Necessary and sufficient conditions such that extended mean values are Schur-convex or Schur-concave

By

Yuming CHU and Xiaoming ZHANG

Abstract

In the article, established are necessary and sufficient conditions such that the extended mean values are Schur-convex and Schur-concave.

1. Introduction

The histories of mean values and inequalities are long [3]. The mean values are related to the Mean Values Theorems for derivative or for integral, which are the bridge between the local and global properties of functions (cf. [4]). The arithmetic-mean-geometric-mean inequality is probably the most important inequality, and certainly a keystone of the theory of inequalities [1]. Inequalities of mean values are one of the main parts of theory of inequalities, they have explicit geometric meanings [4]. The theory of mean values plays an important role in the whole mathematics, since many norms in mathematics are always means (cf. [4]).

In 1975, the extended mean values $E(r, s; x, y)$ were defined in [13] by K. B. Stolarsky as follows.

\[
E(r, s; x, y) = \left( \frac{r}{s} \cdot \frac{y^s - x^s}{y^r - y^r} \right)^{\frac{1}{r-s}}, \quad rs(r-s)(x-y) \neq 0;
\]

\[
E(r, 0; x, y) = \left( \frac{1}{r} \cdot \frac{y^r - x^r}{\log y - \log x} \right)^{\frac{1}{r}}, \quad r(x-y) \neq 0;
\]

\[
E(r, r; x, y) = \left( \frac{x^r}{y^r} \right)^{\frac{1}{r-s}}, \quad r(x-y) \neq 0;
\]

\[
E(0, 0; x, y) = \sqrt{xy}, \quad x \neq y;
\]

\[
E(r, s; x, y) = x, \quad x = y.
\]
Here \( x, y > 0 \) and \( r, s \in \mathbb{R} \).

It is easy to see that the extended mean values \( E(r, s; x, y) \) are continuous on the domain \( \{(r, s; x, y) \mid r, s \in \mathbb{R}; x, y > 0\} \).

They are of symmetry between \( r \) and \( s \) and between \( x \) and \( y \).

Many basic properties have been researched by E. B. Leach and M. C. Sholander in [6].

Study of \( E(r, s; x, y) \) is not only interesting but also important, because most of the two-variables mean values are special cases of \( E(r, s; x, y) \) and it is challenging to study a function whose formulation is so indeterminate [8].

Let \( \Omega \subseteq \mathbb{R}^n \) be a symmetric convex set with nonempty interior. A real-valued function \( f \) on \( \Omega \) is called a Schur-convex function if \( f(x) \leq f(y) \) for each pair \( (x, y) \) of two elements \( x, y \) such that \( x \prec y \), i.e.,

\[
\sum_{i=1}^{k} x[i] \leq \sum_{i=1}^{k} y[i], \quad \sum_{i=1}^{n} x[i] = \sum_{i=1}^{n} y[i],
\]

where \( 1 \leq k \leq n - 1 \) and \( x[i] \) denotes the \( i \)-th largest component in \( x \).

A real-valued function \( f \) is called Schur-concave if \( -f \) is Schur-convex.

The theory of Schur-convex functions is one of the most important theory in the fields of inequalities. It can be used in combinatorial optimization [5], isoperimetric problem for polytopes [14], linear regression [12], graphs and matrices [2] and other related fields.

The Schur-convexity of the extended mean values \( E(r, s; x, y) \) with respect to \( (r, s) \) and \( (x, y) \) are investigated in [9], [10], and [11]. F. Qi first obtained the following result in [9].

**Theorem A.** For fixed \( (x, y) \in (0, \infty) \times (0, \infty) \) with \( x \neq y \), the extended mean values \( E(r, s; x, y) \) are Schur-concave on \( [0, +\infty) \times [0, +\infty) \) and Schur-convex on \( (-\infty, 0] \times (-\infty, 0) \) with respect to \( (r, s) \).

