 8q\Xi_4(q)).\end{aligned}$$

That θ_3 is d-increasing and θ_4 is d-decreasing is observed by $\theta_3'(q) > 0$ and $\Xi_4(q) < 0$ in (7.9) respectively, both for $0 < q < 1$. On the other hand, θ_2 is d-increasing by $\theta_2 = \theta_3\Omega$.

It follows from (8.12) and $\lim_{r\to 1}(\mathcal{K}(r) - \log(4/\sqrt{1-r^2})) = 0$ (see [WW, p. 521]) that $\theta_4(q)$ decreases from 1 to 0 as q increases from 0 to 1. The function $-r\mu'(r)$ is d-increasing because its derivative is $-4\theta_4(q)^{-4}\Xi_4(q)(dq/dr)$ by (9.10), together with $dq/dr = -2q\mu'(r) > 0$; furthermore, it increases from 1 to $+\infty$ because r increases if and only if q increases. One is now able to give

Proof of Theorem 7. Since $-r\mu'(r) > 1$ for all $r \in (0, 1)$, it follows from integration that $\mu(r_1) - \mu(r_2) \geqslant \log r_2 - \log r_1$ for $r_1 \leqslant r_2$. Exchanging r_1 for r_2 in the opposite case one has (9.1).

On the other hand, $-r\mu'(r) \leqslant -a\mu'(a)$ for $r \leqslant a$, whence, by integration, $0 \leqslant \mu(r_1) - \mu(r_2) \leqslant -a\mu'(a)\log(r_2/r_1)$ for $r_1 \leqslant r_2 \leqslant a$. Exchanging r_1 for r_2 in the opposite case one has (9.2). Since

$$-r\mu'(r) = \theta_4(q)^{-4} = 4^{-1}\pi^2(1-r^2)^{-1}\mathcal{K}(r)^{-2} \tag{9.11}$$

for $0 < r < 1$ by (9.10) and (8.12) (see also [BB, p. 137, (4.6.3a)]), one has immediately the expression of $-a\mu'(a)$. One can prove that $\sqrt{1-r^2}\mathcal{K}(r)$ is d-decreasing directly by the formula of $\mathcal{K}(r)$.

It follows from (7.17), together with (8.10), that $S'(x)/S(x) = 4\pi^{-2}\mathcal{K}(r)^2$, where $r = \chi(2^{-1}x)$. By the d-concavity of $\log S$, or by the d-increasing property of $\mathcal{K}$, $S'(x)/S(x)$ is d-decreasing. Thus, $S'(x)/S(x) \leqslant S'(A)/S(A)$ for $x \geqslant A$. The proof of (9.3) is now obvious.

To prove (9.4) the identity $\theta_4(p^2)^4 = \theta_3(p)^2\theta_4(p)^2$ for $0 < p < 1$ resulting from [BB, p. 34, (2.1.7ii)] is of use. It then follows from (7.7) and (7.17) for $q = p^2$ with $p = e^{-x}$ that

$$(\chi'(x)/\chi(x))^2 = \theta_4(p)^4 S'(x)/S(x) < S'(x)/S(x)$$

for $x > 0$. Consequently, the Schwarz inequality for integral gives that

$$\begin{aligned}(\log\chi(x_1) - \log\chi(x_2))^2 &\leqslant |x_1 - x_2|\left|\int_{x_1}^{x_2}(\chi'(x)/\chi(x))^2 dx\right| \\ &\leqslant |x_1 - x_2||\log S(x_1) - \log S(x_2)|,\end{aligned}$$

which, combined with (9.7), proves (9.4). □

10. Grötzsch function μ and Poincaré density

The Poincaré density $P(z)$ in the twice punctured plane $\mathbb{C}^* = \mathbb{C} \setminus \{-1, 0\}$ is the function defined by the equation $P(z)^{-1} = (1 - |w|^2)|\mathcal{M}'(w)|$ at $z = -\mathcal{M}(w)$, where

$$\mathcal{M}(w) = 16q(w) \prod_{n=1}^{+\infty} \left(\frac{1 + q(w)^{2n}}{1 + q(w)^{2n-1}} \right)^8$$

with $q(w) = \exp\{\pi(w+1)/(w-1)\}$, is an elliptic modular function defined in D, which omits precisely three points $0, 1$, and ∞; see [N, p. 319, (76)] and [BB, pp. 112–115].

The Poincaré distance between z and w in $\mathbb{C}^*$ is

$$d(z, w) = \int P(\zeta)|d\zeta|,$$

where the integral is taken along a geodesic connecting z and w in $\mathbb{C}^*$. Furthermore,

$$P(z) = P(-1-z) = |1+z|^{-2} P(-z/(1+z)) \tag{10.1}$$

for $z \in \mathbb{C}^*$ and

$$d(z, w) = d(-1-z, -1-w) = d(-z/(1+z), -w/(1+w)) \tag{10.2}$$

for $z, w \in \mathbb{C}^*$ because the mappings $z \mapsto -1-z$ and $z \mapsto -z/(1+z)$ both are conformal from $\mathbb{C}^*$ onto $\mathbb{C}^*$.

Theorem 8. *For $r_1, r_2 \in (0, 1)$ with $r_1 \neq r_2$,*

$$|\log \mu(r_1) - \log \mu(r_2)| < 4P(1) \left| \log \frac{\sqrt{1-r_1^2}}{r_1} - \log \frac{\sqrt{1-r_2^2}}{r_2} \right|, \tag{10.3}$$

where

$$4P(1) = \frac{8\pi^2}{\Gamma(1/4)^4} = 0.456946\ldots. \tag{10.4}$$

In addition,

$$|\log \mu(r_1) - \log \mu(r_2)| > \frac{\pi}{2} \Upsilon(r_1, r_2) \left| \log \frac{\sqrt{1-r_1^2}}{r_1} - \log \frac{\sqrt{1-r_2^2}}{r_2} \right| \tag{10.5}$$

for $r_1, r_2 \in (0, 1)$ with $r_1 \neq r_2$, where

$$\Upsilon(r_1, r_2) = \frac{1}{\max[\mathcal{K}(r_1)\mathcal{K}(\sqrt{1-r_1^2}),\ \mathcal{K}(r_2)\mathcal{K}(\sqrt{1-r_2^2})]}.$$

