CONCERNING q-SUMMABLE SZLENK INDEX

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ABSTRACT. For each ordinal ξ and each $1 \leqslant q < \infty$, we define the notion of ξ-q-summable Szlenk index. When $\xi = 0$ and $q = 1$, this recovers the usual notion of summable Szlenk index. We define for an arbitrary weak*-compact set a transfinite, asymptotic analogue $\alpha_{\xi,p}$ of the martingale type norm of an operator. We prove that this quantity is determined by norming sets and determines ξ-Szlenk power type and ξ-q-summability of Szlenk index. This fact allows us to prove that the behavior of operators under the $\alpha_{\xi,p}$ seminorms passes in the strongest way to injective tensor products of Banach spaces. Furthermore, we combine this fact with a result of Schlumprecht to prove that a separable Banach space with good behavior with respect to the $\alpha_{\xi,p}$ seminorm can be embedded into a Banach space with a shrinking basis and the same behavior under $\alpha_{\xi,p}$, and in particular it can be embedded into a Banach space with a shrinking basis and the same ξ-Szlenk power type. Finally, we completely elucidate the behavior of the $\alpha_{\xi,p}$ seminorms under ℓ_r direct sums. This allows us to give an alternative proof of a result of Brooker regarding Szlenk indices of ℓ_p and c_0 direct sums of operators.

1. Introduction

Two of the most important renorming theorems in Banach space theory is Enflo's result [12] that a superreflexive Banach space can be renormed to be uniformly convex, and Pisier's result [19] that any superreflexive Banach space can be renormed to be uniformly convex (resp. uniformly smooth) with a power type modulus of uniform convexity (resp. uniform smoothness). Pisier's proof also gave an exact characterization of which power types were possible in

terms of two isomorphic invariants of the space involving Walsh–Paley martingale difference sequences and their domination of (resp. domination by) the canonical ℓ_p basis. Beauzamy [2] proved an operator version of Enflo's result: An operator is super weakly compact if and only if it can be renormed to be uniformly convex. However, there is no operator version of Pisier's result, since the automatic power type obtained by Pisier for Banach spaces has to do with the submultiplicative nature of Haar type/cotype ideal norms for Banach spaces, which are not submultiplicative for operators. The same submultiplicative behavior of ideal norms and seminorms for spaces which are not submultiplicative for operators is by now a well-observed phenomenon, occurring with Rademacher, Gaussian, Haar, martingale, and the recently defined asymptotic basic and asymptotic type and cotype. Since there are super weakly compact operators which do not have non-trivial Haar type or cotype, it has become standard to investigate the notions of subtype and subcotype of operators, defined by whether or not the operator exhibits the worst possible behavior with respect to a sequence of norms or seminorms. This practice has been undertaken by Beauzamy [1] for Radmacher subtype and subcotype to characterize when ℓ_1 or c_0 is crudely finitely representable in an operator; by Hinrichs [14] for Gaussian subcotype to characterize when c_0 is crudely finitely representable in an operator; by Wenzel [22] for martingale and Haar subtype and subcotype to characterize super weak compactness; and Draga, Kochanek, and the author [10] to characterize when an operator is asymptotically uniformly smoothable, when ℓ_1 is asymptotically crudely finitely representable in an operator, and when c_0 is asymptotically crudely finitely representable in an operator.

The notion of asymptotic uniform smoothability is also of significant interest, and is fundamentally connected to the Szlenk index of a Banach space. A Banach space admits an equivalent asymptotically uniformly smooth norm if and only if its Szlenk index does not exceed ω. This was shown by Knaust, Odell, and Schlumprecht [15] for the separable case and Raja [20] for the general case. Lancien, Prochazka, and Raja [17] proved a result analogous to that of Knaust, Odell and Schlumprecht for separable Banach spaces having Szlenk index not exceeding $\omega^{\xi+1}$ for every countable ordinal ξ, and a non-separable, operator version was proved in [9] for every (not necessarily countable) ordinal ξ. Furthermore, Godefroy, Kalton, and Lancien [13] gave a precise renorming theorem for a separable Banach space in terms of the Szlenk power type of the Banach space. This was generalized to non-separable spaces and operators in [7], as well as to higher ordinals in terms of the behavior of special convex combinations of the branches of n-leveled weakly null trees where each level has specified order. Further renorming results were established in [6], analogous to those of Pisier in [19]. More precisely, the existence of equivalent norms with certain asymptotic properties was investigated in terms of special convex combinations of the branches of ω-leveled weakly null trees where each

level has specified order. The downside of the renorming results from [6] is that they are only produce non-trivial equivalent norms when an operator or a space has some power type behavior of the ε-Szlenk indices. That is, for a fixed ξ, the results of [6] produce non-trivial equivalent norms for a Banach space X or an operator $A : X \to Y$ if $\omega^\xi k_\varepsilon \leqslant Sz(X, \varepsilon) \leqslant \omega^\xi(k_\varepsilon + 1)$ (resp. $\omega^\xi k_\varepsilon \leqslant Sz(A, \varepsilon) \leqslant \omega^\xi(k_\varepsilon + 1)$) where $k_\varepsilon \in \mathbb{N}$ satisfies $\lim_{\varepsilon \to 0+} k_\varepsilon = \infty$ but for some $1 \leqslant q < \infty$, $\sup_{\varepsilon > 0} \varepsilon^q k_\varepsilon < \infty$. As was explained in [7], for $\xi = 0$, there exists an operator A with Szlenk index ω which does not have this power type behavior, and for any $\xi > 0$, there is a Banach space X with Szlenk index $\omega^{\xi+1}$ which does not have this power type behavior. This incompleteness in the renorming theorems from [6] is the primary motivation to define the notion of subtype and investigate the behavior of operators with respect to a sequence of seminorms.

For each ordinal ξ, each $1 \leqslant p \leqslant \infty$, each $n \in \mathbb{N}$, and each operator $A : X \to Y$, we define the quantity $\alpha_{\xi,p,n}(A)$. The assignments $\alpha_{\xi,p,n}$ will act as our sequence of ideal seminorms. We will be interested in when an operator satisfies $\sup_n \alpha_{\xi,p,n}(A) < \infty$, which is trivial in the case $p = 1$. For $1 < p \leqslant \infty$, the worst possible behavior would be for the operator A to satisfy $\limsup_n \alpha_{\xi,p,n}(A)/n^{1-1/p} > 0$, so our notion of subtype will be defined by $\lim_n \alpha_{\xi,p,n}(A)/n^{1-1/p} = 0$. For each ordinal ξ and $1 < p \leqslant \infty$, we let $\mathfrak{A}_{\xi,p}$ denote the class of operators for which $\sup_n \alpha_{\xi,p,n}(A) < \infty$.

For each ordinal ξ and each $1 \leqslant q < \infty$, we define what it means for an operator $A : X \to Y$ and a weak*-compact subset K of a dual Banach space X^* to have ξ-q-summable Szlenk index, which generalizes to values of q greater than 1 and higher ordinals the important notion of summable Szlenk index defined and studied in [13]. In what follows, we let $\mathfrak{D}_\xi$ denote the class of operators with Szlenk index not exceeding ω^ξ and for $1 < p \leqslant \infty$, $\mathfrak{T}_{\xi,p}$ denotes the class of ξ-p-asymptotically uniformly smoothable operators.

THEOREM 1.1. *For every ordinal ξ and $1 < p \leqslant \infty$, there exists an ideal norm $\mathfrak{a}_{\xi,p}$ on the class $\mathfrak{A}_{\xi,p}$ making $(\mathfrak{A}_{\xi,p}, \mathfrak{a}_{\xi,p})$ a Banach ideal. Moreover, if $1/p + 1/q = 1$, the class $\mathfrak{A}_{\xi,p}$ coincides with the class of operators having ξ-q-summable Szlenk index.*

An operator A has Szlenk index not exceeding $\omega^{\xi+1}$ if and only if for some (equivalently, every) $1 < p \leqslant \infty$, $\lim_n \alpha_{\xi,p,n}(A)/n^{1-1/p} = 0$.

Furthermore, for any $1 < r < p \leqslant \infty$,

$$\mathfrak{D}_\xi \subsetneqq \mathfrak{A}_{\xi,r} \subsetneqq \mathfrak{T}_{\xi,p} \subset \mathfrak{A}_{\xi,p} \subsetneqq \mathfrak{D}_{\xi+1}.$$

The quantity $\alpha_{\xi,p}(\cdot)$ will be defined not only for operators, but for arbitrary, weak*-compact subsets of the dual of a Banach space. We prove the following regarding the quantities $\alpha_{\xi,p}$ on weak*-compact subsets of the dual of a Banach space.

THEOREM 1.2. *Let X be a Banach space, $K \subset X^*$ weak*-compact, ξ an ordinal, and $1 < p \leqslant \infty$. Then $\alpha_{\xi,p}(K) = \alpha_{\xi,p}(\overline{\text{abs co}}^{\text{weak}^*}(K))$.*

Combining Theorem 1.2 with a result of Schlumprecht, we prove the following.