In [10], F. Qi, J. Sándor, S. S. Dragomir and A. Sofo tried to obtain the Schur-convexity of the extended mean values \( E(r, s; x, y) \) with respect to \( (x, y) \) for fixed \( (r, s) \) and declared an incorrect conclusion as follows: For given \( (r, s) \) with \( r, s \notin \left(0, \frac{1}{2}\right) \) (or \( r, s \in (0, 1) \), resp.), the extended mean values \( E(r, s; x, y) \) are Schur-concave (or Schur-convex, resp.) with respect to \( (x, y) \) on \( (0, \infty) \times (0, \infty) \). H.-N. Shi, Sh.-H. Wu and F. Qi observed that the above conclusion is wrong and obtained the following Theorem B in [11].

**Theorem B.** For fixed \( (r, s) \in \mathbb{R}^2 \),

1. if \( 2 < 2r < s \) or \( 2 \leq 2s \leq r \), then the extended mean values \( E(r, s; x, y) \) are Schur-convex with respect to \( (x, y) \in (0, \infty) \times (0, \infty) \);

2. if \( (r, s) \in \{r < s \leq 2r; 0 < r \leq 1\} \cup \{s < r \leq 2s; 0 < s \leq 1\} \cup \{0 < s < r \leq 1\} \cup \{0 < r < s \leq 1\} \cup \{s \leq 2r < 0\} \cup \{r \leq 2s < 0\} \), then the extended mean values \( E(r, s; x, y) \) are Schur-concave with respect to \( (x, y) \in (0, \infty) \times (0, \infty) \).

The main purpose of this article is to establish the necessary and sufficient conditions such that the extended mean values \( E(r, s; x, y) \) are Schur-convex
or Schur-concave with respect to \((x, y)\) for fixed \((r, s)\). Our main result is the following.

**Theorem 1.1.** For fixed \((r, s) \in \mathbb{R}^2\),

1. the extended mean values \(E(r, s; x, y)\) are Schur-convex with respect to \((x, y) \in (0, \infty) \times (0, \infty)\) if and only if \((r, s) \in \{s \geq 1, r \geq 1, s + r \geq 3\}\);
2. the extended mean values \(E(r, s; x, y)\) are Schur-concave with respect to \((x, y) \in (0, \infty) \times (0, \infty)\) if and only if \((r, s) \in \{r \leq 1, s + r \leq 3\} \cup \{s \leq 1, s + r \leq 3\}\).

2. Lemmas

In this section we introduce and establish several lemmas, which are used in the proof of Theorem 1.1.

**Lemma 2.1 ([7]).** Let \(A \subseteq \mathbb{R}^n\) be a symmetric convex set with non-empty interior \(\text{int} A\), \(\varphi : A \to \mathbb{R}\) is a continuous symmetric function on \(A\). If \(\varphi\) is differentiable on \(\text{int} A\), then \(\varphi\) is Schur-convex on \(A\) if and only if

\[
(x_i - x_j) \left( \frac{\partial \varphi}{\partial x_i} - \frac{\partial \varphi}{\partial x_j} \right) \geq 0
\]

for all \((x_1, x_2, \ldots, x_n) \in \text{int} A\) and \(i, j = 1, 2, \ldots, n\) with \(i \neq j\).

**Lemma 2.2.** Let \(s, r \in \mathbb{R}, s \neq 0\) and \(f(t) = \frac{t}{s}(s - r)(s + r - 1) - s(t^s - 1) - r(t^r - 1)\). Then the following statements hold.

(a) If \(s > r \geq 1\) and \(s + r - 3 \geq 0\), then \(f(t) \geq 0\) for \(t \in [1, \infty)\);
(b) if \(s > r > 1\) and \(s + r - 3 < 0\), then there exist \(t_1, t_2 \in (1, \infty)\) such that \(f(t_1) > 0\) and \(f(t_2) < 0\);
(c) if \(r < 1\), \(r \neq 0\) and \(s + r - 3 > 0\), then there exist \(t_3, t_4 \in (1, \infty)\) such that \(f(t_3) > 0\) and \(f(t_4) < 0\);
(d) if \(s > 0, s > r, r < 1\) and \(s + r - 3 \leq 0\), then \(f(t) \leq 0\) for \(t \in (1, \infty)\);
(e) if \(r < s < 0\), then \(f(t) \leq 0\) for \(t \in [1, \infty)\).