For (10.4) see [Hm, p. 436] where $\rho(-1) = 2P(1)$. The constant $c_H = 2^{-1}P(1)^{-1}$ is important for estimating Poincaré densities in hyperbolic domains; see [Y1, p. 118]. The length l of the lemniscate $(x^2+y^2)^2 = x^2 - y^2$ in the xy-plane is

$$\begin{aligned} l &= 4\int_0^1 (1/\sqrt{1-\tau^4})d\tau = \Gamma(1/4)\Gamma(1/2)\Gamma(3/4)^{-1} \\ &= 2^{-1/2}\pi^{-1/2}\Gamma(1/4)^2 = 5.24411\ldots, \end{aligned}$$

where $l/4 = 1.3110287771460599068\ldots$ is shown by Gauss [Ga, p. 413], so that $P(1) = \pi l^{-2}$, the area of the disk of radius l^{-1}.

For $x_k > 0$, $k = 1,2$, with $x_1 \neq x_2$, set $r_k = \chi(x_k/2)$, $k = 1,2$. Then (10.3) reads that

$$|\log x_1 - \log x_2| < 2P(1)|\log S(x_1) - \log S(x_2)|.$$

Proof of Theorem 8. One first proves that

$$(tP(t))^{-1} = 8\pi^{-1}\mathcal{K}(1/\sqrt{1+t})\mathcal{K}(\sqrt{t/(1+t)}) \tag{10.6}$$

for $t > 0$. This is true for $t = 1$ by (8.1).

Set $q = q(s) \equiv \exp\{\pi(s+1)/(s-1)\}$ for $-1 < s < 1$, so that $-1 < t < 0$ for $t \equiv -\mathcal{M}(s)$. Then $\mu(r) = -2^{-1}\log q$ for $r = r(s) \equiv \Omega(q)^2$ and $t = -r^2$. Since $(1-s^2)q'(s) = 2q\log q$, and since

$$\mathcal{M}'(s) = 4\Omega(q)^4(\Xi_2(q) - \Xi_3(q))q'(s) = -tq^{-1}q'(s)\theta_4(q)^4$$

by (7.6), it follows from (8.12) that

$$-(tP(t))^{-1} = -2\theta_4(q(s))^4\log q(s) = 8\pi^{-1}(1+t)\mathcal{K}(\sqrt{-t})\mathcal{K}(\sqrt{1+t}) \tag{10.7}$$

for $-1 < t < 0$. Identity (10.6) for $t > 0$ follows on replacing t with $-t/(1+t)$ in (10.7) and further, on observing (10.1).

As is seen in the proof of [AVV2, p. 64, Lemma 3.32] the function $\mathcal{K}(r)\mathcal{K}(\sqrt{1-r^2})$ is d-decreasing in $(0, 1/\sqrt{2})$ and d-increasing in $(1/\sqrt{2}, 1)$, so that $(tP(t))^{-1}$ is d-decreasing in $(0,1)$ and d-increasing in $(1,+\infty)$ by (10.6). Consequently, $P(t) < P(1)t^{-1}$ for all $t > 0,\ t \neq 1$. Combining this with the identity

$$d(t,\ \lambda(K,t)) = \int_t^{\lambda(K,t)} P(x)dx = 2^{-1}\log K \qquad (x \in \mathbb{R}) \tag{10.8}$$

for $K > 1$ and $t > 0$ (see [KY, Section 4]), one immediately has

$$2^{-1}\log K < P(1)\log(\lambda(K,t)/t). \tag{10.9}$$

Notice that $\lambda(K,t) > t$. Suppose that $0 < r_1 < r_2 < 1$ and set $t = r_2^{-2} - 1$, and further, $K = \mu(r_1)/\mu(r_2)$. Then $\lambda(K,t) = r_1^{-2} - 1$. Substituting these in (10.9) one immediately obtains (10.3) for $r_1 < r_2$.

The proof of (10.5) begins with the inequality $xP(x) \geqslant C(a,b)$ for $x \in [a,b] \subset (0,+\infty)$, $a \neq b$ with $C(a,b) = \min[aP(a), bP(b)]$. Then for $a = t > 0$ and for $b = \lambda(K,t)$ with $K > 1$ one has in view of (10.8) that

$$2^{-1}\log K \geqslant C(t,\lambda(K,t))\log(\lambda(K,t)/t). \tag{10.10}$$

Given $0 < r_1 < r_2 < 1$, set $t = r_2^{-2} - 1$ and $K = \mu(r_1)/\mu(r_2)$ to have again $\lambda(K,t) = r_1^{-2} - 1$. One then accomplishes the proof by obtaining (10.5) from (10.10) for $C(r_1^{-2} - 1, r_2^{-2} - 1) = 8^{-1}\pi\Upsilon(r_1,r_2)$ with the aid of (10.6). $\square$

Since $\Upsilon(r_1,r_2) \geqslant \Upsilon(a,b)$ for for $r_1,\ r_2 \in [a,b] \subset (0,1)$, it follows from (10.5) that, for $r_1 \neq r_2$,

$$|\log\mu(r_1) - \log\mu(r_2)| > \frac{\pi}{2}\Upsilon(a,b)\left|\log\frac{\sqrt{1-r_1^2}}{r_1} - \log\frac{\sqrt{1-r_2^2}}{r_2}\right|.$$

Set $I_1 = (0,+\infty)$, $I_2 = (-\infty,-1)$, and $I_3 = (-1,0)$. One can then show that

(10.11)
$$d(t_1,t_2) = \frac{1}{2}\left|\log\mu\left(\frac{1}{\sqrt{1+t_1}}\right) - \log\mu\left(\frac{1}{\sqrt{1+t_2}}\right)\right| \qquad \text{for} \quad t_1,\ t_2 \in I_1;$$
(10.12)
$$d(t_1,t_2) = \frac{1}{2}\left|\log\mu\left(\frac{1}{\sqrt{-t_1}}\right) - \log\mu\left(\frac{1}{\sqrt{-t_2}}\right)\right| \qquad \text{for} \quad t_1,\ t_2 \in I_2;$$
(10.13)
$$d(t_1,t_2) = \frac{1}{2}|\log\mu(\sqrt{1+t_1}) - \log\mu(\sqrt{1+t_2})| \qquad \text{for} \quad t_1,\ t_2 \in I_3.$$

Identity (10.11) for $t_1 < t_2$ follows on setting $K = T(t_2)/T(t_1)$, and $t = t_1$ in (10.8), whereas Identities (10.12) and (10.13) both follow from (10.11) with the aid of (10.2).