THEOREM 1.3. *Suppose X is a Banach space having a separable dual. Then there exists a Banach space W with a shrinking basis such that X is isomorphically embeddable in W, for any countable ordinal ξ and $1 \leqslant p \leqslant \infty$, $\mathbf{p}_\xi(W) = \mathbf{p}_\xi(X)$, and if $\alpha_{\xi,p}(X) < \infty$, $\alpha_{\xi,p}(W) < \infty$.*

Given operators $A_0 : X_0 \to X_1$, $A_1 : X_1 \to Y_1$, there is an induced operator $A_0 \otimes A_1 : X_0 \hat{\otimes}_\varepsilon X_1 \to Y_0 \hat{\otimes}_\varepsilon Y_1$ between the injective tensor products. By the geometric Hahn–Banach theorem, $(A_0 \otimes A_1)^*(B_{(Y_0 \hat{\otimes}_\varepsilon Y_1)^*})$ is the weak*-closed, convex hull of $\{x_0^* \otimes x_1^* : x_0^* \in A_0^* B_{Y_0^*}, A_1^* B_{Y_1^*}\}$. This fact combined with Theorem 1.2 and a combinatorial lemma yields the following.

THEOREM 1.4. *Fix any $1 \leqslant q < \infty$, any ordinal ξ, any operators $A_i : X_i \to Y_i$, $i = 0, 1$. Then if A_0, A_1 have ξ-q-summable Szlenk index, so does $A_0 \otimes A_1$, and $\mathbf{p}_\xi(A_0 \otimes A_1) \leqslant \max\{\mathbf{p}_\xi(A_0), \mathbf{p}_\xi(A_1)\}$.*
In the non-trivial case in which $A_0, A_1 \neq 0$, $A_0 \otimes A_1$ has ξ-q-summable Szlenk index if and only if A_0, A_1 do and $\mathbf{p}_\xi(A_0 \otimes A_1) = \max\{\mathbf{p}_\xi(A_0), \mathbf{p}_\xi(A_1)\}$.

We also give a complete description of the behavior of the $\alpha_{\xi,r}$ seminorms under finite ℓ_p direct sums of operators.

THEOREM 1.5. *Fix $1 \leqslant p \leqslant \infty$. Then for any finite set Λ, any ordinal ξ, and any operators $A_\lambda : X_\lambda \to Y_\lambda$,*

$$\alpha_{\xi,r}\left(\bigoplus_{\lambda \in \Lambda} A_\lambda : \left(\bigoplus_{\lambda \in \Lambda} X_\lambda\right)_{\ell_p(\Lambda)} \to \left(\bigoplus_{\lambda \in \Lambda} Y_\lambda\right)_{\ell_p(\Lambda)}\right) = \max_{\lambda \in \Lambda} \alpha_{\xi,r}(A_\lambda)$$

if $1 \leqslant r \leqslant p$ and

$$\alpha_{\xi,r}\left(\bigoplus_{\lambda \in \Lambda} A_\lambda : \left(\bigoplus_{\lambda \in \Lambda} X_\lambda\right)_{\ell_p(\Lambda)} \to \left(\bigoplus_{\lambda \in \Lambda} Y_\lambda\right)_{\ell_p(\Lambda)}\right)^s = \sum_{\lambda \in \Lambda} \alpha_{\xi,r}(A_\lambda)^s$$

if $p < r \leqslant \infty$, where $1/r + 1/s = 1/p$.

We recall that an operator $A : X \to Y$ is *Asplund* if for any finite, positive measure space (Ω, Σ, μ), any $B : Y \to L_\infty(\Omega, \Sigma, \mu)$, and any $\varepsilon > 0$, there exists $S \in \Sigma$ such that $\mu(S) > \mu(\Omega) - \varepsilon$ and $\{f 1_S : f \in BA(B_X)\}$ is relatively weakly compact in $L_\infty(\Omega, \Sigma, \mu)$. Brooker [3] showed that an operator is Asplund if and only if its Szlenk index is an ordinal.

We also investigate the behavior of the $\alpha_{\xi,r}$ seminorms under infinite ℓ_p and c_0 directs sums of operators. In what follows, suppose that Λ is a non-empty set and $\{A_\lambda : X_\lambda \to Y_\lambda : \lambda \in \Lambda\}$ is a uniformly bounded collection of Asplund operators. We offer a new proof of a result due to Brooker concerning the

Szlenk index of diagonal operators between ℓ_p direct sums. The method of our proof is dual to Brooker's method of proof, working with upper estimates on convex combinations of weakly null trees in the spaces rather than derivations in the dual. This result has many cases, so we isolate the following single case, which we believe to be of the greatest interest.

THEOREM 1.6. *Fix $p \in \{0\} \cup (1,\infty)$ and suppose that at least one of the operators $(A_\lambda)_{\lambda \in \Lambda}$ is non-compact. Then if $D^p : (\bigoplus_{\lambda \in \Lambda} X_\lambda)_{\ell_p(\Lambda)} \to (\bigoplus_{\lambda \in \Lambda} Y_\lambda)_{\ell_p(\Lambda)}$ is the diagonal operator such that $D^p|_{X_\lambda} = A_\lambda$, $Sz(D^p) = \sup_{\lambda \in \Lambda} Sz(A_\lambda)$ if and only if for every $\varepsilon > 0$, $\sup_{\lambda \in \Lambda} Sz(A_\lambda, \varepsilon) < \sup_{\lambda \in \Lambda} Sz(A_\lambda)$, and otherwise $Sz(D^p) = (\sup_{\lambda \in \Lambda} Sz(A_\lambda))\omega$.*
  Here, if $p = 0$, we replace the $\ell_p(\Lambda)$ direct sum with the $c_0(\Lambda)$ direct sum.

We also elucidate the values of $\alpha_{\xi,\gamma}(D^p)$ (and therefore compute $\mathbf{p}_\xi(D^p)$) for all values of $p \in \{0\} \cup [1,\infty]$ and $1 \leqslant \gamma \leqslant 1$ only in terms of the behaviors of $Sz(A_\lambda, \varepsilon)$ and $\alpha_{\xi,\gamma}(A_\lambda)$, thus completely describing the behavior of the ideal norms and Szlenk power type of diagonal operators between ℓ_p direct sums in terms of the ideal norms and Szlenk power types of the summands.

THEOREM 1.7. *Fix $p \in \{0\} \cup (1,\infty)$ and suppose $Sz(D_p) = \omega^{\xi+1}$ and $\sup_{\lambda \in \Lambda} Sz(A_\lambda) = \omega^\zeta > 1$. Then by Theorem 1.6, $\zeta + 1 = \xi + 1$ if and only if there exists $\varepsilon > 0$ such that $\sup_{\lambda \in \Lambda} Sz(A_\lambda, \varepsilon) = \omega^\zeta$, and otherwise $\zeta = \xi + 1$.*
  If $\zeta + 1 = \xi + 1$, then for any $1 \leqslant r \leqslant \infty$, then $\alpha_{\xi,r}(D_p) < \infty$ if and only if either $p = 0$ or $1 \leqslant r \leqslant p < \infty$.
  If $\zeta = \xi + 1$, then for any $1 \leqslant r \leqslant \infty$, then $\alpha_{\xi,r}(D_p) < \infty$ if and only of one of the three following conditions hold:

  (i) *$p = 0$ and $\sup_{\lambda \in \Lambda} \alpha_{\xi,r}(A_\lambda) < \infty$.*
  (ii) *$1 \leqslant r \leqslant p < \infty$ and $\sup_{\lambda \in \Lambda} \alpha_{\xi,r}(A_\lambda) < \infty$.*
  (iii) *$1 < p < r \leqslant \infty$, $\sum_{\lambda \in \Lambda} \alpha_{\xi,r}(A_\lambda)^s < \infty$, and for any $\varepsilon > 0$, there exists a finite subset Υ of Λ such that $\sup_{\lambda \in \Lambda \setminus \Upsilon} Sz(A_\lambda, \varepsilon) < \omega^\xi$. Here, s is defined by $1/r + 1/s = 1/p$.*

2. Combinatorial necessities

2.1. Trees of peculiar importance. For a set Λ, we let $\Lambda^{<\mathbb{N}} = \bigcup_{n=0}^\infty \Lambda^n$. We denote the single member of Λ^0 by $\varnothing$. We order $\Lambda^{<\mathbb{N}} \setminus \Lambda^0$ by initial segments, denoted by $\leqslant$. That is, $s \leqslant (\lambda_i)_{i=1}^n$ if $s = (\lambda_i)_{i=1}^m$ for some $1 \leqslant m \leqslant n$. A subset T of $\Lambda^{<\mathbb{N}} \setminus \Lambda^0$ will be called a *tree* if $\varnothing \neq s \leqslant t \in T$ implies $s \in T$. Given a subset U of $\Lambda^{<\mathbb{N}}$, we let $\mathrm{MAX}(U)$ denote the subset of U consisting of those members of U which are maximal with respect to the initial segment relation. Given two members s, t of $\Lambda^{<\mathbb{N}}$, we let $s \frown t$ denote the concatenation of s and t.