**Proof.** (a) Let \(g(t) = t^{2-r} f'(t)\) and \(h(t) = t^{2+r-s} g''(t)\), then simple computation yields

\[
(2.1) \quad f(1) = 0,
\]

\[
(2.2) \quad f'(t) = \frac{r}{s}(s - r)(s + r - 1)t^{s-r-2} - r(s - 1)t^{s-2} + r^2t^{r-1} + \frac{r^2}{s}(r - 1)t^{r-2} - r^2t^{r-1},
\]

\[
(2.3) \quad g(1) = f'(1) = 0,
\]

\[
(2.4) \quad g'(t) = r(s - r)(s + r - 1)t^{s-1} - r(s - 1)(s - r)t^{s-r-1} + r^2 - r^2(s - r + 1)t^{r-r},
\]

\[
(2.5) \quad g'(1) = 0,
\]
\[(2.6)\]
\[g''(t) = r(s - r)(s + r - 1)(s - 1)t^{s-2} - r(s-1)(s-r)(s-r-1)t^{s-r-2} - r^2(s-r+1)(s-r)t^{s-r-1} \]

and

\[(2.7)\]
\[h'(t) = r^2(s-r)(s+r-1)(s-1)t^{s-1} - r^2(s-r+1)(s-r).\]

If \( s > r \geq 1, s+r-3 \geq 0 \), then from (2.6) and (2.7) we see that

\[(2.8)\]
\[h(1) = g''(1) = r^2(s-r)(s+r-3) \geq 0 \]

and

\[(2.9)\]
\[h'(t) \geq h'(1) = r^2s(s-r)(s+r-3) \geq 0 \]

for \( t \geq 1 \). Then Lemma 2.2(a) follows from (2.1)-(2.9).

(b) If \( s > r > 1 \) and \( s+r-3 < 0 \), then \( h'(1) = r^2s(s-r)(s+r-3) < 0 \) by (2.7), this and the continuity of \( h'(t) \) imply that there exists \( \delta_0 > 0 \) such that \( h'(t) < 0 \) for \( t \in [1, 1 + \delta_0] \). Hence \( h(t) \leq h(1) = r^2(s-r)(s+r-3) < 0 \) for \( t \in [1, 1 + \delta_0] \), from (2.1)-(2.5) we clearly see that \( f(t) < 0 \) for \( t \in (1, 1 + \delta_0) \).

On the other hand, it is easy to see that \( \lim_{t \to \infty} f(t) = +\infty \). Hence Lemma 2.2(b) is true.

(c) If \( r < 1, r \neq 0 \) and \( s+r-3 > 0 \), then \( s > r, s > 0 \) and \( h'(1) = r^2(s-r)(s+r-3) > 0 \) by (2.7). The continuity of \( h'(t) \) implies that there exists \( \delta_2 > 0 \) such that \( h'(t) > 0 \) for \( t \in [1, 1 + \delta_2] \), this leads to \( h(t) > h(1) = g''(1) = r^2(s-r)(s+r-3) > 0 \) for \( t \in (1, 1 + \delta_2] \), from (2.1)-(2.5) we see that \( f(t) > 0 \) for \( t \in (1, 1 + \delta_2) \).

On the other hand, it is easy to see that \( \lim_{t \to +\infty} f(t) = -\infty \). Hence Lemma 2.2(c) is true.

(d) If \( s > 0, s > r, r < 1, s+r-3 \leq 0 \) and \( t \in [1, \infty) \). Then we claim that \( h'(t) \leq 0 \), and from this we can get Lemma 2.2(d) by a similar argument as in Lemma 2.2(a). In fact, if \((s+r-1)(s-1) \geq 0\), then clearly (2.7) gives that

\[h'(t) \leq h'(1) = r^2s(s-r)(s+r-3) \leq 0;\]

if \((s+r-1)(s-1) < 0\), then again (2.7) yields that

\[h'(t) \leq -r^2(s-r+1)(s-r) \leq 0.\]

(e) If \( r < s < 0, t \geq 1 \). Let \( f_1(t) = t^{-s-r+1}f(t), f_2(t) = t^{1+s}f_1'(t) \) and
Now, Lemma 2.2(e) follows from (2.10)–(2.17).

