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Some infinite elements in the Adams spectral sequence for the sphere spectrum*

By

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Abstract

In the stable homotopy group $\pi_{p^nq+(p+1)q-1}(V(1))$ of the Smith-Toda spectrum $V(1)$, the author constructed an essential element ϖ_n for $n \geq 3$ at the prime greater than three. Let $\beta_s^* \in [V(1), S]_{spq+(s-1)q-2}$ denote the dual of the generator $\beta_s'' \in \pi_{s(p+1)q}(V(1))$, which defines the β-element β_s. In this paper, the author shows that the composite $\alpha_1\beta_1\xi_s \in \pi_{p^nq+(s+1)pq+sq-6}(S)$ for $1 < s < p-2$ is non-trivial, where $\xi_s = \beta_{s-1}^*\varpi_n \in \pi_{p^nq+spq+(s-1)q-3}(S)$ and $q = 2(p-1)$. As a corollary, ξ_s, $\alpha_1\xi_s$ and $\beta_1\xi_s$ are also non-trivial for $1 < s < p-2$.

1. Introduction and statement of results

We are interested in the problem of detecting nontrivial elements in the stable homotopy groups of spheres.

Throughout this paper, we fix a prime $p > 3$ and $q = 2(p-1)$. Let S denote the sphere spectrum localized at the prime p. Let M be the Moore spectrum modulo the prime p given by the cofibration

$$S \xrightarrow{p} S \xrightarrow{i_0} M \xrightarrow{j_0} \Sigma S. \tag{1.1}$$

Let $\alpha : \Sigma^q M \to M$ be the Adams map and $V(1)$ be its cofibre given by the cofibration

$$\Sigma^q M \xrightarrow{\alpha} M \xrightarrow{i_1} V(1) \xrightarrow{j_1} \Sigma^{q+1} M. \tag{1.2}$$

Let $V(2)$ be the cofibre of $\beta : \Sigma^{(p+1)q} V(1) \to V(1)$ given by the cofibration

$$\Sigma^{(p+1)q} V(1) \xrightarrow{\beta} V(1) \xrightarrow{i_2} V(2) \xrightarrow{j_2} \Sigma^{(p+1)q+1} V(1). \tag{1.3}$$

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To determine the stable homotopy groups of spheres $\pi_*(S)$ is one of the central problems in the stable homotopy theory. So far, several methods have been found to determine the stable homotopy groups of spheres. For example we have the classical Adams spectral sequence (ASS, for short)(cf. [1]) based on the Eilenberg-MacLane spectrum KZ_p, whose E_2-term is $\mathrm{Ext}_A^{s,t}(Z_p, Z_p)$ and the Adams differential is given by

$$d_r : E_r^{s,t} \longrightarrow E_r^{s+r,t+r-1},$$

where A denote the mod p Steenrod algebra. We also have the Adams-Novikov spectral sequence (ANSS, for short) based on the Brown-Peterson spectrum BP (cf. [2], [3], [4]).

From [5], we know that $\mathrm{Ext}_A^{1,*}(Z_p, Z_p)$ has Z_p-basis consisting of $a_0 \in \mathrm{Ext}_A^{1,1}(Z_p, Z_p)$, $h_i \in \mathrm{Ext}_A^{1,p^i q}(Z_p, Z_p)$ for all $i \geq 0$ and $\mathrm{Ext}_A^{2,*}(Z_p, Z_p)$ has Z_p-basis consisting of α_2, a_0^2, $a_0 h_i (i > 0)$, $g_i (i \geq 0)$, $k_i (i \geq 0)$, $b_i (i \geq 0)$, and $h_i h_j (j \geq i+2, i \geq 0)$ whose internal degrees are $2q+1$, 2, $p^i q+1, p^{i+1}q+2p^i q$, $2p^{i+1}q+p^i q$, $p^{i+1}q$ and $p^i q+p^j q$ respectively.

So far, not so many families of homotopy elements in $\pi_*(S)$ have been detected.

In [6], R. Cohen constructed a certain infinite family of elements denoted by $\zeta_k \in \pi_{q(p^{k+1}+1)-3}(S), k \geq 1$. ζ_k is represented by $h_0 b_k \in \mathrm{Ext}_A^{3,p^{k+1}q+q}(Z_p, Z_p)$ in the ASS. Using the method of ANSS, Chun-Nip Lee [4] proved that $\beta_1^{p-1}\zeta_k$ is non-trivial for all k, i.e., $b_0^{p-1}h_0 b_k$ is a permanent cycle in the ASS and converges non-trivially to $\beta_1^{p-1}\zeta_k$. This result gave another infinite family of elements in the stable homotopy of spheres.

In [7], a result similar to Lee's was proved using the method of ASS, and Zhou X. showed the non-triviality of $\beta_2\zeta_k$.