We first define some trees which will be of significant importance for us. These trees and the associated convex weights defined in the next paragraphs

were first defined in [5]. We refer the interested reader to [5] for a more complete presentation. Here we are interested only in presenting the details of the construction necessary to prove our results. Given a sequence $(\zeta_i)_{i=1}^n$ of ordinals and an ordinal ζ, we let $\zeta + (\zeta_i)_{i=1}^n = (\zeta + \zeta_i)_{i=1}^n$. Given a set G of sequences of ordinals and an ordinal ζ, we let $\zeta + G = \{\zeta + t : t \in G\}$. For each $\xi \in \mathbf{Ord}$ and $n \in \mathbb{N}$, we define a tree $\Gamma_{\xi,n}$ which consists of finite, decreasing sequences of ordinals in the interval $[0, \omega^\xi n)$. We let

$$\Gamma_{0,1} = \{(0)\}.$$

If $\Gamma_{\xi,1}$ has been defined, we let $\Lambda_{\xi,1,1} = \Gamma_{\xi,1}$ and for $1 < n \in \mathbb{N}$ and $1 \leqslant i \leqslant n$, we let

$$\Lambda_{\xi,n,i} = \big\{ \big(\omega^\xi(n-1) + t_1\big) \frown \cdots \frown \big(\omega^\xi(n-i) + t_i\big) : t_i \in \Gamma_{\xi,1},$$
$$t_1, \ldots, t_{i-1} \in \mathrm{MAX}(\Gamma_{\xi,1})\big\}$$

and $\Gamma_{\xi,n} = \bigcup_{i=1}^n \Lambda_{\xi,n,i}$. If for some ξ, $\Gamma_{\xi,n}$ has been defined for each $n \in \mathbb{N}$, we let $\Gamma_{\xi+1,1} = \bigcup_{n=1}^\infty \Gamma_{\xi,n}$. If ξ is a limit ordinal and $\Gamma_{\zeta,1}$ has been defined for every $\zeta < \xi$, we let

$$\Gamma_{\xi,1} = \bigcup_{\zeta<\xi} \big(\omega^\zeta + \Gamma_{\zeta+1,1}\big).$$

Let us observe that for each ordinal ξ and each $n \in \mathbb{N}$, the first member of each sequence in $\Gamma_{\xi,n}$ is an ordinal which lies in the interval $[\omega^\xi(n-1), \omega^\xi n)$. Therefore the union $\Gamma_{\xi+1,1} = \bigcup_{n=1}^\infty \Gamma_{\xi,n}$ is a totally incomparable union. We refer to the sets $\Lambda_{\xi,n,1}, \ldots, \Lambda_{\xi,n,n}$ as the *levels* of $\Gamma_{\xi,n}$. The union $\Gamma_{\xi,1} = \bigcup_{\zeta<\xi}(\omega^\zeta + \Gamma_{\zeta+1,1})$ when ξ is a limit ordinal is also a totally incomparable union.

We provide a brief, intuitive description of the trees constructed above. We begin with a tree $\Gamma_{0,1}$, which consists of a single node. Once we have constructed $\Gamma_{\xi,1}$, the tree $\Gamma_{\xi,n}$ consists of n levels. The first level of $\Gamma_{\xi,n}$ is isomorphic to $\Gamma_{\xi,1}$. Then beneath each maximal node on a level $i < n$, we place another copy of $\Gamma_{\xi,1}$. The tree $\Gamma_{\xi+1,1}$ is a totally incomparable union of the n-leveled trees $\Gamma_{\xi,n}$. Finally, when ξ is a limit ordinal, $\Gamma_{\xi,1}$ is just a totally incomparable union of a cofinal subset of smaller trees.

We also recall the existence of functions $\mathbb{P}_{\xi,n} : \Gamma_{\xi,n} \to [0,1]$ having the property that for each ordinal ξ, each $n \in \mathbb{N}$, each $1 \leqslant i \leqslant n$, and $t \in \mathrm{MAX}(\Gamma_{\xi,n})$,

$$1 = \sum_{\Lambda_{\xi,n,i} \ni s \leqslant t} \mathbb{P}_{\xi,n}(s).$$

The functions $\mathbb{P}_{\xi,n}$ are quite easy to understand. The function $\mathbb{P}_{0,1}$ is identically 1 on the single node contained in $\Gamma_{0,1}$. Since $\Gamma_{\xi,n}$ is obtained by "gluing" together many copies of trees which are canonically isomorphic to $\Gamma_{\xi,1}$, we define $\mathbb{P}_{\xi,n}$ so that the canonical identification of each of these copies of $\Gamma_{\xi,1}$ also identifies the values of $\mathbb{P}_{\xi,n}$ on isomorphic copies of $\Gamma_{\xi,1}$ with the values of $\mathbb{P}_{\xi,1}$ on $\Gamma_{\xi,1}$. For $\Gamma_{\xi+1,1} = \bigcup_{n=1}^\infty \Gamma_{\xi,n}$, we normalize the branches (that is

$\Gamma_{\xi+1,1}|_{\Gamma_{\xi,n}} = n^{-1}\mathbb{P}_{\xi,n}$). For a limit ordinal ξ, $\mathbb{P}_{\xi,1}$ is defined in such a way that its values on $\omega^\zeta + \Gamma_{\zeta+1,1}$ are equal to the values of $\mathbb{P}_{\zeta+1,1}$ on $\Gamma_{\zeta+1,1}$ via the canonical identification.

For a directed set D, a set Λ, and a subset S of $\Lambda^{<\mathbb{N}} \setminus \Lambda^0$, we let

$$S.D = \big\{(\lambda_j, u_j)_{j=1}^k : (\zeta_j)_{j=1}^k \in S, u_j \in D\big\}.$$

In particular, for an ordinal ξ, $n \in \mathbb{N}$, and $1 \leqslant i \leqslant n$,

$$\Lambda_{\xi,n,i}.D = \big\{(\zeta_j, u_j)_{j=1}^k : (\zeta_j)_{j=1}^k \in \Lambda_{\xi,n,i}, u_j \in D\big\}.$$

We define functions (which we also denote by $\mathbb{P}_{\xi,n}$) on $\Gamma_{\xi,n}.D$ by letting

$$\mathbb{P}_{\xi,n}\big((\zeta_j, u_j)_{j=1}^k\big) = \mathbb{P}_{\xi,n}\big((\zeta_j)_{j=1}^k\big).$$

We remark that for each ζ and any directed set D, $(\omega^\zeta + \Gamma_{\zeta+1,1}).D$ is canonically identifiable with $\Gamma_{\zeta+1,1}.D$. For any ξ and any $n \in \mathbb{N}$, $\Lambda_{\xi,n,1}.D$ is canonically identifiable with $\Gamma_{\xi,1}.D$. We often implicitly use these canonical identifications without giving them specific names. Moreover, these canonical identifications preserve the values of the functions $\mathbb{P}_{\xi,n}$, as was shown in [5].

We last define what it means for a subset of $\Gamma_{\xi,n}.D$ to be a *unit*. For any ordinal ξ and any $n \in \mathbb{N}$, $\Lambda_{\xi,n,1}.D$ is a unit. If for some $n \in \mathbb{N}$, every ordinal ξ, and every $1 \leqslant k \leqslant n$, the units in $\Gamma_{\xi,k}.D$ are defined, we say a subset U of $\Gamma_{\xi,n+1}.D$ is a unit if either $U = \Lambda_{\xi,n+1,1}.D$ or if there exists $t \in \mathrm{MAX}(\Lambda_{\xi,n+1,1}.D)$ such that, if

$$j : \{s \in \Gamma_{\xi,n+1}.D : t < s\} \to \Gamma_{\xi,n}.D$$

is the canonical identification, $j(U)$ is a unit in $\Gamma_{\xi,n}.D$

2.2. Cofinal and eventual sets. For a fixed directed set D, we now define sets $\Omega_{\xi,n}$. Each set $\Omega_{\xi,n}$ will be a subset of the power set of $\mathrm{MAX}(\Gamma_{\xi,n}.D)$. Given $\mathcal{E} \subset \Gamma_{0,1}.D$, we can write

$$\mathcal{E} = \big\{(0, u) : u \in D_0\big\}$$

for some $D_0 \subset D$. Then we say $\mathcal{E} \in \Omega_{0,1}$ if D_0 is cofinal in D.

Now suppose that for a limit ordinal ξ and every $\zeta < \xi$, $\Omega_{\zeta+1,1}$ has been defined. For each $\zeta < \xi$, let $j_\zeta : (\omega^\zeta + \Gamma_{\zeta+1}).D \to \Gamma_{\zeta+1,1}.D$ be the canonical identification. Then a subset $\mathcal{E} \subset \mathrm{MAX}(\Gamma_{\xi,1})$ lies in $\Omega_{\xi,1}$ if there exists a cofinal subset M of $[0, \xi)$ such that for every $\zeta \in M$, $j_\zeta(\mathcal{E} \cap \mathrm{MAX}((\omega^\zeta + \Gamma_{\zeta+1}.D)) \in \Omega_{\zeta+1,1}$.

Now suppose that for an ordinal ξ and every $n \in \mathbb{N}$, $\Omega_{\xi,n}$ has been defined. Then we say $\mathcal{E} \subset \mathrm{MAX}(\Gamma_{\xi+1,1}.D)$ is a member of $\Omega_{\xi+1,1}$ if there exists a cofinal subset M of $\mathbb{N}$ such that for every $n \in \mathbb{N}$, $\mathcal{E} \cap \Gamma_{\xi,n}.D \in \Omega_{\xi,n}$.