As a corollary of Theorem 8 the following six inequalities are listed. Three upper estimates of $d(t_1,t_2)$ are first exhibited.

$$d(t_1,t_2) < P(1)\left|\log\frac{t_1}{t_2}\right| \qquad \text{for} \quad t_1,\ t_2 \in I_1, \quad t_1 \neq t_2;$$
$$d(t_1,t_2) < P(1)\left|\log\frac{1+t_1}{1+t_2}\right| \qquad \text{for} \quad t_1,\ t_2 \in I_2, \quad t_1 \neq t_2;$$
$$d(t_1,t_2) < P(1)\left|\log\frac{t_1(1+t_2)}{t_2(1+t_1)}\right| \qquad \text{for} \quad t_1,\ t_2 \in I_3, \quad t_1 \neq t_2.$$

Three lower estimates of $d(t_1, t_2)$ are the following.

$$d(t_1,t_2) > \frac{\pi}{8}\Upsilon\left(\frac{1}{\sqrt{1+A}}, \frac{1}{\sqrt{1+B}}\right)\left|\log\frac{t_1}{t_2}\right| \quad \text{for} \quad t_1,\ t_2 \in [A,B] \subset I_1, \quad t_1 \neq t_2;$$

$$d(t_1,t_2) > \frac{\pi}{8}\Upsilon\left(\frac{1}{\sqrt{-A}}, \frac{1}{\sqrt{-B}}\right)\left|\log\frac{1+t_1}{1+t_2}\right| \quad \text{for} \quad t_1,\ t_2 \in [A,B] \subset I_2, \quad t_1 \neq t_2;$$

$$d(t_1,t_2) > \frac{\pi}{8}\Upsilon(\sqrt{1+A}, \sqrt{1+B})\left|\log\frac{t_1(1+t_2)}{t_2(1+t_1)}\right| \quad \text{for} \quad t_1,\ t_2 \in [A,B] \subset I_3, \quad t_1 \neq t_2,$$

where $A \neq B$ in all cases.

11. Function μ and iteration σ_n

Two expressions of μ in terms of σ_n are summarized in

Proposition. *For* $0 < r < 1$,

$$\mu(r) = \log\frac{1}{r} + \sum_{n=0}^{+\infty} 2^{-n}\log(1+\sigma_n(\sqrt{1-r^2})), \tag{11.1}$$

$$\mu(r) = \frac{\pi}{2}\prod_{n=0}^{\infty}\frac{1+\sigma_n(\sqrt{1-r^2})}{1+\sigma_n(r)}. \tag{11.2}$$

The expansion (11.1) can be read about in [QV, p. 1059, Theorem 1.1]. It will be shown, nevertheless, that (11.1) follows from Gauss's identity explained below.

Proof of the Proposition. Set $a_0(r) = 1$, $b_0(r) = \sqrt{1-r^2}$; and inductively, $a_{n+1}(r) = (a_n(r) + b_n(r))/2$, $b_{n+1}(r) = \sqrt{a_n(r)b_n(r)}$ for $0 < r < 1$ and for $n \geqslant 0$. Then one obtains that

$$b_n(r)/a_n(r) = \sigma_n(\sqrt{1-r^2}) \tag{11.3}$$

for $n \geqslant 0$ and for $0 < r < 1$, which may be proved by making use of the recursion formula $b_n(r)/a_n(r) = \sigma(b_{n-1}(r)/a_{n-1}(r))$ for $n \geqslant 1$.

The Gauss identity [BB, p. 50, (2.5.14)] states that

$$\mu(r) = \log(4/r) + \sum_{n=0}^{+\infty} 2^{-n}\log(a_{n+1}(r)/a_n(r)) \tag{11.4}$$

for $0 < r < 1$; the cited identity of Gauss is the case $a = 1$, $b = \sqrt{1-r^2}$, and $c = r$ in the formula in the second line in [Ga, p. 388]. On the other hand, the recursion formula

$$\frac{a_{n+1}(r)}{a_n(r)} = \frac{a_n(r)}{a_{n-1}(r)}\cdot\frac{1+\sigma_n(\sqrt{1-r^2})}{1+\sigma_{n-1}(\sqrt{1-r^2})}$$

for $n \geqslant 1$ and $0 < r < 1$ following from (11.3) demonstrates that

$$a_{n+1}(r)/a_n(r) = (1 + \sigma_n(\sqrt{1-r^2}))/2. \tag{11.5}$$

Substituting this in (11.4) one obtains (11.1).

To prove (11.2) the celebrated limit formula [BB, p. 5, Theorem 1.1]

$$1/\lim_{n\to\infty} a_n(r) = 1/\lim_{n\to\infty} b_n(r) = (2/\pi)\mathcal{K}(r)$$

due to Gauss should be recalled. Meanwhile, the expression

$$a_n(r) = 2^{-n} \prod_{k=0}^{n-1} (1 + \sigma_k(\sqrt{1-r^2})) \tag{11.6}$$

for $n \geqslant 2$ and $0 < r < 1$, immediately follows from (11.5), which, together with the Gauss limit formula for $\mathcal{K}$, proves that

$$\mathcal{K}(r) = \frac{\pi}{2} \prod_{n=0}^{\infty} \frac{2}{1 + \sigma_n(\sqrt{1-r^2})}. \tag{11.7}$$

Hence (11.2) follows. Formula (11.7) is equivalent to [BB, p. 14, Algorithm 1.1, (a)] on replacing k_0 with $\sqrt{1-r^2}$ there. □

Incidentally, (11.6), combined with (11.3), shows that

$$b_n(r) = 2^{-n} \sigma_n(\sqrt{1-r^2}) \prod_{k=0}^{n-1} (1 + \sigma_k(\sqrt{1-r^2}))$$

for $n \geqslant 2$ and $0 < r < 1$.