In [8], I detected a new family in the stable homotopy groups of spheres and obtained the following theorem.

Theorem 1.1 ([8, Theorem 1.4]). *Let $p > 3$, $n \geq 3$, then*

$$k_0 h_n \neq 0 \in \mathrm{Ext}_A^{3,p^n q+2pq+q}(Z_p, Z_p)$$

is a permanent cycle in the ASS and converges to an element of order p in $\pi_{p^n q+2pq+q-3}(S)$.

On the way of proving the above theorem, I detected a new family in the stable homotopy groups of $V(1)$ which is a spectrum closely related to S. I gave the following theorem.

Theorem 1.2 ([8, Theorem 1.5]). *Let $p > 3$, $n \geq 3$ and $h_n \in \mathrm{Ext}_A^{1,p^n q}(Z_p, Z_p)$, be the known generator in* [5]. *Then*

$$(\beta i_1 i_0)_*(h_n) \in \mathrm{Ext}_A^{2,p^n q+(p+1)q+1}(H^*V(1), Z_p)$$

is a permanent cycle in the ASS and converges to a nontrivial element $\varpi_n \in \pi_{p^n q+(p+1)q-1}(V(1))$.

Definition 1.1. We define, for $t \geq 1$, the $\beta-$element $\beta_t = j_0 j_1 \beta^t i_1 i_0 \in \pi_{tpq+(t-1)q-2}(S)$. Here the maps i_0, j_0, i_1, j_1 and β are given in (1.1)–(1.3) respectively.

From [2, Theorem 2.12], $\beta_t \neq 0$ in $\pi_*(S)$ for $p \geq 5$ and $t \geq 1$.

Theorem 1.3 ([9, Theorem 2.2]). *For $p > 3$ and $1 < s < p$, there exists the second Greek letter element $\tilde{\beta}_s \in \mathrm{Ext}_A^{s,spq+(s-1)q+s-2}(Z_p, Z_p)$, and $\tilde{\beta}_s$ converges to the β-element $\beta_s \in \pi_{spq+(s-1)q-2}(S)$ in the ASS.*

In this paper, we will base on the family of homotopy elements in $\pi_*(V(1))$ in [8] to detect a ϖ_n-related family of filtration $s+4$ in the stable homotopy groups of spheres.

Let $\beta_s^* = j_0 j_1 \beta^s \in [V(1), S]_{spq+(s-1)q-2}$ denote the dual of the generator $\beta_s'' = \beta^s i_1 i_0 \in \pi_{s(p+1)q}(V(1))$, which defines the β-element β_s. Let $\xi_s = \beta_{s-1}^* \varpi_n \in \pi_{p^n q+spq+(s-1)q-3}(S)$. Our main result can be stated as follows.

Theorem 1.4. *Let $p > 3$, $n \geq 3$ and $1 < s < p-2$, then the product*

$$b_0 h_0 h_n \tilde{\beta}_s \neq 0 \in \mathrm{Ext}_A^{s+4,p^n q+(s+1)pq+sq+s-2}(Z_p, Z_p)$$

is a permanent cycle in the ASS and converges to a nontrivial element $\alpha_1 \beta_1 \xi_s \in \pi_{p^n q+(s+1)pq+sq-6}(S)$, where $\alpha_1 = j_0 \alpha i_0$.

In this paper, the May spectral sequence (MSS) and the ASS play very important roles in the proof of the main theorem. We want to emphasize that the proof of our theorem is completely elementary. This paper gives a good example for detecting non-trivial elements in the stable homotopy groups of spheres.

The paper is arranged as follows: after giving some useful lemmas on the MSS in Section 2, we make use of the MSS to obtain two Ext groups in Section 3. Then the proof of Theorem 1.4 is given in this section.

2. Two spectral sequences: the ASS and the MSS

For the sake of completeness, in this section we first review some knowledge on the ASS and the MSS. Then we show some important lemmas on the MSS which will be often used in Section 3.

One of the main tools to determine the stable homotopy groups of spheres $\pi_* S$ is the ASS. In 1957, Adams, adapting the methods of homological algebra, constructed such a machine in the form of a spectral sequence leading from the doubly graded group $\mathrm{Ext}_A^{*,*}(Z_p, Z_p)$ to the p-primary components of the stable homotopy groups of spheres. From then on, the ASS has been an invaluable tool in studying stable homotopy theory.