(2.10) \[ f_1(1) = f(1) = 0, \]

(2.11) \[ f'_1(t) = -\frac{r}{s}(s - r)(-s - r + 1)t^{-s-r} + r^2 t^{-r-1} \]
\[ + r(-s + 1)t^{-s} - r^2 t^{-s-1} - \frac{r^2}{s}(-r + 1)t^{-r}, \]

(2.12) \[ f_2(1) = f'_2(1) = 0, \]

(2.13) \[ f'_2(t) = -\frac{r}{s}(s - r)(-s - r + 1)(1 - r)t^{-r} + r^2(s - r)t^{s-r-1} \]
\[ + r(-s + 1) - \frac{r^2}{s}(-r + 1)(1 + s - r)t^{-r}, \]

(2.14) \[ f_3(1) = f''_2(1) = 0, \]

(2.15) \[ f''_2(t) = \frac{r^2}{s}(s - r)(-s - r + 1)(1 - r)t^{s-r-2} \]
\[ + r^2(s - r)(s + r - 1)(1 + s - r)t^{s-r-1}, \]

(2.16) \[ f_3(1) = f''_2(1) = r^2(s - r)(s + r - 3) < 0, \]

and

(2.17) \[ f'_3(t) = \frac{r^2}{s}(s - r)(-s - r + 1)(1 - r)(1 - s)t^{s-r} - \frac{r^2}{s}(-r + 1)(1 + s - r)(s - r) \]
\[ \leq \frac{r^2}{s}(s - r)(-s - r + 1)(1 - r)(1 - s) - \frac{r^2}{s}(-r + 1)(1 + s - r)(s - r) \]
\[ = r^2(s - r)(1 - r)(s + r - 3) < 0. \]

Now, Lemma 2.2(e) follows from (2.10)–(2.17).

Lemma 2.3. For \( r \in \mathbb{R} \) and \( t \geq 1 \), let \( h(t) = -r(t^{-1} + t')\log t + (t'^{-1} + 1)(t' - 1) \). If \( 1 < r < \frac{3}{2} \), then there exists \( t_1, t_2 \in (1, \infty) \) such that \( h(t_1) < 0 \) and \( h(t_2) > 0 \).

Proof. For \( t \geq 1 \), let \( h_1(t) = t^{2-r}h'(t) \). If \( 1 < r < \frac{3}{2} \), then simple computation yields

(2.18) \[ h'(t) = -r[(r - 1)t^{-2} + rt'^{-1}] \log t + (2r - 1)t^{2r-2} - (2r - 1)t'^{-2}, \]
\[ h_1(1) = h'(1) = 0, \]

(2.19) \[ h''_1(t) = -\frac{r^2}{t} + \frac{r(r - 1)}{t^2} + r(2r - 1)(r - 1)t'^{-2} \]
and

\[(2.20)\quad h''_1(1) = r^2(2r - 3) < 0.\]

By (2.20) and the continuity of \(h'_1(t)\) we know that there exists \(\delta_1 > 0\) such that \(h''_1(t) < 0\) for \(t \in [1, 1 + \delta_1]\), this together with (2.19) imply that \(h'_1(t) < h'_1(1) = 0\) for \(t \in (1, 1 + \delta_1)\). Then (2.18) and \(h(1) = 0\) lead to \(h(t) < 0\) for \(t \in (1, 1 + \delta_1)\).

On the other hand, it is easy to see that \(\lim_{t \to +\infty} f(t) = +\infty\). This completes the proof of Lemma 2.3. \(\square\)

**Lemma 2.4.** For \(t \geq 1\), let \(f(t) = r(1 + t^{r-1}) \log t - t^r - t^{r-1} + 1 + \frac{1}{r}\). If \(r > 3\), then there exist \(t_1, t_2 \in (1, \infty)\) such that \(f(t_1) > 0\) and \(f(t_2) < 0\).