Last, suppose that for an ordinal ξ, a natural number n, and each $1 \leqslant i \leqslant n$, $\Omega_{\xi,i}$ has been defined. Suppose that $\mathcal{E} \subset \mathrm{MAX}(\Gamma_{\xi,n+1}.D)$ is given. For each $t \in \mathrm{MAX}(\Lambda_{\xi,n,1})$, let $P_t = \{s \in \Gamma_{\xi,n+1}.D : t < s\}$, let $j_t : P_t \to \Gamma_{\xi,n}.D$ be

the canonical identification, and let $j : \Lambda_{\xi,n,1}.D \to \Gamma_{\xi,1}.D$ be the canonical identification. Let

$$\mathcal{F} = \big\{ t \in \mathrm{MAX}(\Lambda_{\xi,n+1,1}.D) : j_t\big(\mathcal{E} \cap \mathrm{MAX}(P_t)\big) \in \Omega_{\xi,n} \big\}.$$

Then we say $\mathcal{E} \in \Omega_{\xi,n+1}$ if $j(\mathcal{F}) \in \Omega_{\xi,1}$.

We remark that an easy induction proof shows that $\mathrm{MAX}(\Gamma_{\xi,n}.D) \in \Omega_{\xi,n}$ for every ξ an n, and if $\mathcal{F} \subset \mathcal{E} \subset \mathrm{MAX}(\Gamma_{\xi,n}.D)$ and $\mathcal{F} \in \Omega_{\xi,n}$, then $\mathcal{E} \in \Omega_{\xi,n}$.

We refer to the sets in $\Omega_{\xi,n}$ as *cofinal* in $\Gamma_{\xi,n}.D$. We say a subset $\mathcal{E}$ of $\mathrm{MAX}(\Gamma_{\xi,n}.D)$ is *eventual* if $\mathrm{MAX}(\Gamma_{\xi,n}.D) \setminus \mathcal{E}$ fails to be cofinal. Each unit $U \subset \Gamma_{\xi,n}.D$ is canonically identifiable with $\Gamma_{\xi,1}.D$, and as such we can define what it means for a subset of $\mathrm{MAX}(U)$ to be cofinal or eventual using the identification with $\Gamma_{\xi,1}.D$.

We next recall some results from [6]. For the following results, we say $d : \Gamma_{\xi,n}.D \to \Gamma_{\xi,n}.D$ is a *level map* if

(i) for any $\varnothing < s < t \in \Gamma_{\xi,n}.D$, $d(s) < d(t)$,

(ii) if $U \subset \Lambda_{\xi,n,i}.D$ is a unit, then there exists a unit $V \subset \Lambda_{\xi,n,i}.D$ such that $d(U) \subset V$.

Note that since $\Gamma_{\xi,1}.D$ is a single unit, (ii) is vacuous in the case $n = 1$. Given a level map $d : \Gamma_{\xi,n}.D \to \Gamma_{\xi,n}.D$, we say $e : \mathrm{MAX}(\Gamma_{\xi,n}.D) \to \mathrm{MAX}(\Gamma_{\xi,n}.D)$ is an *extension* of d if for any $t \in \mathrm{MAX}(\Gamma_{\xi,n}.D)$, $d(t) \leqslant e(t)$. Since $\Gamma_{\xi,n}.D$ is well-founded, any level map d admits some extension. We define an extension of a monotone map in the same way we define an extension of a level map.

We let $\Pi(\Gamma_{\xi,n}.D) = \{(s,t) \in \Gamma_{\xi,n}.D \times \mathrm{MAX}(\Gamma_{\xi,n}.D) : s \leqslant t\}$.

LEMMA 2.1. *Suppose that ξ is an ordinal, $n \in \mathbb{N}$, X is a Banach space, and $(x_t)_{t \in \Gamma_{\xi,n}.D} \subset X$ is weakly null.*

(i) *If $\mathcal{E} \subset \mathrm{MAX}(\Gamma_{\xi,n}.D)$ is cofinal, there exists a level map $d : \Gamma_{\xi,n}.D \to \Gamma_{\xi,n}.D$ with extension e such that $(x_{d(t)})_{t \in \Gamma_{\xi,n}.D}$ is weakly null and $e(\mathrm{MAX}(\Gamma_{\xi,n}.D)) \subset \mathcal{E}$.*

(ii) *For any $k \in \mathbb{N}$, if $\mathrm{MAX}(\Gamma_{\xi,n}.D) \supset \mathcal{E} = \bigcup_{i=1}^{k} \mathcal{E}_i \in \Omega_{\xi,n}$, then there exists $1 \leqslant j \leqslant k$ such that $\mathcal{E}_j \in \Omega_{\xi,n}$.*

(iii) *If F is a finite set and $\chi : \Pi(\Gamma_{\xi,n}.D) \to F$ is a function, then there exist a level map $d : \Gamma_{\xi,n}.D \to \Gamma_{\xi,n}.D$ with extension e and $\alpha_1, \ldots, \alpha_n \in F$ such that for any $1 \leqslant i \leqslant n$ and any $\Lambda_{\xi,n,i}.D \ni s \leqslant \in \mathrm{MAX}(\Gamma_{\xi,n}.D)$, $\alpha_i = F(d(s), e(t))$, and such that $(x_{d(t)})_{t \in \Gamma_{\xi,n}.D}$ is weakly null.*

(iv) *If $h : \Pi(\Gamma_{\xi,n}.D) \to \mathbb{R}$ is bounded and if $\mathcal{E} \subset \mathrm{MAX}(\Gamma_{\xi,n}.D)$ is cofinal, then for any $\delta > 0$, there exist $a_1, \ldots, a_n \in \mathbb{R}$ and a level map $d : \Gamma_{\xi,n}.D \to \Gamma_{\xi,n}.D$ with extension e such that $e(\mathrm{MAX}(\Gamma_{\xi,n}.D)) \subset \mathcal{E}$, for each $1 \leqslant i \leqslant n$ and each $\Lambda_{\xi,n,i}.D \ni s \leqslant t \in \mathrm{MAX}(\Gamma_{\xi,n}.D)$, $h(d(s), e(t)) \geqslant a_i - \delta$, and for any $t \in \mathrm{MAX}(\Gamma_{\xi,n}.D)$,*

$$\sum_{\varnothing < s \leqslant e(t)} \mathbb{P}_{\xi,n}(s) h\big(s, e(t)\big) \leqslant \delta + \sum_{i=1}^{n} a_i.$$

REMARK 2.2. Items (i) and (ii) together yield that if $\mathrm{MAX}(\Gamma_{\xi,n}.D) = \bigcup_{i=1}^{k} \mathcal{E}_i$, then there exist $1 \leqslant j \leqslant k$ and a level map $d : \Gamma_{\xi,n}.D \to \Gamma_{\xi,n}.D$ with extension e such that $(x_{d(t)})_{t \in \Gamma_{\xi,n}.D}$ is weakly null and $e(\mathrm{MAX}(\Gamma_{\xi,n}.D)) \subset \mathcal{E}_j$. A typical application of this result will be to have a real-valued function $h : \mathrm{MAX}(\Gamma_{\xi,n}.D) \to C \subset \mathbb{R}$, where C is compact. We may then fix $\delta > 0$ and a finite cover $F_1, \ldots, F_k$ of C by sets of diameter less than δ. We then let $\mathcal{E}_i$ denote those $t \in \mathrm{MAX}(\Gamma_{\xi,n}.D)$ such that $h(t) \in F_i$. We may then find d, e, and j as above and obtain $(x_{d(t)})_{t \in \Gamma_{\xi,n}.D}$ weakly null such that for every $t \in \mathrm{MAX}(\Gamma_{\xi,1}.D)$, $h(e(t)) \in F_j$.

Similarly, we will often apply (iii) to a function $h_1 : \Pi(\Gamma_{\xi,n}.D) \to C \subset \mathbb{R}$, where C is compact, by first covering C by $F_1, \ldots, F_k$ of sets of diameter less than δ. We then define $h(s,t)$ to be the minimum $j \leqslant k$ such that $h_1(s,t) \in F_j$.

3. Szlenk index

Given a Banach space X, a weak*-compact subset K of X^*, and $\varepsilon > 0$, we let $s_\varepsilon(K)$ denote the set of those $x^* \in K$ such that for every weak*-neighborhood V of x^*, $\mathrm{diam}(V \cap K) > \varepsilon$. We then define the transfinite derivations by

$$s_\varepsilon^0(K) = K,$$
$$s_\varepsilon^{\xi+1}(K) = s_\varepsilon\big(s_\varepsilon^\xi(K)\big),$$

and if ξ is a limit ordinal,

$$s_\varepsilon^\xi(K) = \bigcap_{\zeta < \xi} s_\varepsilon^\zeta(K).$$

For convenience, we define $s_\varepsilon^\xi(K) = K$ for each $\varepsilon \leqslant 0$. We let $Sz(K, \varepsilon)$ be the minimum ξ such that $s_\varepsilon^\xi(K) = \varnothing$, assuming such an ordinal exists. If no such ordinal exists, we write $Sz(K, \varepsilon) = \infty$. We let $Sz(K) = \sup_{\varepsilon > 0} Sz(K, \varepsilon)$, with the agreement that $Sz(K) = \infty$ if $Sz(K, \varepsilon) = \infty$ for some $\varepsilon > 0$. Given an operator $A : X \to Y$, we let $Sz(A, \varepsilon) = Sz(A^* B_{Y^*}, \varepsilon)$ and $Sz(A) = Sz(A^* B_{Y^*})$. Given a Banach space X, we let $Sz(X, \varepsilon) = Sz(I_X, \varepsilon)$ and $Sz(X) = Sz(I_X)$.