Let p be a prime, X a spectrum of finite type and Y a finite dimensional spectrum. Then there is a natural spectral sequence $\{E_r^{s,t}, d_r\}$, which is called Adams spectral sequence(ASS)

$$E_2^{s,t} = \mathrm{Ext}_A^{s,t}((H^* X; Z_p), H^*(Y; Z_p)) \Rightarrow ([Y, X]_{t-s})_p, \tag{2.1}$$

where the differential is

$$d_r : E_r^{s,t} \to E_r^{s+r,t+r-1}. \tag{2.2}$$

If X and Y are sphere spectra S, then the ASS

$$E_2^{s,t} = \mathrm{Ext}_A^{s,t}(Z_p, Z_p) \Rightarrow (\pi_{t-s}(S))_p, \tag{2.3}$$

the p-primary components of the group $\pi_{t-s}(S)$.

There are three problems in using the ASS: calculation of the E_2-term, computation of the differentials and determination of the nontrivial extensions from E_∞ to $\pi_* S$. So, for computing the stable homotopy groups of spheres with the ASS, we must compute the E_2-term of the ASS, $\mathrm{Ext}_A^{*,*}(Z_p, Z_p)$. The most successful method for computing $\mathrm{Ext}_A^{*,*}(Z_p, Z_p)$ is the MSS.

From [3], there is a MSS $\{E_r^{s,t,*}, d_r\}$ which converges to $\mathrm{Ext}_A^{s,t}(Z_p, Z_p)$ with E_1-term

$$E_1^{*,*,*} = E(h_{m,i} | m > 0, i \geq 0) \bigotimes P(b_{m,i} | m > 0, i \geq 0) \bigotimes P(a_n | n \geq 0), \tag{2.4}$$

where E is the exterior algebra, P is the polynomial algebra, and

$$h_{m,i} \in E_1^{1,2(p^m-1)p^i,2m-1}, b_{m,i} \in E_1^{2,2(p^m-1)p^{i+1},p(2m-1)}, a_n \in E_1^{1,2p^n-1,2n+1}.$$

The May differential is

$$d_r : E_r^{s,t,u} \to E_r^{s+1,t,u-r} \tag{2.5}$$

and if $x \in E_r^{s,t,*}$ and $y \in E_r^{s',t',*}$, then

$$d_r(x \cdot y) = d_r(x) \cdot y + (-1)^s x \cdot d_r(y).$$

There exists a graded commutativity of the MSS:

$$x \cdot y = (-1)^{ss'+tt'} y \cdot x$$

for $x, y = h_{m,i}, b_{m,i}$ or a_n. The first May differential d_1 is given by

$$\begin{cases} d_1(h_{i,j}) = \sum\limits_{0<k<i} h_{i-k,k+j} h_{k,j}, \\ d_1(a_i) = \sum\limits_{0 \leq k<i} h_{i-k,k} a_k, \\ d_1(b_{i,j}) = 0. \end{cases} \tag{2.6}$$

For each element $x \in E_1^{s,t,*}$, we define *filt* $x = s$, *deg* $x = t$. Then we have:

$$\begin{cases} \text{filt } h_{i,j} = \text{filt } a_i = 1, \text{filt } b_{i,j} = 2, \\ \deg h_{i,j} = 2(p^i - 1)p^j = q(p^{i+j-1} + \cdots + p^j), \\ \deg b_{i,j} = 2(p^i - 1)p^{j+1} = q(p^{i+j} + \cdots + p^{j+1}), \\ \deg a_i = 2p^i - 1 = q(p^{i-1} + \cdots + 1) + 1, \\ \deg a_0 = 1, \end{cases} \tag{2.7}$$

where $i \geq 1$, $j \geq 0$.

In Section 3, we need the following three lemmas on the MSS.

By the knowledge on p-adic expression in number theory, we have that for each integer $t \geq 0$, it can be always expressed uniquely as

$$t = q(c_n p^n + c_{n-1}p^{n-1} + \cdots + c_1 p + c_0) + e,$$

where $0 \leq c_i < p$ $(0 \leq i < n)$, $p > c_n > 0$, $0 \leq e < q$.

Lemma 2.1 ([10, Proposition 1.1])**.** *Let $t = q(c_n p^n + c_{n-1}p^{n-1} + \cdots + c_1 p + c_0) + e$, where $0 \leq c_i < p$ $(0 \leq i < n)$, $p > c_n > 0$, $0 \leq e < q$. Let s_1 be a positive integer with $0 < s_1 < p$. If there exists some j $(0 \leq j \leq n)$ such that $c_j > s_1$, then in the MSS*

$$E_1^{s_1,t,*} = 0.$$

Let s_2 be an arbitrary positive integer. For the above t, we consider the structure of $E_1^{s_2,t,*}$ in the MSS. Suppose that in the MSS $w = x_1 x_2 \cdots x_m \in E_1^{s_2,t,*}$, where $m \leq s_2$, x_i is one of a_k, $h_{l,j}$ or $b_{u,z}$, $0 \leq k \leq n{+}1$, $0 < l{+}j \leq n{+}1$, $0 < u + z \leq n$, $l > 0$, $j \geq 0$, $u > 0$, $z \geq 0$. By (2.7) we may assume that $\deg x_i = q(c_{i,n}p^n + \cdots + c_{i,1}p + c_{i,0}) + e_i$, where $c_{i,j} = 0$ or 1, $e_i = 1$ if $x_i = a_{k_i}$, or $e_i = 0$. Then we have