It would be interesting that, as a consequence of (11.3), the function $\sigma_n(4e^{-2^n x})$ which appears in (3.1) is the quotient $b_n(\sqrt{1 - 16e^{-2^{n+1}x}})/a_n(\sqrt{1 - 16e^{-2^{n+1}x}})$ for $n \geqslant 1$ and for $x > 2^{1-n}\log 2$.

Since

$$1 + \sigma_n(\sqrt{1-r^2}) = 2\sigma_n(\sqrt{1-r^2})^{1/2} \sigma_{n+1}(\sqrt{1-r^2})^{-1},$$

it follows from (11.1) that

$$\mu(r) = \log \frac{4(1-r^2)}{r} - (3/2) \sum_{n=0}^{+\infty} 2^{-n} \log \sigma_n(\sqrt{1-r^2}). \tag{11.8}$$

Substituting $b_n(r)/a_n(r)$ instead of $\sigma_n(\sqrt{1-r^2})$ in (11.8) which is possible by (11.3) one has the Jacobi expansion [BB, p. 52, (2.5.15)] which is equivalent to

$$\mu(r) = \log(4\sqrt[4]{1-r^2}/r) + (3/2) \sum_{n=1}^{+\infty} 2^{-n} \log(a_n(r)/b_n(r)) \tag{11.9}$$

for $0 < r < 1$. One can reverse this procedure, so that the Jacobi expansion (11.9) follows from the Gauss expansion (11.4), and *vice versa.*

Setting $r = 1/\sqrt{2}$ in (11.1) one has

$$\sum_{n=0}^{+\infty} 2^{-n} \log(1 + \psi_{-n}) = \pi/2 - \log\sqrt{2} = 1.2242\ldots$$

and setting $r = 1/\sqrt{2}$ in (11.8) one further has has

$$\sum_{n=0}^{+\infty} 2^{-n} \log\psi_{-n} = (2/3)\log(2\sqrt{2}) - \pi/3 = -0.35405\ldots.$$

Upper and lower bounds of $\mu(r)$ can here be studied. The expression (11.1) is transformed into

$$\mu(r) = \log\frac{2(1+\sqrt{1-r^2})}{r} + \sum_{n=1}^{+\infty} 2^{-n}\log\frac{1+\sigma_n(\sqrt{1-r^2})}{2},$$

whence for $n \geqslant 1$,

$$\begin{aligned}
(11.10)\qquad \mu(r) &< \log\frac{2(1+\sqrt{1-r^2})}{r} + \sum_{k=1}^{n} 2^{-k}\log\frac{1+\sigma_k(\sqrt{1-r^2})}{2} \\
&\leqslant \log\frac{2(1+\sqrt{1-r^2})}{r} + 2^{-1}\log\frac{1+\sigma_1(\sqrt{1-r^2})}{2} \\
&= \log\{2^{1/2}r^{-1}(1+\sqrt{1-r^2})^{1/2}(1+\sqrt[4]{1-r^2})\} \\
&< \log\{2r^{-1}(1+\sqrt{1-r^2})\} < \log(4/r).
\end{aligned}$$

Furthermore, the expression (11.1) is equivalent to

$$\mu(r) = \log\frac{(1+\sqrt{1-r^2})^2}{r} + \sum_{n=1}^{+\infty} 2^{-n}\log\frac{1+\sigma_n(\sqrt{1-r^2})}{1+\sqrt{1-r^2}}.$$

On the other hand, since $\sigma(r) > r$ for $0 < r < 1$, it follows that $\sigma_n(\sqrt{1-r^2}) > \sqrt{1-r^2}$, where $n \geqslant 1$. Hence for $n \geqslant 1$,

$$\begin{aligned}
(11.11)\qquad \mu(r) &> \log\frac{(1+\sqrt{1-r^2})^2}{r} + \sum_{k=1}^{n}\frac{1}{2^k}\log\frac{1+\sigma_k(\sqrt{1-r^2})}{1+\sqrt{1-r^2}} \\
&> \log\frac{(1+\sqrt{1-r^2})^2}{r}.
\end{aligned}$$

Let us treat the case $n = 2$ in (11.11). Since

$$\frac{1+\sigma_2(r)}{1+r} = \left(\frac{1+\sqrt{\sigma(r)}}{1+\sigma(r)}\right)^2\left(\frac{1+\sqrt{r}}{1+r}\right)^2,$$

it follows that

$$\frac{1}{4}\log\frac{1+\sigma_2(\sqrt{1-r^2})}{1+\sqrt{1-r^2}} = \frac{1}{2}\log\frac{1+\sigma(\sqrt{1-r^2})^{1/2}}{1+\sigma(\sqrt{1-r^2})} + \frac{1}{2}\log\frac{1+\sqrt[4]{1-r^2}}{1+\sqrt{1-r^2}}$$

for $0 < r < 1$. Consequently,

(11.12)

$$\begin{aligned}\mu(r) &> \log\frac{(1+\sqrt{1-r^2})(1+\sqrt[4]{1-r^2})^{1/2}}{r} + \frac{1}{2}\log(1+\sigma(\sqrt{1-r^2})^{1/2}) \\ &> \log\frac{(1+\sqrt{1-r^2})(1+\sqrt[4]{1-r^2})}{r} > \log\frac{(1+\sqrt{1-r^2})^2}{r}.\end{aligned}$$

Both inequalities in (4.1) are actually established with the aid of a conformal mapping in [H, p. 318] and [LV, p. 61]. On the other hand, improvements of (4.1) are obtained by (11.10) and (11.12) both of which follow essentially from (11.4) due to Gauss.