Given an ordinal ξ, $1 \leqslant q < \infty$, and $M \geqslant 0$, we say K has *M-ξ-q-summable Szlenk index* if for any $\varepsilon_1, \ldots, \varepsilon_n \geqslant 0$ such that $s_{\varepsilon_1}^{\omega^\xi} \cdots s_{\varepsilon_n}^{\omega^\xi}(K) \neq \varnothing$, $\sum_{i=1}^{n} \varepsilon_i^q \leqslant M^q$. We say K has *ξ-q-summable Szlenk index* if it has M-ξ-q-summable Szlenk index for some $M \geqslant 0$. Given an operator $A : X \to Y$, we say A has ξ-q-summable Szlenk index if $A^* B_{Y^*}$ does, and we say the Banach space X has ξ-summable Szlenk index if I_X does. The notion of 0-1-summable Szlenk index has been previously defined in [13], and it is quite important to the nonlinear theory of Banach spaces and renorming theory. For $\xi > 0$ or $1 < q < \infty$, the notion of ξ-q-summable Szlenk index is new.

Suppose X is a Banach space and $K \subset X^*$ is weak*-compact and nonempty. By a standard compactness argument, $Sz(K, \varepsilon) > \omega^\xi n$ for each $n \in \mathbb{N}$

would imply $Sz(K, \varepsilon) > \omega^\xi n$. Therefore, $Sz(K) \leqslant \omega^{\xi+1}$ implies that for every $\varepsilon > 0$, we have $Sz(K, \varepsilon) \leqslant \omega^\xi n$ for some $n \in \mathbb{N}$. Then if $Sz(K) \leqslant \omega^{\xi+1}$, we let $Sz_\xi(K, \varepsilon)$ be the smallest $n \in \mathbb{N}$ such that $Sz(K, \varepsilon) \leqslant \omega^\xi n$. For this K, we define the ξ-*Szlenk power type* $\mathbf{p}_\xi(K)$ by

$$\mathbf{p}_\xi(K) := \limsup_{\varepsilon \to 0^+} \frac{\log Sz_\xi(K, \varepsilon)}{|\log(\varepsilon)|}.$$

This value need not be finite. We note that if $Sz(K) \leqslant \omega^\xi$, $Sz_\xi(K, \varepsilon) = 1$ for all $\varepsilon > 0$, whence $\mathbf{p}_\xi(K) = 0$. For completeness, we write $\mathbf{p}_\xi(K) = \infty$ if $Sz(K) > \omega^{\xi+1}$.

We remark that if K has M-ξ-q-summable Szlenk index, then $\mathbf{p}_\xi(K) \leqslant q$. Indeed, for any $\varepsilon > 0$, if $n < Sz_\xi(K, \varepsilon)$,

$$\varnothing \neq s_\varepsilon^{\omega^\xi n}(K) = s_\varepsilon^{\omega^\xi} \cdots s_\varepsilon^{\omega^\xi}(K),$$

and $\varepsilon^q n = \sum_{i=1}^n \varepsilon_i^q \leqslant M^q$. From this it follows that

$$\log Sz_\xi(K, \varepsilon) \leqslant \log(1 + M^q / \varepsilon^q),$$

whence

$$\limsup_{\varepsilon \to 0^+} \frac{\log Sz_\xi(K, \varepsilon)}{|\log(\varepsilon)|} \leqslant q.$$

We also recall the following fact. This fact was shown by Lancien [16] in the case that $K = B_{X^*}$.

PROPOSITION 3.1 ([7, LEMMA 3.8]). *Let X be a Banach space, $K \subset X^*$ weak*-compact and convex, ξ an ordinal.*

(i) *If for some $\varepsilon > 0$, $Sz(K, \varepsilon) > \omega^\xi$, then $Sz(K, \varepsilon/n) > \omega^\xi n$.*
(ii) *If $\omega^\xi < Sz(K) \leqslant \omega^{\xi+1}$, then $Sz(K) = \omega^{\xi+1}$.*
(iii) *If $Sz(K) > \omega^\xi$ then $\mathbf{p}_\xi(K) \geqslant 1$.*

COROLLARY 3.2. *If X is a Banach space and $K \subset X^*$ is a weak*-compact, convex, non-empty set, then either $Sz(K) = \infty$ or there exists an ordinal ξ such that $Sz(K) = \omega^\xi$.*

4. The quantities $\alpha_{\xi,p,n}(K)$

Given a directed set D and $(x_t)_{t \in \Gamma_{\xi,n}.D} \subset X$, if $t \in \mathrm{MAX}(\Gamma_{\xi,n}.D)$, there exist $\varnothing = t_0 < \cdots < t_n = t$ such that $t_i \in \mathrm{MAX}(\Lambda_{\xi,n,i}.D)$ for each $1 \leqslant i \leqslant n$. We then let

$$z_i^t = \sum_{t_{i-1} < s \leqslant t_i} \mathbb{P}_{\xi,n}(s) x_s \in \mathrm{co}(x_s : t_{i-1} < s \leqslant t_i).$$

This notation should reference the underlying collection $(x_t)_{t \in \Gamma_{\xi,n}.D}$, but the notation will not cause confusion.

Let X be a Banach space and let $K \subset X^*$ be weak*-compact. For $x \in X$, let $r_K(x) = 0$ if $K = \varnothing$, and otherwise let $r_K(x) = \max_{x^* \in K} \mathrm{Re}\, x^*(x)$. For an

ordinal ξ, $1 \leqslant p \leqslant \infty$, and $n \in \mathbb{N}$, we let $\alpha_{\xi,p,n}(K)$ be the infimum of those $\alpha > 0$ such that for any directed set D, any $(a_i)_{i=1}^n \in \mathbb{K}^n$, and any weakly null $(x_t)_{t \in \Gamma_{\xi,n}.D} \subset B_X$,

$$\inf_{t \in \mathrm{MAX}(\Gamma_{\xi,n}.D)} r_K\left(\sum_{i=1}^n a_i z_i^t\right) \leqslant \alpha \left\| (a_i)_{i=1}^n \right\|_{\ell_p^n}.$$

We let $\alpha_{\xi,p}(K) = \sup_n \alpha_{\xi,p,n}(K)$. Let $\theta_{\xi,n}(K)$ be the infimum of those $\theta > 0$ such that for any directed set D and any weakly null $(x_t)_{t \in \Gamma_{\xi,n}.D} \subset B_X$,

$$\inf_{t \in \mathrm{MAX}(\Gamma_{\xi,n}.D)} r_K\left(\sum_{i=1}^n \frac{1}{n} z_i^t\right) \leqslant \theta.$$

REMARK 4.1. It is an easy consequence of [4, Theorem 2.2] is that if X is a Banach space, $K \subset X^*$ is weak*-compact, ξ is an ordinal, $n \in \mathbb{N}$, $\varepsilon > 0$, $Sz(K, \varepsilon) > \omega^\xi n$, and D is any weak neighborhood basis at 0 in X, then for any $0 < \delta < \varepsilon/4$, there exist a weakly null collection $(x_t)_{t \in \Gamma_{\xi,n}.D} \subset B_X$ and $(x_t^*)_{t \in \mathrm{MAX}(\Gamma_{\xi,n}.D)} \subset K$ such that for each $\varnothing < s \leqslant t \in \mathrm{MAX}(\Gamma_{\xi,n}.D)$, $\mathrm{Re}\, x_t^*(x_s) \geqslant \delta$. In particular, $\theta_{\xi,n}(K) \geqslant \varepsilon/4$.

Conversely, it follows from [4, Theorem 2.2] and [5, Corollary 5.3] that if $\theta_{\xi,1}(K) \geqslant \varepsilon$, $s_{\varepsilon_0}^{\omega^\xi}(K) \neq \varnothing$ for any $0 < \varepsilon_0 < \varepsilon$.