(2.8)

$$\begin{aligned}\deg w &= \sum_{i=1}^{m} \deg x_i \\ &= q\left(\left(\sum_{i=1}^{m} c_{i,n}\right)p^n + \cdots + \left(\sum_{i=1}^{m} c_{i,1}\right)p + \left(\sum_{i=1}^{m} c_{i,0}\right)\right) + \left(\sum_{i=1}^{m} e_i\right).\end{aligned}$$

Denote $\sum\limits_{i=1}^{m} c_{i,j}$ and $\sum\limits_{i=1}^{m} e_i$ by $\bar{c}_j$ and $\bar{e}$, $0 \leq j \leq n$, respectively. Then we have the following two lemmas.

Lemma 2.2 ([11, Theorem 2.4])**.** *If $\bar{c}_0 - \bar{e} > n + 1$, then in the MSS*

$$E_1^{s_2,t,*} = 0.$$

Lemma 2.3. *With notation as above. If there exist two integers i_1 and i_2 satisfying the following conditions*:

(1) $i_1 > i_2 \geq 0$;

(2) $\bar{c}_{i_1} = m$;

(3) $\bar{c}_{i_2} < \bar{e} \leq \bar{c}_{i_1} = m$,

then w is impossible to exist.

Proof. The lemma is easily obtained by (2.7), and omitted here. □

3. Proof of Theorem 1.4

In this section, we first make use of the MSS to determine two Ext groups which will be used in the proof of Theorem 1.4.

Lemma 3.1. *Let $p > 3$, $n \geq 3$, $1 < s < p-2$ and $1 \leq r \leq s+4$, then in the MSS,*

$$E_1^{s+4-r, p^n q+(s+1)pq+sq+s-r-1,*} = 0.$$

Proof. For convenience, we let $t = p^n q + (s+1)pq + sq + s - r - 1$. If r is equal to $s+4$, $s+3$ or $s+2$, Lemma 3.1 is obvious. Consequently, in the rest of the proof we always assume that $1 \leq r < s+2$.

Consider $w = x_1 x_2 \cdots x_m \in E_1^{s+4-r,t,*}$ in the MSS, where x_i is one of a_k, $h_{l,j}$ or $b_{u,z}$, $0 \leq k \leq n+1$, $0 \leq l+j \leq n+1$, $0 \leq u+z \leq n$, $l > 0$, $j \geq 0$, $u > 0$, $z \geq 0$. By (2.7), we may assume that deg $x_i = q(c_{i,n}p^n + c_{i,n-1}p^{n-1} + \cdots + c_{i,0}) + e_i$, where $c_{i,j} = 0$ or 1, $e_i = 1$ if $x_i = a_{k_i}$, or $e_i = 0$. It follows that

$$\text{filt } w = \sum_{i=1}^{m} \text{filt } x_i = s+4-r \tag{3.1}$$

and

(3.2)

$$\begin{aligned}
\deg w &= \sum_{i=1}^{m} \deg x_i \\
&= q\left(\left(\sum_{i=1}^{m} c_{i,n}\right)p^n + \cdots + \left(\sum_{i=1}^{m} c_{i,1}\right)p + \left(\sum_{i=1}^{m} c_{i,0}\right)\right) + \left(\sum_{i=1}^{m} e_i\right) \\
&= q(p^n + (s+1)p + s) + s - r - 1.
\end{aligned}$$

Note that filt $x_i = 1$ or 2, $1 < s < p-2$ and $1 \leq r < s+2$. We have that $m \leq s+4-r \leq s+3 < p+1$ from (3.1).

Assertion. $s - r - 1 \geq 0$.

Otherwise, we would get $\sum_{i=1}^{m} e_i = s-r-1+q = 2p-3+s-r > 2p-3-2 = 2p-5$ by $1 \leq r < s+2$, since $e_i = 0$ or 1. Thus $\sum_{i=1}^{m} e_i > 2p-5 \geq p$ by $p \geq 5$. On the other hand, by $e_i = 0$ or 1, we have $\sum_{i=1}^{m} e_i \leq m < p+1$ which contradicts to $\sum_{i=1}^{m} e_i > 2p-5 \geq p$. The assertion is proved.