**Proof.** Let \(g(t) = tf(t)\) and \(h(t) = tg''(t)\). If \(r > 3\), then simple computation yields

\[(2.21)\quad g(1) = f(1) = 0,\]
\[(2.22)\quad g'(1) = 0,\]
\[(2.23)\quad g''(t) = r^2(r - 1) t^{r-2} \log t + (1 + r^2) t^{r-1},\]
\[(2.24)\quad h'(1) = g''(1) = 0,\]

and

\[(2.25)\quad h(t) = r^2(r - 1) t^{r-2} \log t - 2r^2(r - 1) t^{r-2} - r^2(r + 1) t^{r-1},\]

and

\[(2.26)\quad h'(1) = r^2(r - 3) > 0.\]

From (2.26) and the continuity of \(h'(t)\) we see that there exists \(\delta > 0\) such that \(h'(t) > 0\) for \(t \in [1, 1 + \delta]\), this together with (2.21)–(2.25) imply that \(f(t) > 0\) for \(t \in (1, 1 + \delta)\).

On the other hand, it is easy to see that \(\lim_{t \to +\infty} f(t) = -\infty\). This completes the proof of Lemma 2.4. \(\square\)

3. **Proof of Theorem 1.1**

**Proof.** For fixed \(r, s \in R\), it is easy to see that \(E(r, s; x, y)\) is differentiable with respect to \((x, y) \in (0, \infty) \times (0, \infty)\) by the elementary theory of differential and integral calculus. We use Lemma 2.1 to discuss the nonpositivity and nonnegativity of \((y - x)(\frac{\partial^2 E}{\partial y^2} - \frac{\partial^2 E}{\partial x^2})\) for all \((x, y) \in (0, \infty) \times (0, \infty)\) and for fixed \((r, s) \in R^2\). Since \((y - x)(\frac{\partial^2 E}{\partial y^2} - \frac{\partial^2 E}{\partial x^2}) = 0\) for \(x = y\) and \((y - x)(\frac{\partial^2 E}{\partial y^2} - \frac{\partial^2 E}{\partial x^2})\) is symmetric with respect to \(x\) and \(y\), without loss of generality we assume \(y > x\) in the following discussion.
Let

\[ E_1 = \{(r, s) : r \geq 1, s \geq 1, r + s \geq 3\}, \]
\[ E_2 = \{(r, s) : r > 1, s > 1, r + s < 3\} \cup \{(r, s) : r < 1, s + r > 3\} \]
\[ \cup \{(r, s) : s < 1, s + r > 3\} \]

and

\[ E_3 = \{(r, s) : r \leq 1, s + r \leq 3\} \cup \{(r, s) : s \leq 1, s + r \leq 3\}. \]

Then \( E_1 \cup E_2 \cup E_3 = R^2, E_1 \cap E_2 = \emptyset, E_3 \cap E_2 = \emptyset \) and \( intE_1 \cap intE_3 = \emptyset \), where \( intE_1 \) and \( intE_3 \) are the interior of \( E_1 \) and \( E_3 \), respectively.

It is obvious that Theorem 1.1 is true if once we prove that \( E(r, s; x, y) \) is Schur-convex, Schur-concave, and neither Schur-convex nor Schur-concave with respect to \( (x, y) \in (0, \infty) \times (0, \infty) \) for \((r, s) \in E_1, E_3 \) and \( E_2 \), respectively. We divide our proof into three cases.

**Case 1.** \((r, s) \in E_1\). Let \( E_{11} = \{(r, s) : s + r \geq 3, s > r \geq 1\}, E_{12} = \{(r, s) : s + r \geq 3, r > s \geq 1\} \) and \( F(r, s; x, y) = \frac{x^r y^s}{x^r + y^s} \), then

\[ E_1 = E_{11} \cup E_{12}. \]

and

\[ (y - x) \left( \frac{\partial E}{\partial y} - \frac{\partial E}{\partial x} \right) \]

\[ = \frac{1}{s - r} \left( \frac{y - x}{y^r - x^r} \right)^{s + r - 1} \left( \frac{1}{s - r} \right)^{s + r - 1} \]