12. Nine remarks

The following remarks might serve for further studies.

Remark 1. Let $\mathcal{F}_K$ be the family of K-quasiconformal mappings f from $\mathbb{C}$ onto $\mathbb{C}$ with $f(0) = f(1) - 1 = 0$, $K \geqslant 1$. Set $P_2(t,K) = \sup_{f\in\mathcal{F}_K}\max_{|z|=t}|f(z)|$ for $t > 0$. S. Agard established in [A, p. 10, (3.11)] that $P_2(t,K) = \lambda(K,t)$ for $t \geqslant 1$. Although Agard assumes that $t \geqslant 1$, this is also true for $0 < t < 1$. In reality, it is verified that $\lambda(K,t) = \max_{f\in\mathcal{F}_K}\max_{|z|=t}|f(z)|$ for all $t > 0$; see [Y2, Theorem 1]. Let $\mathcal{G}_S$ be the family of functions f holomorphic in D with $f(D) \subset \mathbb{C}\setminus\{0,1\}$. For $t > 0$ let $\mathcal{G}_{S,t}$ be the family of $f \in \mathcal{G}_S$ with $|f(0)| = t$. Martin [Ma, Theorem 1.1] claims that $\sup_{f\in\mathcal{G}_{S,t}}|f(z)| = P_2(t,(1+|z|)/(1-|z|))$ for $z \in D$. Since $1/f \in \mathcal{G}_{S,1/t}$ for $f \in \mathcal{G}_{S,t}$, it follows that $\inf_{f\in\mathcal{G}_{S,t}}|f(z)| = 1/\lambda(K,1/t)$ for $K = (1+|z|)/(1-|z|)$ with $z \in D$.

For extensive treatment of $\lambda(K,t)$ which is defined even for $t < 0$, see [KY]; the starting definition of $\lambda(K,t)$ in [KY] is different but natural and it coincides with $S(KT(t))$ for $t > 0$. Also the function $\nu(K,t)$ for real t is defined in [KY]; in particular, $\nu(K,t) = S(T(t)/K) = 1/\lambda(K,1/t)$ for $t > 0$.

Remark 2. Obviously $\chi(\pi/2) = 1/\sqrt{2}$ and $S(\pi) = 1$. First, for $x > 0$,

$$\chi(x) = \sqrt{1-\chi(4^{-1}\pi^2/x)^2}. \tag{12.1}$$

For the proof, let us set $r = \chi(x)$ in the formula $\pi^2/4 = \mu(r)\mu(\sqrt{1-r^2})$ which directly follows from the definition of μ. Analogously,

$$S(x) = S(\pi^2/x)^{-1} \tag{12.2}$$

for $x > 0$. For the proof, replace x with $x/2$ in (12.1) and eliminate χ to have the equality only for S, from which (12.2) follows. One then has $T(t)T(t^{-1}) = \pi^2$

for $t > 0$. Consequently, $S(\kappa^{-1}T(t^{-1})) = S(\kappa T(t))^{-1}$, whence it follows that

$$\begin{aligned}\eta_\kappa(t) &\equiv (\varphi_\kappa(\sqrt{t/(1+t)})/\varphi_{1/\kappa}(1/\sqrt{1+t}))^2 \\ &= (S(\kappa T(t))+1)/(S(\kappa^{-1}T(t^{-1}))+1) = S(\kappa T(t))\end{aligned}$$

for $\kappa > 0$ and $t > 0$, where $\varphi_\kappa(r) = \chi(\kappa^{-1}\mu(r))$ for $\kappa > 0$ and $0 < r < 1$.

Let us be concerned with the case $0 < x \leqslant \pi$ for $S(x)$. First, (2.6) reads that

$$S(x) = 16^{-1}e^x - 2^{-1} + (1 + \Delta_{S,1}(x))e^{-x}$$

for $x \geqslant \pi$. Hence, for $0 < x \leqslant \pi$, one has

$$S(x)^{-1} = S(\pi^2/x) = 16^{-1}e^{\pi^2/x} - 2^{-1} + (1+\Delta_{S,1}(\pi^2/x))e^{-\pi^2/x} \quad \text{with} \quad 0 < \Delta_{S,1}(\pi^2/x) < (1+\sqrt{1-16e^{-2\pi}})^{-1}.$$

A consequence is that $\lim(S(x)e^{\pi^2/x}) = 16$ as $x \to +0$.

Remark 3. Recall that $\mu(1) = 0$. Hence $0 \leqslant \mu(r) + \alpha_n \log \omega_n(r) < \alpha_n \log 4$ for all $r \in (0,1]$ by (4.2). Consequently, the sequence of functions $-\alpha_n \log \omega_n$ converges to μ as $n \to +\infty$ uniformly on $(0,1]$. The k-th derivative of $-\alpha_n \log \omega_n$, therefore, converges to $\mu^{(k)}$ uniformly on each closed interval $[p,q] \subset (0,1)$. Particularly, $-\alpha_n \omega_n'/\omega_n \to \mu'$. It then follows from (11.7) and (9.11) that

$$2^{n/2}(r(1-r^2))^{-1/2}(\omega_n(r)/\omega_n'(r))^{1/2} \to (2/\pi)\mathcal{K}(r) = \prod_{n=0}^{\infty} \frac{2}{1+\sigma_n(\sqrt{1-r^2})}$$

as $n \to +\infty$ uniformly on every closed interval $[p,q] \subset (0,1)$.

An exercise is to prove that $-2^{-n}\log \psi_n \to \pi/2$ as $n \to +\infty$.