Using $0 \leq s, s+1, s-r-1 < p$ and the knowledge on the p-adic expression in number theory, from (3.2) we have that

$$
(3.3)\qquad \begin{cases}
\sum_{i=1}^{m} e_i = s-r-1+\lambda_{-1}q, & \lambda_{-1}\geq 0;\\
\sum_{i=1}^{m} c_{i,0}+\lambda_{-1} = s+\lambda_0 p, & \lambda_0\geq 0;\\
\sum_{i=1}^{m} c_{i,1}+\lambda_0 = s+1+\lambda_1 p, & \lambda_1\geq 0;\\
\sum_{i=1}^{m} c_{i,2}+\lambda_1 = 0+\lambda_2 p, & \lambda_2\geq 0;\\
\cdots & \cdots\\
\sum_{i=1}^{m} c_{i,n-1}+\lambda_{n-2} = 0+\lambda_{n-1} p, & \lambda_{n-1}\geq 0;\\
\sum_{i=1}^{m} c_{i,n}+\lambda_{n-1} = 1.
\end{cases}
$$

Case 1. $1 < s < p-3$. Then $m \leq s+3 < p$. It follows that

$$0 \leq \sum_{i=1}^{m} e_i, \sum_{i=1}^{m} c_{i,j} < p.$$

It is easy to get that the sequence $(\lambda_{-1}, \lambda_0, \lambda_1, \lambda_2, \cdots, \lambda_{n-2}, \lambda_{n-1})$ must equal the sequence $(0,0,0,0,\cdots,0,0)$. By (2.7), there exists a factor $h_{1,n}$ or $b_{1,n-1}$ among w. By the graded commutativity of $E_1^{*,*,*}$, we denote by x_m the factor $h_{1,n}$ or $b_{1,n-1}$. Then we put $w = w_1 x_m$ for $w_1 = x_1 x_2 \cdots x_{m-1} \in E_1^{l,t-p^n q,*}$, where $l = s+3-r$ (if $x_m = h_{1,n}$) or $s+2-r$ (if $x_m = b_{1,n-1}$). At the same time we have

$$(3.4)\qquad \sum_{i=1}^{m-1} e_i = s-r-1,\ \sum_{i=1}^{m-1} c_{i,0} = s,\ \sum_{i=1}^{m-1} c_{i,1} = s+1.$$

Subcase 1.1. $1 < r < s+2$. Since $\sum_{i=1}^{m-1} c_{i,0} - \sum_{i=1}^{m-1} e_i = r+1 > 1+1$ by $r > 1$, we can get that $E_1^{s+3-r,t-p^n q,*} = 0$ and $E_1^{s+2-r,t-p^n q,*} = 0$ by Lemma 2.2, i.e., w_1 cannot exist. Thus in this case w cannot exist

Subcase 1.2. $r = 1$. It follows that $m \geq s+2$ from $\sum_{i=1}^{m-1} c_{i,1} = s+1$. Meanwhile, we know $m \leq s+3$. Thus m can equal $s+2$ or $s+3$. Since $\sum_{i=1}^{m-1} e_i = s-2$, $\deg h_{i,j} \equiv 0 (\mathrm{mod}\ q)$ $(i>0, j\geq 0)$, $\deg a_i \equiv 1 (\mathrm{mod}\ q)$ $(i \geq 0)$ and $\deg b_{i,j} \equiv 0 (\mathrm{mod}\ q)$ $(i>0, j\geq 0)$, then by the graded commutativity of $E_1^{*,*,*}$, up to sign w_1 must have a factor $a_{j_1} a_{j_2} \cdots a_{j_{s-2}}$ $(0 \leq j_1 \leq j_2 \leq \cdots \leq j_{s-2})$. Noticing the degrees of a_i's, we assume that $w_1 = a_0^x a_1^y a_2^z x_{s-1} \cdots x_{m-1}$, where $x+y+z = s-2$ for $x,y,z \geq 0$. Then from (3.4) we put $w_2 = x_{s-1} \cdots x_{m-1} \in E_1^{l-s+2,t',*}$ where $t' = (s+1-z)pq + (s-y-z)q$, and

$$(3.5)\qquad \sum_{i=s-1}^{m-1} e_i = 0,\ \sum_{i=s-1}^{m-1} c_{i,0} = s-y-z,\ \sum_{i=s-1}^{m-1} c_{i,1} = s+1-z.$$

(I). If $w = x_1x_2\cdots x_{m-1}h_{1,n}$, then $w_1 = a_0^x a_1^y a_2^z w_2 \in E_1^{s+2,t-p^nq,*}$ for $w_2 = x_{s-1}\cdots x_{m-1} \in E_1^{4,t',*}$.

When $m = s+2$, from

$$\sum_{i=s-1}^{s+1} c_{i,1} = s+1-z,$$

we have that

$$z = s+1-\sum_{i=s-1}^{s+1} c_{i,1} \geq s+1-3 = s-2.$$

Note that $z \leq s-2$. It follows that

$$z = s-2, x = y = 0.$$

Thus in this case, $w_2 = x_{s-1}x_sx_{s+1} \in E_1^{4,t',*} = E_1^{4,3pq+2q,*} = 0$.