REMARK 4.2. For later convenience, the definition of $\alpha_{\xi,p,n}$ considers all weakly null collections indexed by any directed set D. However, in the definition of $\alpha_{\xi,p,n}$ could be taken to include only weakly null collections indexed by $\Gamma_{\xi,n}.D_1$, where D_1 is some fixed weak neighborhood basis at 0 in X. We will freely use this fact throughout. In order to see why this holds, fix a scalar sequence $(a_i)_{i=1}^n$ and a positive number α. If we have

$$\inf_{t \in \mathrm{MAX}(\Gamma_{\xi,n}.D)} r_K\left(\sum_{i=1}^n a_i z_i^t\right) \leqslant \alpha \left\| (a_i)_{i=1}^n \right\|_{\ell_p^n}$$

for every directed set D and every weakly null collection $(x_t)_{t \in \Gamma_{\xi,n}.D} \subset B_X$, then we obviously have it whenever $D = D_1$. For the converse, if there exists a directed set D and a weakly null collection $(x_t)_{t \in \Gamma_{\xi,n}.D} \subset B_X$ such that $\inf_{t \in \mathrm{MAX}(\Gamma_{\xi,n}.D)} r_K(\sum_{i=1}^n a_i z_i^t) > \alpha \|(a_i)_{i=1}^n\|_{\ell_p^n}$, we may define a map $\phi : \Gamma_{\xi,n}.D_1 \to \Gamma_{\xi,n}.D$ such that, with $u_t = x_{\phi(t)}$, $(u_t)_{t \in \Gamma_{\xi,n}.D_1} \subset B_X$ is weakly null and for every $t \in \mathrm{MAX}(\Gamma_{\xi,n}.D_1)$,

$$\sum_{i=1}^n a_i \sum_{\Lambda_{\xi,n,i}.D_1} \mathbb{P}_{\xi,n}(s) u_s = \sum_{i=1}^n a_i z_i^{\phi(t)},$$

whence

$$\inf_{t \in \mathrm{MAX}(\Gamma_{\xi,n}.D_1)} r_K\left(\sum_{i=1}^n a_i \sum_{\Lambda_{\xi,n,i}.D_1} \mathbb{P}_{\xi,n}(s) u_s\right)$$

$$\geqslant \inf_{t\in\text{MAX}(\Gamma_{\xi,n}.D)} r_K\left(\sum_{i=1}^n a_i z_i^t\right)$$

$$> \alpha\left\|(a_i)_{i=1}^n\right\|_{\ell_p^n}.$$

The previous remark will be useful, for example, when considering direct sums. If $A_1 : X_1 \to Y_1$ and $A_2 : X_2 \to Y_2$ are operators, we may wish to consider $\alpha_{\xi,p,n}(A_1 \oplus A_2 : X_1 \oplus_r X_2 \to Y_1 \oplus_r Y_2)$. If $\alpha_1 < \alpha_{\xi,p,n}(A_1)$, we will find a directed set D_1 and a weakly null collection $(x_t^1)_{t\in\Gamma_{\xi,n}.D_1} \subset B_{X_1}$ to witness that $\alpha_1 < \alpha_{\xi,p,n}(A_1)$. Similarly, if $\alpha_2 < \alpha_{\xi,p,n}(A_2)$, we will find a directed set D_2 and a weakly null collection $(x_t^2)_{t\in\Gamma_{\xi,n}.D_2} \subset B_{X_2}$ to witness that $\alpha_2 < \alpha_{\xi,p,n}(A_2)$. By the definition, we do not have control over D_1 or D_2, and the fact that D_1 need not be equal to D_2 is problematic. We will use Remark 4.2 to deduce that, in both cases, we can take $D_1 = D_2 = D$, where D is a fixed weak neighborhood basis at 0 in $X_1 \oplus_r X_2$.

One inconvenience of the definitions of $\alpha_{\xi,p,n}$ and $\theta_{\xi,n}$ is that they involve special convex combinations, and the convex coefficient on a vector depends upon its position in the tree. Therefore even if we know that

$$\alpha < \inf_{t\in\text{MAX}(\Gamma_{\xi,n}.D)} r_K\left(\sum_{i=1}^n a_i z_i^t\right),$$

if $d : \Gamma_{\xi,n}.D \to \Gamma_{\xi,n}.D$ is a level map such that $(x_{d(t)})_{t\in\Gamma_{\xi,n}.D}$ is weakly null, we do not know that

$$\alpha < \inf_{t\in\text{MAX}(\Gamma_{\xi,n}.D)} r_K\left(\sum_{i=1}^n \sum_{t\geqslant s\in\Lambda_{\xi,n,i}.D} a_i \mathbb{P}_{\xi,n}(s)x_{d(x)}\right).$$

Initially this prevents us from using our combinatorial lemmas to stabilize certain quantities for members of a given tree $(x_t)_{t\in\Gamma_{\xi,n}.D}$ which was chosen to witness that $\alpha < \alpha_{\xi,p,n}(K)$. But suppose that $(a_i)_{i=1}^n$ are non-negative reals and that we have numbers $b_1,\ldots,b_n$ such that $\sum_{i=1}^n a_i b_i > \alpha$, a weakly null collection $(x_t)_{t\in\Gamma_{\xi,n}.D} \subset B_X$, and a collection $(x_t^*)_{t\in\text{MAX}(\Gamma_{\xi,n}.D)} \subset K$ such that for every $1 \leqslant i \leqslant n$ and $\Lambda_{\xi,n,i}.D \ni s \leqslant t \in \text{MAX}(\Gamma_{\xi,n}.D)$, $\text{Re}\, x_t^*(x_s) \geqslant b_i$. Then for any level map $d : \Gamma_{\xi,n}.D \to \Gamma_{\xi,n}.D$, any extension e of d, any $1 \leqslant i \leqslant n$, and $\Lambda_{\xi,n,i}.D \ni s \leqslant t \in \text{MAX}(\Gamma_{\xi,n}.D)$, $\text{Re}\, x_{e(t)}^*(x_{d(s)}) \geqslant b_i$. Therefore for any $t \in \text{MAX}(\Gamma_{\xi,n}.D)$,

$$r_K\left(\sum_{i=1}^n \sum_{e(t)\geqslant s\in\Lambda_{\xi,n,i}.D} a_i \mathbb{P}_{\xi,n}(s)x_{d(s)}\right)$$

$$\geqslant x_{e(t)}^*\left(\sum_{i=1}^n \sum_{e(t)\geqslant s\in\Lambda_{\xi,n,i}.D} a_i \mathbb{P}_{\xi,n}(s)x_{d(s)}\right)$$

$$\geqslant \sum_{i=1}^{n} a_i b_i > \alpha.$$

Therefore if we have a map $f : B_X \to M$ into a compact metric space M, we may apply Lemma 2.1 to the function $F : \Pi(\Gamma_{\xi,n}.D) \to M$ given by $F(s,t) = f(x_s)$ to deduce that for any $\delta > 0$, there exist $\varpi_1, \ldots, \varpi_n \in M$, a level map $d : \Gamma_{\xi,n}.D \to \Gamma_{\xi,n}.D$, and an extension e of d such that $d_M(F(d(s),e(t)), \varpi_i) = d_M(f(x_{d(s)}), \varpi_i) < \delta$ for each $1 \leqslant i \leqslant n$ and each $\Lambda_{\xi,n,i}.D \ni s \leqslant t \in \mathrm{MAX}(\Gamma_{\xi,n}.D)$, and the collections $(x_{d(t)})_{t \in \Gamma_{\xi,n}.D}$, $(x^*_{e(t)})_{t \in \mathrm{MAX}(\Gamma_{\xi,n}.D)} \subset K$ can be used as described above to deduce that

$$\alpha < \inf_{t \in \mathrm{MAX}(\Gamma_{\xi,n}.D)} r_K \left(\sum_{i=1}^{n} \sum_{e(t) \geqslant s \in \Lambda_{\xi,n,i}.D} a_i \mathbb{P}_{\xi,n}(s) x_{d(s)} \right).$$

Thus we can replace the collection $(x_t)_{t \in \Gamma_{\xi,n}.D}$ with $(x_{d(t)})_{t \in \Gamma_{\xi,n}.D}$ without losing the inequality coming from the definition of $\alpha_{\xi,p,n}$ or $\theta_{\xi,n}$ while stabilizing the function f on each level of the collection. This is the primary motivation for the next theorem.

THEOREM 4.3. *Let X be a Banach space, $K \subset X^*$ weak*-compact, $1 \leqslant p \leqslant \infty$.*

(i) *For $\alpha \in \mathbb{R}$, the following are equivalent:*
  (a) $\alpha_{\xi,p,n}(K) > \alpha$.
  (b) *There exist non-negative scalars $(a_i)_{i=1}^{n} \in B_{\ell_p^n}$, a directed set D, a weakly null collection $(x_t)_{t \in \Gamma_{\xi,n}.D} \subset B_X$, and $\alpha' > \alpha$ such that*
  $$\left\{ t \in \mathrm{MAX}(\Gamma_{\xi,n}.D) : r_K \left(\sum_{i=1}^{n} a_i z_i^t \right) > \alpha' \right\}$$
  is cofinal.
  (c) *There exist non-negative scalars $(a_i)_{i=1}^{n} \in B_{\ell_p^n}$, a directed set D, a weakly null collection $(x_t)_{t \in \Gamma_{\xi,n}.D} \subset B_X$, $(x_t^*)_{t \in \mathrm{MAX}(\Gamma_{\xi,n}.D)} \subset K$, and non-negative real numbers $b_1, \ldots, b_n$ such that $\alpha < \sum_{i=1}^{n} a_i b_i$ and for each $1 \leqslant i \leqslant n$ and each $\Lambda_{\xi,n,i}.D \ni s \leqslant t \in \mathrm{MAX}(\Gamma_{\xi,n}.D)$, $\mathrm{Re}\, x_t^*(x_s) \geqslant b_i$.*

(ii) $\alpha_{\xi,p,n}(K) = 0$ *for some $n \in \mathbb{N}$ if and only if $\alpha_{\xi,p,n}(K) = 0$ for all $n \in \mathbb{N}$ if and only if $Sz(K) \leqslant \omega^\xi$.*

(iii) *If $R \geqslant 0$ is such that $K \subset RB_{X^*}$, $\alpha_{\xi,p,n}(K) \leqslant n^{1-1/p} R$.*

(iv) $\alpha_{\xi,p,n}(K) = \alpha_{\xi,p,n}(\overline{\mathrm{abs\, co}}^{\mathrm{weak}^*}(K))$.