Remark 4. Let $\beta \neq 0$ and $\beta \geqslant -2$. For each $p > 0$, the function $\sigma_n(4e^{-2^n x})^\beta$ in (3.1) uniformly converges to $\chi(x)^\beta$ as $n \to +\infty$ on the interval $[p, +\infty)$. Actually, let us choose $N \geqslant 1$ such that $p > 2^{1-N}\log 2$, so that $2^{N+1} > 2 \geqslant -\beta$. Then, for all $n > N$, and for all $x \in [p, +\infty)$, it follows from (3.3) that $|\chi(x)^\beta - \sigma_n(4e^{-2^n x})^\beta| < |\Delta_{n,\beta}(x)| < |\beta| 2^{2\beta - n + 4}$; the right-most tends to 0 as $n \to +\infty$. Since $\chi(x)^\beta$ and $\sigma_n(4e^{-2^n x})^\beta$ both are real-analytic in $(0, +\infty)$, the k-th derivative of $\sigma_n(4e^{-2^n x})^\beta$ converges to that of $\chi(x)^\beta$ uniformly on each $[p, +\infty)$, $p > 0$. A conjecture is that the conclusion were valid for all $\beta \neq 0$.

The function $\sigma(r)$ is d-increasing and d-concave for $0 < r < 1$, so that the same is true of $\sigma_n(r)$, and furthermore, of $\log \sigma_n(r)$. For a constant $\beta < 0$ the function $\sigma_n^\beta = \exp(\beta \log \sigma_n)$ is therefore d-decreasing and d-convex in $(0,1)$. Since $4\exp(-2^n x)$ is d-decreasing and d-convex for $x > 0$, the function $\sigma_n(4e^{-2^n x})^\beta$, with a constant $\beta < 0$, is d-increasing and d-convex for $x > 2^{2-n}\log 2$. As was observed in Section 7, the function $\chi(x)^\beta$ with $\beta < 0$ is d-increasing and d-convex for $x > 0$.

Remark 5. The constant $\sigma_n(\sqrt{2})$ in (2.4), (3.4), and (3.6) can be replaced with any algebraic number N_a satisfying

$$\sigma_n(\sqrt{2}) < N_a < \chi(2^{1-n}\log 2).$$

Obviously N_a becomes better as N_a becomes nearer to $\chi(2^{1-n}\log 2)$. For a rational number $p > 0$ there exists a unique algebraic number k_p with $0 < k_p < 1$ and $\mu(k_p) = \pi\sqrt{p}/2$ ([BB, p. 139 *et seqq.*] and [BB, p. 156]). If a natural number m is found so that

$$\log 2 < 2^{-m-1}\pi\sqrt{p} < \pi/4, \tag{12.3}$$

or equivalently, if $\log 2 < \mu(\sigma_m(k_p)) < \pi/4$, then $N_a = \sigma_{n+m-1}(k_p)$ will do. Actually, the inequality $\alpha_{n-1}\log 2 < \mu(\sigma_{n+m-1}(k_p))$ implies that $\sigma_{n+m-1}(k_p) < \chi(\alpha_{n-1}\log 2)$. On the other hand, $\mu(\sigma_n(\sqrt{2})) = \alpha_{n-1}\mu(\sigma(\sqrt{2})) = \alpha_{n-1}\pi/4 > \alpha_{n-1}\mu(\sigma_m(k_p)) = \mu(\sigma_{n+m-1}(k_p))$, whence $\sigma_n(\sqrt{2}) < \sigma_{n+m-1}(k_p)$.

The algebraic number $\sigma(\sqrt{2}) = 0.98517\ldots$ appearing in (6.15) may be replaced with $\sigma_m(k_p) > \sigma(\sqrt{2})$ for m and p satisfying (12.3).

Let ε be rational with $0 < \varepsilon < 64(1-(4\pi^{-1}\log 2)^2) = 14.151\ldots$. Then, (12.3) is true for $m = 4$ and $p = 64-\varepsilon$. For instance, $\varepsilon = 6$ will do for which $k_{58} = (13\sqrt{58}-99)(\sqrt{2}-1)^6$ by $k_{58} = \lambda^*(58)$ in [BB, p. 299, Exercise 9.d).iii)]. Here $\mu(\sigma_4(k_{58})) = \pi\sqrt{58}/32 = 0.75409\ldots$.

Suppose that $t \geqslant \sigma_4(k_{64-\varepsilon})^{-2} - 1$. Then $\mu(1/\sqrt{1+t}) \geqslant \mu(\sigma_4(k_{64-\varepsilon})) = \pi\sqrt{64-\varepsilon}/32$, so that $L(2,t) \geqslant \exp(\pi\sqrt{64-\varepsilon}/16)$. It then follows from (6.13) that

$$\begin{aligned}\frac{5}{4} - \frac{4}{e^{\pi\sqrt{64-\varepsilon}/8}+4} &< \delta_{LVV}(K,t)\exp\{2K\mu(1/\sqrt{1+t})\} \\ &< \frac{5}{4} + \frac{e^{-\pi\sqrt{64-\varepsilon}/8}}{2(1+\sqrt{1-16e^{-\pi\sqrt{64-\varepsilon}/4}})}\end{aligned}$$

for t with $\sigma_4(k_{64-\varepsilon})^{-2} - 1 \leqslant t < \sigma(\sqrt{2})^{-2} - 1$.

It is remarkable that there exists k_p with $p \neq 64-\varepsilon$ for which $\log 2 < \mu(\sigma_m(k_p)) < \pi/4$ with $m \neq 4$, or (12.3) is still valid. Notice that $49 < 64-\varepsilon < 64$.

As a first example, let us choose k_{13} which satisfies the equation $4k_{13}^2(1-k_{13}^2) = G_{13}^{-24} = 649-180\sqrt{13}$; see [BB, p. 172, Table 5.2a] where $G_N^{-12} = 2k_N k_N'$. Calculation with the aid of [BB, p. 161, Exercise 2.a).ii)], together with $G_{13}^{-12} = 5\sqrt{13}-18$, then reveals that $k_{13} = 2^{-1}(\sqrt{5\sqrt{13}-17} - \sqrt{19-5\sqrt{13}}) = 0.01387\ldots$ and $\mu(\sigma_3(k_{13})) = \pi\sqrt{13}/2^4 = 0.70794\ldots$, so that (12.3) is valid for $m = 3$.