When $m = s+3$, by an argument as above we get $z = s-3$ or $z = s-2$. Since $E_1^{4,r_1pq+r_2q,*} = 0$ for $(r_1, r_2) = (4,3)$, $(4,2)$ and $(3,2)$, we can have that when $z = s-3, x = 1, y = 0$, $z = s-3, y = 1, x = 0$ and $z = s-2, x = y = 0$, w_2 cannot exist respectively.

(II). If $w = x_1x_2\cdots x_{m-1}b_{1,n}$, then $w_1 = a_0^x a_1^y a_2^z w_2 \in E_1^{s+1,t-p^nq,*}$ for $w_2 = x_{s-1}\cdots x_{m-1} \in E_1^{3,t',*}$.

When $m = s+3$, $w_2 = x_{s-1}\cdots x_{s+2}$. From filt $w_2 = \sum_{i=s-1}^{s+2}$ filt $x_i \geq 4 > 3 =$ filt w_2, we get that in this case w_2 cannot exist.

When $m = s+2$, it is easy to show that $z = s-2$, $x = y = 0$ by an argument similar to that used in (I), and $w_2 = x_{s-1}x_sx_{s+1} \in E_1^{3,t',*} = E_1^{3,3pq+2q,*} = 0$.

From Subcases 1.1 and 1.2, we have that when $1 < s < p-3$, w is impossible to exist. Thus $E_1^{s+4-r,t,*} = 0$.

Case 2. $s = p-3$. Note that filt $w = \sum_{i=1}^{m}$ filt $x_i = p+1-r$. Then $m \leq p+1-r$ by filt $x_i = 1$ or 2.

Subcase 2.1. $1 < r < s-2$. Then $m \leq p+1-r \leq p-1$. We easily get that the sequence $(\lambda_{-1}, \lambda_0, \lambda_1, \lambda_2, \cdots, \lambda_{n-2}, \lambda_{n-1})$ must equal the sequence $(0,0,0,0,\cdots,0,0)$. A similar proof as in Subcase 1.1 shows that w is impossible to exist either.

Subcase 2.2. $r = 1$. Then $m \leq p$. By the knowledge of p-adic expression in number theory it is easy to get that the sequence $(\lambda_{-1}, \lambda_0, \lambda_1)$ must equal $(0,0,0)$. From (3.3) we have

$$\sum_{i=1}^{m} c_{i,2} = \lambda_2 p.$$

Note that $0 \leq \sum_{i=1}^{m} c_{i,2} \leq m \leq p$. It follows that λ_2 may be equal to 0 or 1.

(I). If $\lambda_2 = 0$, then $\sum_{i=1}^{m} c_{i,2} = 0$.

When $n = 3$, we have $\sum_{i=1}^{m} c_{i,3} = 1$. By (2.7), there exists a factor $h_{1,3}$ or $b_{1,2}$ in w.

When $n > 3$, we may similarly discuss and obtain that λ_3 may be equal to 0 or 1. We claim that

$$\lambda_3 = 0.$$

Otherwise, we would have that $\lambda_3 = 1$ and $\sum_{i=1}^{m} c_{i,3} = p$, then $m = p$. Meanwhile, note that $\sum_{i=1}^{m} c_{i,2} = 0$ and $\sum_{i=1}^{p} e_i = p - 5$. By Lemma 2.3, in this case w is impossible to exist. Thus λ_3 cannot be equal to 1 and the claim is proved. By induction on j we get that

$$\lambda_j = 0 (3 \le j \le n-1),$$

so $\sum_{i=1}^{m} c_{i,n} = 1$. By (2.7), there is a factor $h_{1,n}$ or $b_{1,n-1}$ in w.

In a word, for $n \ge 3$, there always exists a factor $h_{1,n}$ or $b_{1,n-1}$ in w. By an argument similar to that used in the proof in Subcase 1.2, it is easily obtained that w is impossible to exist.

(II). If $\lambda_2 = 1$, then $\sum_{i=1}^{m} c_{i,2} = p$. By $c_{i,2} = 0$ or 1 and $m \le p$, we get that $m = p$. Note that filt $w = p$. By (2.7), we have that, for each i, filt $x_i = 1$ and

$$w = x_1 x_2 \cdots x_p \in E(h_{m,i} | m > 0, i \ge 0) \bigotimes P(a_n | n \ge 0).$$

When $n = 3$, it is easy to get that $\sum_{i=1}^{p} c_{i,3} = \sum_{i=1}^{p} c_{i,n} = 0$. Then we have that

$$\sum_{i=1}^{p} e_i = p - 5, \sum_{i=1}^{p} c_{i,0} = p - 3, \sum_{i=1}^{p} c_{i,1} = p - 2, \sum_{i=1}^{p} c_{i,2} = p.$$

By an argument similar to that used in Subcase 1.2 it is easy to show that $w = x_1 x_2 \cdots x_p$ cannot exist either.