\[ \times \frac{r}{s} \left[ (s - r) \left( \left( \frac{y}{x} \right)^{s + r - 1} - 1 \right) - s \left( \left( \frac{y}{x} \right)^{s - 1} - \left( \frac{y}{x} \right)^r \right) + r \left( \left( \frac{y}{x} \right)^{r - 1} - \left( \frac{y}{x} \right)^s \right) \right] \]

for \((r, s) \in E_{11}\). From Lemma 2.1, Lemma 2.2 \( (a), (3.1) \) and the assumption \( y > x \) we see that \( E(r, s; x, y) \) is Schur-convex with respect to \( (x, y) \in (0, \infty) \times (0, \infty) \) for \((r, s) \in E_{11}\). Then the continuity and symmetry of \( E(r, s; x, y) \) with respect to \((r, s)\) imply that \( E(r, s; x, y) \) is Schur-convex with respect to \( (x, y) \in (0, \infty) \times (0, \infty) \) for \((r, s) \in E_1\).

**Case 2.** \((r, s) \in E_2\). We divide the discussion of this case into seven subcases. Let

\[ E_{21} = \{(r, s) : s > r \geq 1, s + r < 3\}, \]
\[ E_{22} = \{(r, s) : r > s > 1, s + r < 3\}, \]
\[ E_{23} = \left\{(r, s) : 1 < s = r < \frac{3}{2}\right\}, \]
\[ E_{24} = \{(r, s) : 1 > r \neq 0, s + r > 3\}, \]
\[ E_{25} = \{(r, s) : 1 > s \neq 0, s + r > 3\}, \]
\[ E_{26} = \{(r, s) : s = 0, r > 3\} \]
and

\[ E_{27} = \{(r, s) : r = 0, s > 3\}. \]

Then

\[ E_2 = E_{21} \cup E_{22} \cup E_{23} \cup E_{24} \cup E_{25} \cup E_{26} \cup E_{27}. \tag{3.2} \]

**Subcase 2.1.** If \((r, s) \in E_{21}\), then Lemma 2.1, Lemma 2.2 (b), (3.1) and the assumption \(y > x\) imply that \(E(r, s; x, y)\) is neither Schur-convex nor Schur-concave with respect to \((x, y) \in (0, \infty) \times (0, \infty)\).

**Subcase 2.2.** If \((r, s) \in E_{22}\), then the symmetry of \(E(r, s; x, y)\) with respect to \((r, s)\) and subcase 2.1 show that \(E(r, s; x, y)\) is neither Schur-convex nor Schur-concave with respect to \((x, y) \in (0, \infty) \times (0, \infty)\).

**Subcase 2.3.** If \((r, s) \in E_{23}\), then

\[
(y - x) \left( \frac{\partial E(r, r; x, y)}{\partial y} - \frac{\partial E(r, r; x, y)}{\partial x} \right)
\]

\[
= \frac{y - x}{(x^r - y^r)^2} \frac{(y - x)^r}{x} E(r, r; x, y) x^{r-1}
\]

\[
\times \left\{ -r \left[ \left( \frac{y}{x} \right)^{r-1} + \left( \frac{y}{x} \right)^{r} \right] \log \frac{y}{x} + \left[ \left( \frac{y}{x} \right)^{r-1} + 1 \right] \left[ \left( \frac{y}{x} \right)^{r} - 1 \right] \right\}. \tag{3.3} \]

Now, Lemma 2.1, Lemma 2.3, (3.3) together with the assumption \(y > x\) imply that \(E(r, s; x, y)\) is neither Schur-convex nor Schur-concave with respect to \((x, y) \in (0, \infty) \times (0, \infty)\).

**Subcase 2.4.** If \((r, s) \in E_{24}\), then Lemma 2.1, Lemma 2.2 (c), (3.1) and the assumption \(y > x\) imply that \(E(r, s; x, y)\) is neither Schur-convex nor Schur-concave with respect to \((x, y) \in (0, \infty) \times (0, \infty)\).

**Subcase 2.5.** If \((r, s) \in E_{25}\), then the symmetry of \(E(r, s; x, y)\) with respect to \((r, s)\) and subcase 2.4 imply that \(E(r, s; x, y)\) is neither Schur-convex nor Schur-concave with respect to \((x, y) \in (0, \infty) \times (0, \infty)\).