Another example for large p is $\sigma_5(k_{210}) = 0.99266\ldots$ for S. Ramanujan's celebrated

$$\begin{aligned}k_{210} &= (\sqrt{2}-\sqrt{1})^2(\sqrt{4}-\sqrt{3})(\sqrt{7}-\sqrt{6})^2(\sqrt{10}-\sqrt{9})^2 \\ &\quad\times(\sqrt{15}-\sqrt{14})(\sqrt{16}-\sqrt{15})^2(\sqrt{36}-\sqrt{35})(\sqrt{64}-\sqrt{63}) \\ &= 10^{-10}\times 5.2025\ldots\end{aligned}$$

because $\log 2 < \mu(\sigma_5(k_{210})) = \pi\sqrt{210}/2^6$ ($= 0.71134\ldots$) $< \pi/4$; see [BB, p. 141, (4.6.12)] for k_{210}. Since $\mu(\sigma_3(k_{13})) = \pi\sqrt{208}/2^6 < \pi\sqrt{210}/2^6 = \mu(\sigma_5(k_{210}))$, it exactly follows that $\sigma_3(k_{13}) > \sigma_5(k_{210})$.

Finally, for the non-integer $31/2$ one has $\mu(\sigma_3(k_{31/2})) = \pi 2^{-4}\sqrt{31/2} = 0.77302\ldots$, so that $p = 31/2$ with $m = 3$ is an example.

Remark 6. For a fixed $K \geqslant 1$ the functions $\delta_{LVV}(K,t)$ and $\zeta_K(t)$ in (6.5) and (6.6), respectively, are functions of $t > 0$. Set $\Delta(K,t) = \delta_{LVV}(K,t)\zeta_K(t)^{-1} - 1$. Then

$$\lambda(K,t) = 16^{-1}\zeta_K(t)^{-1} - 2^{-1} + \zeta_K(t) + \Delta(K,t)\zeta_K(t). \tag{12.4}$$

For $\Delta(K,t)$ one observes in [KY, Theorem 6.2, (6.7), (6.6)] that

$$0 < \Delta(K,t) < 8 \tag{12.5}$$

for $t \geqslant t_o \equiv S(K^{-1}\log 4)$, or equivalently, $K \geqslant T(t)^{-1}\log 4$, whereas

$$-5/2 < \Delta(K,t) < 5/2 \tag{12.6}$$

for $0 < t < t_o$, or equivalently, $K < T(t)^{-1}\log 4$.

Set $n = 1$ and $x = 2K\mu(1/\sqrt{1+t}) = -\log\zeta_K(t)$ in Theorem 1. Then Formula (2.1) in this case is exactly Formula (12.4) with $\Delta_{S,1}(x) = \Delta(K,t)$. It then follows from (2.3) that $0 < \Delta(K,t) < 1$ for $t \geqslant t_o$, a result better than (12.5). On the other hand, it follows from (2.4) that

(12.7)
$$-0.5625 = 1 - \sigma(4)^{-2} < \Delta(K,t) < 4(\sigma(\sqrt{2})^{-2} - 1) = 3/\sqrt{2} - 2 = 0.12132\ldots$$

for $0 < t \leqslant t_o$. Estimation (12.6) is thus improved in (12.7).

One can replace $\sigma(\sqrt{2})$ in (12.7) with $\sigma_4(k_{64-\varepsilon})$; see Remark 5. One cannot set $t = 1$ in (12.7) because $T(1)^{-1}\log 4 = \pi^{-1}\log 4 = 0.44127\ldots < 1$. Hence (12.7) does not serve for estimating $\delta_{LVV}(K)e^{\pi K} = \Delta(K,1) + 1$.

Finally, (6.14) yields that $0.05 < \Delta(K,t) < (6 - \sqrt{15})/8 = 0.2658\ldots$ for t with $t > S(\log 4) \geqslant t_o$.

Remark 7. Particular values of $\lambda(K,t)$ and $\varphi_K(r)$ for $K \geqslant 1$ are obtained:

$$\lambda(2^m\sqrt{p/q},\ \sigma_{n+m}(k_q)^{-2} - 1) = \sigma_n(k_p)^{-2} - 1;$$
$$\varphi_K(\sigma_n(k_p)) = \sigma_{n+m}(k_q), \qquad K = 2^m\sqrt{p/q},$$

where p and q are rational numbers with $0 < q \leqslant p$ and n and m are integers with $m \geqslant 0$. First, $\lambda(K, r_2^{-2} - 1) = r_1^{-2} - 1$ and $\varphi_K(r_1) = r_2$ for $0 < r_1 \leqslant r_2 < 1$ and $K = \mu(r_1)/\mu(r_2)$. Next, $\mu(\sigma_n(k_p))/\mu(\sigma_{n+m}(k_q)) = 2^m\sqrt{p/q}$ for rational numbers p, q with $0 < q \leqslant p$, and for integers n and m with $m \geqslant 0$. On the other hand, it follows from $k_p \leqslant k_q$ that $r_1 \equiv \sigma_n(k_p) \leqslant \sigma_n(k_q) \leqslant \sigma_{n+m}(k_q) \equiv r_2$. Hence the requested formulae follow.

Remark 8. Identity (7.9) can be rewritten as

$$q\Xi_4(q) = \pi^{-2}\mathcal{K}(r)(\mathcal{E}(r) - \mathcal{K}(r)),$$

for $r = \theta_2(q)^2\theta_3(q)^{-2}$ with $0 < q < 1$, which, combined with (7.16) and (8.10), yields

$$q\Xi_2(q) = \pi^{-2}\mathcal{K}(r)\mathcal{E}(r),$$

whereas, combined with (7.11) and (8.11), yields

$$q\Xi_3(q) = \pi^{-2}\mathcal{K}(r)(\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)).$$

Since

$$(1-r^2)(\mathcal{K}(r) - \mathcal{E}(r)) < \mathcal{E}(r) - (1-r^2)\mathcal{K}(r) < \mathcal{K}(r) - \mathcal{E}(r)$$

for $0 < r < 1$ ([AVV2, p. 53, Theorem 3.21, (6)]), it follows from $1 - r^2 = \theta_4(q)^4\theta_3(q)^{-4}$ that

$$(0 <) - \theta_4(q)^4\theta_3(q)^{-4}\Xi_4(q) < \Xi_3(q) < -\Xi_4(q)$$

and since

$$4^{-1}\pi^2 < \mathcal{E}(r)\mathcal{K}(r) < 4^{-1}\pi^2(1-r^2)^{-1/4}$$

for $0 < r < 1$ ([AVV2, p. 62, Theorem 3.31, (1)]), it follows further that

$$4^{-1} < q\Xi_2(q) < 4^{-1}\theta_4(q)^{-1}\theta_3(q)$$

for $0 < q < 1$.