When $n > 3$, from (3.3) we have

$$\sum_{i=1}^{p} c_{i,3} + 1 = 0 + \lambda_3 p.$$

By $c_{1,3} = 0$ or 1, we have

$$\lambda_3 = 1.$$

By induction on j, we have

$$\lambda_j = 1 (3 \le j \le n-1).$$

Using (2.7) and the facts that $\sum_{i=1}^{p} c_{i,2} = p$, $\sum_{i=1}^{p} c_{i,3} = \cdots = \sum_{i=1}^{p} c_{i,n-1} = p-1$, we divide the p x_i's into two disjoint classes S_1 and S_2. The two disjoint classes are given by

$$\begin{cases} S_1 = \{x | \deg x = q(p^{n-1} + p^{n-2} + \cdots + p^2) + \text{lower terms}\}, \\ S_2 = \{x | \deg x = qp^2 + \text{lower terms}\}. \end{cases}$$

For a class S in this paper, denote by $N(S)$ the number of elements in S, then we get that $N(S_1) = p-1$ and $N(S_2) = 1$. Similarly, Using the facts that $\sum_{i=1}^{p} e_i = p-5$, $\sum_{i=1}^{p} c_{i,0} = p-3$, $\sum_{i=1}^{p} c_{i,1} = p-2$, $\sum_{i=1}^{p} c_{i,2} = p$ and (2.7), we can also divide the p x_i's into four disjoint classes. The four classes are given by

$$\begin{cases} S_3 = \{x | \deg x = q(\text{higher terms} + p^2 + p + 1) + 1\}, N(S_3) = p-5; \\ S_4 = \{x | \deg x = q(\text{higher terms} + p^2 + p + 1)\}, N(S_4) = 2; \\ S_5 = \{x | \deg x = q(\text{higher terms} + p^2 + p)\}, N(S_5) = 1; \\ S_6 = \{x | \deg x = q(\text{higher terms} + p^2)\}, N(S_6) = 2. \end{cases}$$

Since $S_1 \bigcup S_2 = S_3 \bigcup S_4 \bigcup S_5 \bigcup S_6$, S_6 or S_4 must be in S_1. For example, all elements of S_4 are in S_1. Then there would be at least two $h_{n,0}$'s among w with $\deg h_{n,0} = q(p^{n-1} + \cdots + 1)$. This is impossible since $h_{i,j}^2 = 0$ ($i > 0$, $j \geq 0$). Thus in this case w is impossible to exist either.

From Subcases 2.1 and 2.2, we get that when $s = p-3$, $E_1^{s+4-r,t,*} = 0$.

From Cases 1 and 2, the lemma follows. □

Theorem 3.1. *Let $p > 3$, $n \geq 3$, $1 < s < p-2$, then the product*

$$b_0 h_0 h_n \tilde{\beta}_s \neq 0 \in \mathrm{Ext}_A^{s+4, p^n q + (s+1)pq + sq + s - 2}(Z_p, Z_p).$$

Proof. Since $b_{1,0}$, $h_{1,n}$ and $a_2^{s-2} h_{2,0} h_{1,1} \in E_1^{*,*,*}$ are permanent cycles in the MSS and converge nontrivially to b_0, h_n, $\tilde{\beta}_s \in \mathrm{Ext}_A^{*,*}(Z_p, Z_p)$ for $n \geq 0$ respectively (see Theorem 1.3),

$$b_{1,0} h_{1,0} h_{1,n} a_2^{s-2} h_{2,0} h_{1,1} \in E_1^{s+4, p^n q + (s+1)pq + sq + s - 2, *}$$

is a permanent cycle in the MSS and converges to

$$b_0 h_0 h_n \tilde{\beta}_s \in \mathrm{Ext}_A^{s+4, p^n q + (s+1)pq + sq + s - 2}(Z_p, Z_p).$$

From the case $r = 1$ in Lemma 3.1, we see that

$$E_1^{s+3, p^n q + (s+1)pq + sq + s - 2, *} = 0,$$

then for $r \geq 1$,

$$E_r^{s+3, p^n q + (s+1)pq + sq + s - 2, *} = 0.$$

Thus the permanent cycle

$$b_{1,0} h_{1,0} h_{1,n} a_2^{s-2} h_{2,0} h_{1,1} \in E_r^{s+4, p^n q + (s+1)pq + sq + s - 2, *}$$

cannot be hit by any differential in the MSS. Then

$$b_0h_0h_n\tilde{\beta}_s \neq 0 \in \mathrm{Ext}_A^{s+4,*}(Z_p, Z_p).$$

This completes the proof of Theorem 3.1. □

Theorem 3.2. *Let $p > 3$, $n \geq 3$, $1 < s < p-2$, $2 \leq r \leq s+4$, then*

$$\mathrm{Ext}_A^{s+4-r,q(p^n+(s+1)p+s)+s-r-1}(Z_p, Z_p) = 0.$$

Proof. From the case $2 \leq r \leq s+4$ in Lemma 3.1, we have

$$E_1^{s+4-r,q(p^n+(s+1)p+s)+s-r-1,*} = 0.$$

By the MSS, Theorem 3.2 follows immediately. □

Now we give the main theorem in this paper.