Proof. (i) First assume there exist $\alpha' > \alpha$, a directed set D, non-negative scalars $(a_i)_{i=1}^{n} \in B_{\ell_p^n}$, and a weakly null collection $(x_t)_{t \in \Gamma_{\xi,n}.D} \subset B_X$ such that

$$\mathcal{E} := \left\{ t \in \mathrm{MAX}(\Gamma_{\xi,n}.D) : r_K \left(\sum_{i=1}^{n} a_i z_i^t \right) > \alpha' \right\}$$

is cofinal. For every $t \in \mathrm{MAX}(\Gamma_{\xi,n}.D)$, fix $x_t^* \in K$ such that

$$\mathrm{Re}\, x_t^* \left(\sum_{i=1}^{n} a_i z_i^t \right) = r_K \left(\sum_{i=1}^{n} a_i z_i^t \right).$$

Define $h : \Pi(\Gamma_{\xi,n}.D) \to \mathbb{R}$ by $h(s,t) = \mathrm{Re}\, x_t^*(a_i x_s)$, where $\Lambda_{\xi,n,i} \ni s \leqslant t \in \mathrm{MAX}(\Gamma_{\xi,n}.D)$. Fix a positive number δ such that $\delta(n+1) < \alpha' - \alpha$. By Lemma 2.1(iv), there exists a level map $d : \Gamma_{\xi,n}.D \to \Gamma_{\xi,n}.D$ with extension $e : \mathrm{MAX}(\Gamma_{\xi,n}.D) \to \mathcal{E}$ and scalars $b_1', \ldots, b_n' \in \mathbb{R}$ such that $(x_{d(t)})_{t \in \Gamma_{\xi,n}.D}$ is weakly null, for each $1 \leqslant i \leqslant n$ and each $\Lambda_{\xi,n,i}.D \ni s \leqslant t \in \mathrm{MAX}(\Gamma_{\xi,n}.D)$, $\mathrm{Re}\, x_{e(t)}^*(a_i x_{d(t)}) \geqslant b_i' - \delta$, and for each $t \in \mathrm{MAX}(\Gamma_{\xi,n}.D)$,

$$\alpha' < x_{e(t)}^* \left(\sum_{i=1}^{n} a_i z_i^{d(t)} \right) = \sum_{\varnothing < s \leqslant e(t)} \mathbb{P}_{\xi,n}(s) h\big(s, e(t)\big) < \delta + \sum_{i=1}^{n} b_i'.$$

By relabeling, we assume $d(t) = t$ and $e(t) = t$. Furthermore, by replacing x_t with 0 for any $1 \leqslant i \leqslant n$ such that $b_i' - \delta \leqslant 0$ and $t \in \Lambda_{\xi,n,i}.D$, we may assume $x_t = 0$ for any such t and i. We then let $b_i = 0$ provided $b_i' - \delta \leqslant 0$, and $b_i = (b_i' - \delta)/a_i$ if $b_i' - \delta > 0$. Note that the condition $b_i' - \delta > 0$ implies $b_i > 0$. Then

$$\inf_{t \in \mathrm{MAX}(\Gamma_{\xi,n}.D)} r_K \left(\sum_{i=1}^{n} a_i z_i^t \right) \geqslant \mathrm{Re}\, x_t^* \left(\sum_{i=1}^{n} a_i z_i^t \right)$$

$$\geqslant \sum_{i=1}^{n} \sum_{\Lambda_{\xi,n,i} \ni s \leqslant t} \mathbb{P}_{\xi,n}(s)\, \mathrm{Re}\, x_t^*(a_i x_s)$$

$$\geqslant \sum_{i=1}^{n} (b_i' - \delta) \geqslant \alpha' - (n+1)\delta > \alpha.$$

Now assume there exist a directed set D, non-negative scalars $(a_i)_{i=1}^{n} \in B_{\ell_p^n}$, and a weakly null collection $(x_t)_{t \in \Gamma_{\xi,n}.D} \subset B_X$ such that

$$\inf_{t \in \Gamma_{\xi,n}.D} r_K \left(\sum_{i=1}^{n} a_i z_i^t \right) > \alpha.$$

Then for any α' such that

$$\inf_{t \in \Gamma_{\xi,n}.D} r_K \left(\sum_{i=1}^{n} a_i z_i^t \right) > \alpha' > \alpha,$$

$$\left\{ t \in \mathrm{MAX}(\Gamma_{\xi,n}.D) : r_K \left(\sum_{i=1}^{n} a_i z_i^t \right) > \alpha' \right\} = \mathrm{MAX}(\Gamma_{\xi,n}.D)$$

is cofinal.

Obviously the condition in the previous paragraph implies that $\alpha_{\xi,p,n}(K) > \alpha$. Conversely, suppose there exists $(a_i)_{i=1}^n \in \mathbb{K}^n$, a directed set D, and a weakly null collection $(x_t)_{t \in \Gamma_{\xi,n}.D} \subset B_X$ such that

$$\inf_{t \in \Gamma_{\xi,n}.D} r_K\left(\sum_{i=1}^n a_i z_i^t\right) > \alpha \left\|(a_i)_{i=1}^n\right\|_{\ell_p^n}.$$

By positive homogeneity of r_K, we may assume $(a_i)_{i=1}^n \in S_{\ell_p^n}$. For each $1 \leqslant i \leqslant n$, fix a unimodular scalar ε_i such that $a_i/\varepsilon_i = |a_i|$. For each $1 \leqslant i \leqslant n$ and each $t \in \Lambda_{\xi,n,i}.D$, let $x_i^\varepsilon := \varepsilon_i x_t$. For $t \in \mathrm{MAX}(\Gamma_{\xi,n}.D)$, let

$$z_i^{\varepsilon,t} = \sum_{\Lambda_{\xi,n,i}.D \ni s \leqslant t} \mathbb{P}_{\xi,n}(s) x_s^\varepsilon.$$

Then for each $t \in \mathrm{MAX}(\Gamma_{\xi,n}.D)$,

$$r_K\left(\sum_{i=1}^n a_i z_i^t\right) = r_K\left(\sum_{i=1}^n |a_i| z_i^{\varepsilon,t}\right).$$

Since $(x_t^\varepsilon)_{t \in \Gamma_{\xi,n}.D} \subset B_X$ is weakly null, this finishes (i).

(ii) The main theorem of [4] can be stated as: $\alpha_{\xi,p,1}(K) = 0$ if and only if $Sz(A) \leqslant \omega^\xi$. The rest of (ii) follows from the fact that that $(\alpha_{\xi,p,n}(K))_{n=1}^\infty$ is non-decreasing and the obvious fact that $\alpha_{\xi,p,n}(K) \leqslant n\alpha_{\xi,p,1}(K)$.

(iii) This is obvious from Hölder's theorem.

(iv) We let $\mathbb{T} = \{\varepsilon \in \mathbb{K} : |\varepsilon| = 1\}$ and $\mathbb{T}K = \{\varepsilon x^* : \varepsilon \in \mathbb{T}, x^* \in K\}$. We first show that $\alpha_{\xi,p,n}(K) = \alpha_{\xi,p,n}(\mathbb{T}K)$. Since $r_K \leqslant r_{\mathbb{T}K}$, $\alpha_{\xi,p,n}(K) \leqslant \alpha_{\xi,p,n}(\mathbb{T}K)$. Fix $\alpha < \alpha_{\xi,n,p}(\mathbb{T}K)$, non-negative scalars $(a_i)_{i=1}^n \in B_{\ell_p^n}$, $b_1, \ldots, b_n \in \mathbb{R}$ such that $\alpha < \sum_{i=1}^n a_i b_i$, a directed set D, a weakly null collection $(x_t)_{t \in \Gamma_{\xi,n}.D} \subset B_X$, and a collection $(\varepsilon_t x_t^*)_{t \in \mathrm{MAX}(\Gamma_{\xi,n}.D)} \subset \mathbb{T}K$ such that for every $1 \leqslant i \leqslant n$ and every $\Lambda_{\xi,n,i}.D \ni s \leqslant t \in \mathrm{MAX}(\Gamma_{\xi,n}.D)$,

$$b_i \leqslant \mathrm{Re}\, \varepsilon_t x_t^*(x_s).$$

We may fix such constants, vectors, and functionals by (i). Here, the collection

$$\left(\varepsilon_t x_t^*\right)_{t \in \mathrm{MAX}(\Gamma_{\xi,n}.D)} \subset \mathbb{T}K$$

is written such that $|\varepsilon_t| = 1$ and $x_t^* \in K$ for all $t \in \mathrm{MAX}(\Gamma_{\xi,n}.D)$. Fix $R > 0$ such that $K \subset R B_{X^*}$ and $\delta > 0$ such that $nR\delta + \alpha < \sum_{i=1}^n a_i b_i$. Fix a finite δ-net $(\varepsilon_i)_{i=1}^k$ of $\mathbb{T}$. For each $1 \leqslant i \leqslant k$, let $\mathcal{E}_i = \{t \in \mathrm{MAX}(\Gamma_{\xi,n}.D) : |\varepsilon_i - \varepsilon_t| \leqslant \delta\}$. By Lemma 2.1, we may relabel and assume there exists a single $\varepsilon \in \mathbb{T}$ such that $|\varepsilon - \varepsilon_t| \leqslant \delta$ for all $t \in \mathrm{MAX}(\Gamma_{\xi,n}.D)$. Let $c_i = b_i - R\delta$, so that

$$\sum_{i=1}^n a_i c_i \geqslant \sum_{i=1}^n a_i b_i - nR\delta > \alpha.$$

Note that for each $1 \leqslant i \leqslant n$ and each $\Lambda_{\xi,n,i}.D \ni s \leqslant t \in \mathrm{MAX}(\Gamma_{\xi,n}.D)$,

$$\mathrm{Re}\, x_t^*(\varepsilon x_t) \geqslant \mathrm{Re}\, \varepsilon_t x_t^*(x_t) - |\varepsilon - \varepsilon_t| \|x_t\| \geqslant b_i - R\delta = c_i.$$