**Subcase 2.6.** If \((r, s) \in E_{26}\), then

\[
(y - x) \left( \frac{\partial E(r, 0; x, y)}{\partial y} - \frac{\partial E(r, 0; x, y)}{\partial x} \right)
\]

\[
= \left( \frac{1}{r} \frac{y^{r-1}}{\log y - \log x} \right)^{\frac{1}{r-1}} (y - x)^{r-1}
\]

\[
\times \left\{ r \left[ 1 + \left( \frac{y}{x} \right)^{r-1} \right] \log \frac{y}{x} - \left( \frac{y}{x} \right)^{r} + \left( \frac{y}{x} \right)^{r-1} + 1 + \frac{1}{r} \right\}. \tag{3.4} \]

So, Lemma 2.1, Lemma 2.4, (3.4) together with the assumption \(y > x\) show that \(E(r, s; x, y)\) is neither Schur-convex nor Schur-concave with respect to \((x, y) \in (0, \infty) \times (0, \infty)\).
Subcase 2.7. If \((r, s) \in E_2\). Then the symmetry of \(E(r, s; x, y)\) with respect to \((r, s)\) and subcase 2.6 imply that \(E(r, s; x, y)\) is neither Schur-convex nor Schur-concave with respect to \((x, y) \in (0, \infty) \times (0, \infty)\).

Now, (3.2) and subcases 2.1-2.7 show that \(E(r, s; x, y)\) is neither Schur-convex nor Schur-concave with respect to \((x, y) \in (0, \infty) \times (0, \infty)\) for \((r, s) \in E_2\).

Case 3. \((r, s) \in E_3\). We divide the discussion of this case into four subcases. Let

\[
E_{31} = \{(r, s) : s > 0, s > r, r < 1, r \neq 0, s + r < 3\},
E_{32} = \{(r, s) : r > 0, r > s, s < 1, s \neq 0, s + r < 3\},
E_{33} = \{(r, s) : 0 > s > r\}
\]

and

\[
E_{34} = \{(r, s) : 0 > r > s\}.
\]

Then

(3.5) \(E_{31} \cup E_{32} \cup E_{33} \cup E_{34} = E_3\).

Subcase 3.1. If \((r, s) \in E_{31}\). Then Lemma 2.1, Lemma 2.2 (d), (3.1) together with the assumption \(y > x\) imply that \(E(r, s; x, y)\) is Schur-concave with respect to \((x, y) \in (0, \infty) \times (0, \infty)\).

Subcase 3.2. If \((r, s) \in E_{32}\). Then the symmetry of \(E(r, s; x, y)\) with respect to \((r, s)\) and subcase 3.1 lead to that \(E(r, s; x, y)\) is Schur-concave with respect to \((x, y) \in (0, \infty) \times (0, \infty)\).

Subcase 3.3. If \((r, s) \in E_{33}\). Then Lemma 2.1, Lemma 2.2 (e), (3.1) and the assumption \(y > x\) imply that \(E(r, s; x, y)\) is Schur-concave with respect to \((x, y) \in (0, \infty) \times (0, \infty)\).

Subcase 3.4. If \((r, s) \in E_{34}\). Then the symmetry of \(E(r, s; x, y)\) with respect to \((r, s)\) and subcase 3.3 lead to that \(E(r, s; x, y)\) is Schur-concave with respect to \((x, y) \in (0, \infty) \times (0, \infty)\).

Now, the continuity of \(E(r, s; x, y)\), (3.5) together with subcases 3.1-3.4 imply that \(E(r, s; x, y)\) is Schur-concave with respect to \((x, y) \in (0, \infty) \times (0, \infty)\) for \((r, s) \in E_3\).

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Yuming Chu and Xiaoming Zhang

Department of Mathematics
Huzhou Teachers College
313000 Huzhou, China
e-mail: chuyuming@hutc.zj.cn

Haining Radio and TV University
314400 Haining, China
e-mail: zjzxm79@tom.com

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