Theorem 1.4. *Let $p > 3$, $n \geq 3$ and $1 < s < p-2$, then the product*

$$b_0h_0h_n\tilde{\beta}_s \neq 0 \in \mathrm{Ext}_A^{s+4,p^nq+(s+1)pq+sq+s-2}(Z_p, Z_p)$$

is a permanent cycle in the ASS and converges to a nontrivial element $\alpha_1\beta_1\xi_s \in \pi_{p^nq+(s+1)pq+sq-6}(S)$, where $\alpha_1 = j_0\alpha i_0$.

Proof. It's well known that $b_0 \in \mathrm{Ext}_A^{2,pq}(Z_p, Z_p)$ and $h_0 \in \mathrm{Ext}_A^{1,q}(Z_p, Z_p)$ both are permanent cycles in the ASS, and converges nontrivially to the β-element $\beta_1 = j_0j_1\beta i_1i_0 \in \pi_{pq-2}(S)$ and α-element $\alpha_1 = j_0\alpha i_0 \in \pi_{q-1}(S)$ respectively. From Theorem 1.2,

$$(\beta i_1i_0)_*(h_n) \in \mathrm{Ext}_A^{2,p^nq+(p+1)q+1}(H^*V(1), Z_p)$$

is a permanent cycle in the ASS and converges to a nontrivial element $\varpi_n \in \pi_{p^nq+(p+1)q-1}(V(1))$.

Now consider the following composition of maps

$$\alpha_1\beta_1\xi_s: \quad \Sigma^{p^nq+spq+(s-1)q-3}S \xrightarrow{\xi_s=\beta_{s-1}^*\varpi_n} S \xrightarrow{\beta_1} \Sigma^{-pq+2}S \xrightarrow{\alpha_1} \Sigma^{-pq-q+3}S.$$

Since $\varpi_n \in \pi_{p^nq+(p+1)q-1}(V(1))$ is represented up to nonzero scalar by

$$(\beta i_1i_0)_*(h_n) \in \mathrm{Ext}_A^{2,p^nq+(p+1)q+1}(H^*V(1), Z_p)$$

in the ASS, then the map $\alpha_1\beta_1\xi_s$ is represented up to nonzero scalar by

$$(j_0j_1\beta^si_1i_0)_*(b_0h_0h_n) \in \mathrm{Ext}_A^{s+4,p^nq+(s+1)pq+sq+s-2}(Z_p, Z_p)$$

in the ASS. By Theorem 1.3 and the knowledge of Yoneda products we know that the composition

$$\begin{aligned}(j_0j_1\beta^si_1i_0)_*: \mathrm{Ext}_A^{0,*}(Z_p, Z_p) &\xrightarrow{(i_1i_0)_*} \mathrm{Ext}_A^{0,*}(H^*V(1), Z_p)\\ &\xrightarrow{(j_0j_1\beta^s)_*} \mathrm{Ext}_A^{s,*+spq+(s-1)q+(s-2)}(Z_p, Z_p)\end{aligned}$$

is a multiplication up to nonzero scalar by

$$\tilde{\beta}_s \in \mathrm{Ext}_A^{s,spq+(s-1)q+(s-2)}(Z_p, Z_p).$$

Hence, $\alpha_1\beta_1\xi_s$ is represented up to nonzero scalar by

$$b_0h_0h_n\tilde{\beta}_s \neq 0 \in \mathrm{Ext}_A^{s+4,p^nq+(s+1)pq+s)q+s-2}(Z_p, Z_p)$$

in the ASS (see Theorem 3.1).

Moreover, from Theorem 3.2 and (2.2), it follows that $b_0h_0h_n\tilde{\beta}_s$ cannot be hit by any differential in the ASS. Thus the corresponding homotopy element $\alpha_1\beta_1\xi_s$ is nontrivial. This finishes the proof of Theorem 1.4. □

From Theorem 1.4, the following consequence is immediate.

Corollary 3.1. *Let $p > 3$, $n \geq 3$ and $1 < s < p-2$. Then ξ_s, $\beta_1\xi_s$ and $\alpha_1\xi_s$ are non-trivial.*

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