Remark 9. The doubly connected domain which is the plane $\mathbb{C}$ slit along the interval $(-\infty, 0]$ and the circular arc $\{e^{i\theta};\ |\theta| \leqslant \alpha\}$ for $0 < \alpha < \pi$ can be mapped conformally onto the ring domain $\{z;\ 1 < |z| < \exp\mu(\sin(\alpha/2))\}$. Calculation with the aid of [AVV2, p. 82, (5.9)] yields that

$$(d^2/d\alpha^2)\mu(\sin(\alpha/2)) = 16^{-1}\pi^2 r^{-2}(1-r^2)^{-1}\mathcal{K}(r)^{-3}(2\mathcal{E}(r) - \mathcal{K}(r))$$

for $r = \sin(\alpha/2)$. Since

$$2\mathcal{E}(r) - \mathcal{K}(r) = \int_0^{\pi/2} \frac{1 - 2r^2\sin^2\theta}{\sqrt{1 - r^2\sin^2\theta}}\,d\theta,$$

it follows that $2\mathcal{E}(r) - \mathcal{K}(r) > 0$ for $0 < r \leqslant 1/\sqrt{2}$. Consequently, $\mu(\sin(\alpha/2))$ is d-decreasing and d-convex as a function of α, $0 < \alpha < \pi/2$. For $0 < \alpha < \pi/2$, the described doubly connected domain is known as Mori's extremal domain. See [Mo] and [LV, p. 59].

Department of Mathematics
Tokyo Metropolitan University
Minami-Osawa 1-1, Hachioji
Tokyo 192-0397, Japan
e-mail: yamashin@comp.metro-u.ac.jp

References

[A] S. Agard, *Distortion theorems for quasiconformal mappings*, Ann. Acad. Sci. Fenn. Ser. A I. Math. **413** (1968), 1–12.

[AVV1] G. D. Anderson, M. K. Vamanamurthy, and M. Vuorinen, *Distortion functions for plane quasiconformal mappings*, Israel J. Math. **62** (1988), 1–16.

[AVV2] G. D. Anderson, M. K. Vamanamurthy and M. K. Vuorinen, *Conformal invariants, inequalities, and quasiconformal maps*, John Wiley & Sons, Inc., New York *et al.*, 1997.

[BB] J. M. Borwein and P. B. Borwein, *Pi and the AGM*, John Wiley & Sons, New York *et al.*, 1987.

[Ga] C. F. Gauss, *Werke,* vol. III, Königl. Gesel. Wissens. Göttingen, 1866, reprinted by Georg Olms, Hildesheim & New York, 1973.

[Gr] H. Grötzsch, *Über einige Extremalprobleme der konformen Abbildung*, Ber. Verh. Säcs. Akad. Wiss. Leibzig **80** (1928), 367–376.

[Hm] J. A. Hempel, *The Poincaré metric on the twice punctured plane and the theorems of Landau and Schottky*, J. London Mth. Soc. **20** (1979), 435–445.

[Hr] J. Hersch, *Longueurs extrémales et théorie des fonctions*, Comm. Math. Helv. **29** (1955), 301–337.

[J] C. G. J. Jacobi, *Fundamenta nova theoriae functionum ellipticarum*, Bornträger, Königsberg, 1829. (pp. 49–239 *in* C. G. J. Jacobi's Gesammelte Werke, vol. I, edited by C. W. Borchardt, Berlin, 1881; second edition by Chelsea, New York, 1969)

[KY] S. Kurihara and S. Yamashita, *Extremal functions for plane quasiconformal mappings*, J. Math. Kyoto Univ. **42** (2002), 71–99.

[LV] O. Lehto and K. I. Virtanen, *Quasiconformal mappings in the plane*, Springer-Verlag, Berlin-Heidelberg-NewYork, 1973.

[LVV] O. Lehto, K. I. Virtanen and J. Väisälä, *Contributions to the distortion theory of quasiconformal mappings*, Ann. Acad. Sci. Fenn. Ser. A I. Math. **273** (1959), 1–14.

[Ma] G. J. Martin, *The distortion theorem for quasiconformal mappings, Schottky's theorem and holomorphic motions*, Proc. Amer. Math. Soc. **125** (1997), 1095–1103.

[Mo] A. Mori, *On an absolute constant in the theory of quasi-conformal mappings*, J. Math. Soc. Japan **8** (1956), 156–166.

[N] Z. Nehari, *Conformal mapping*, McGraw-Hill, NewYork, 1952.

[QV] S.-L. Qiu and M. Vuorinen, *Infinite products and normalized quotients of hypergeometric functions*, SIAM J. Math. Anal. **30** (1999), 1057–1075.

[T] O. Teichmüller, *Untersuchungen über konforme und quasikonforme Abbildung*, Deutsche Mathematik **3** (1938), 621–678. (*in* pp. 205–262 *in* "Gesammelte Abhandlungen", edited by L. A. Ahlfors and F. W. Gehring; Springer-Verlag, Berlin-Heidelberg-New York, 1982)

[WW] E. T. Whittaker and G. N. Watson, *Modern analysis*, 4th edition (1927), Cambridge Univ. Press, reprinted (1992), Cambridge Univ. Press, Cambridge-NewYork-Oakleigh.

[Y1] S. Yamashita, *The derivative of a holomorphic function and estimates of the Poincaré density*, Kodai Math. J. **15** (1992), 102–121.

[Y2] ———, *Extremals for families of plane quasiconformal mappings*, Nihonkai Math. J. **14** (2003), 31–42.