We now appeal to (i) to deduce that $\alpha_{\xi,p,n}(K) > \alpha$. This shows that $\alpha_{\xi,p,n}(K) = \alpha_{\xi,p,n}(\mathbb{T}K)$.

We now note that for any weak*-compact set L of X^*, $r_L = r_{\overline{\mathrm{co}}^{\,\mathrm{weak}^*}(L)}$, whence

$$\alpha_{\xi,p,n}(L) = \alpha_{\xi,p,n}\big(\overline{\mathrm{co}}^{\,\mathrm{weak}^*}(L)\big).$$

If $L = \mathbb{T}K$, then

$$\overline{\mathrm{abs\ co}}^{\,\mathrm{weak}^*}(K) = \overline{\mathrm{co}}^{\,\mathrm{weak}^*}(L),$$

and

$$\alpha_{\xi,p,n}\big(\overline{\mathrm{abs\ co}}^{\,\mathrm{weak}^*}(K)\big) = \alpha_{\xi,p,n}\big(\overline{\mathrm{co}}^{\,\mathrm{weak}^*}(L)\big) = \alpha_{\xi,p,n}(L)$$
$$= \alpha_{\xi,p,n}(\mathbb{T}K) = \alpha_{\xi,p,n}(K). \qquad \square$$

REMARK 4.4. In Theorem 4.3(i), we showed that in order to check the values of $\alpha_{\xi,p,n}(K)$, it is sufficient to check only over sequences $(a_i)_{i=1}^n \in S_{\ell_p^n}$ of non-negative, real scalars. We now note that it is sufficient to check only over positive, real scalars by density of $\{(a_i)_{i=1}^n \in \ell_p^n : a_i > 0\}$ in $\{(a_i)_{i=1}^n \in \ell_p^n : a_i \geqslant 0\}$.

REMARK 4.5. We note that a weak*-compact set $K \subset X^*$ is norm compact if and only if $\alpha_{0,p,1}(K) = 0$ for some $1 \leqslant p \leqslant \infty$ if and only if $\alpha_{0,p,1}(K) = 0$ for every $1 \leqslant p \leqslant \infty$ if and only if for any weakly null net $(x_\lambda) \subset B_X$, $r_K(x_\lambda) \to 0$.

The results contained in Theorem 4.6 have similar proofs to those contained in Theorem 4.3, so we omit them.

THEOREM 4.6. Let X be a Banach space, $K \subset X^*$ weak*-compact.
(i) $\theta_{\xi,n}(K) = 0$ for some $n \in \mathbb{N}$ if and only if $\theta_{\xi,n}(K) = 0$ for all $n \in \mathbb{N}$ if and only if $Sz(K) \leqslant \omega^\xi$.
(ii) If $R \geqslant 0$ is such that $K \subset RB_{X^*}$, $\theta_{\xi,n}(K) \leqslant R$.
(iii) $\theta_{\xi,n}(K) = \theta_{\xi,n}(\overline{\mathrm{abs\ co}}^{\,\mathrm{weak}^*}(K))$.
(iv) For $\theta \in \mathbb{R}$, $\theta_{\xi,n}(K) > \theta$ if and only if there exist a directed set D, a weakly null collection $(x_t)_{t \in \Gamma_{\xi,n}.D} \subset B_X$, $(x_t^*)_{t \in \mathrm{MAX}(\Gamma_{\xi,n}.D)} \subset K$, and real numbers $b_1, \ldots, b_n$ such that $\theta n < \sum_{i=1}^n b_i$ and for each $1 \leqslant i \leqslant n$ and each $\Lambda_{\xi,n,i}.D \ni s \leqslant t \in \mathrm{MAX}(\Gamma_{\xi,n}.D)$, $\mathrm{Re}\, x_t^*(x_s) \geqslant b_i$.

PROPOSITION 4.7. Let X be a Banach space and let $K \subset X^*$ be weak*-compact. The assignment $\omega^\xi n \mapsto \theta_{\xi,n}(K)$ is continuous and non-increasing from $\{\omega^\xi n : \xi \in \mathbf{Ord}, n \in \mathbb{N}\}$ into $\mathbb{R}$. That is, if $\omega^\zeta m \leqslant \omega^\xi n$ (which happens if and only if $\zeta \leqslant \xi$ and either $\zeta < \xi$ or $\zeta = \xi$ and $m \leqslant n$), then $\theta_{\xi,n}(K) \leqslant \theta_{\zeta,m}(K)$, and for any ordinal ξ,

$$\theta_{\xi+1,1}(K) = \inf\big\{\theta_{\xi,n}(K) : n \in \mathbb{N}\big\}$$

and if ξ is a limit ordinal,

$$\theta_{\xi,1}(K) = \inf\big\{\theta_{\zeta,n}(K) : \zeta < \xi, n \in \mathbb{N}\big\} = \inf\big\{\theta_{\zeta+1,1}(K) : \zeta < \xi\big\}.$$

Proof. In order to see that $\omega^\xi n \mapsto \theta_{\xi,n}(K)$ is non-increasing, it is sufficient to show that for any ordinal ξ and $n \in \mathbb{N}$,

(i) $\theta_{\xi,n+1}(K) \leqslant \theta_{\xi,n}(K)$,

(ii) $\theta_{\xi+1,1}(K) \leqslant \theta_{\xi,n}(K)$,

(iii) for any limit ordinal $\lambda > \xi$, $\theta_{\lambda,1}(K) \leqslant \theta_{\xi+1,1}(K)$.

Let us show (i). Suppose $\vartheta < \theta_{\xi,n+1}(K)$, a directed set D, a weakly null collection $(x_t)_{t \in \Gamma_{\xi,n+1}.D} \subset B_X$, and non-negative numbers $b_1, \ldots, b_{n+1}$ such that $\sum_{i=1}^{n+1} \frac{b_i}{n+1} > \vartheta$ and for each $1 \leqslant i \leqslant n+1$ and $\Lambda_{\xi,n,i}.D \ni s \leqslant t \in \mathrm{MAX}(\Gamma_{\xi,n+1}.D)$, $\mathrm{Re}\, x_t^*(x_s) \geqslant b_i$. Now for each $1 \leqslant i \leqslant n+1$, let $T_i = \{1, \ldots, n+1\} \setminus \{i\}$. We note that for any $1 \leqslant i \leqslant n+1$, $\sum_{j \in T_i} \frac{b_j}{n} \leqslant \theta_{\xi,n}(K)$. Indeed, by Lemma 2.1, there exists a level map $d : \Gamma_{\xi,n}.D \to \Gamma_{\xi,n+1}.D$ such that $d(\Lambda_{\xi,n,j}.D) \subset \Lambda_{\xi,n+1,j}.D$ for each $j < i$ and $d(\Lambda_{\xi,n,j}.D) \subset \Lambda_{\xi,n+1,j+1}.D$ for $j \geqslant i$. Then if e is any extension of d,

$$\sum_{j \in T_i} \frac{b_j}{n} \leqslant \inf_{t \in \mathrm{MAX}(\Gamma_{\xi,n}.D)} \mathrm{Re}\, x_{e(t)}^* \left(\sum_{j=1}^n \sum_{e(t) \geqslant s \in \Lambda_{\xi,n,j}.D} n^{-1} x_{d(s)} \right) \leqslant \theta_{\xi,n}(K).$$

Then

$$\vartheta < \sum_{j=1}^{n+1} \frac{b_j}{n+1} = \sum_{i=1}^{n+1} \sum_{j \in T_i} \frac{b_j}{(n+1)n}$$

$$= \frac{1}{n+1} \sum_{i=1}^{n+1} \sum_{j \in T_i} \frac{b_j}{n} \leqslant \frac{1}{n+1} \sum_{i=1}^{n+1} \theta_{\xi,n}(K)$$

$$= \theta_{\xi,n}(K).$$

This yields (i).

Now if $\vartheta < \theta_{\xi+1,1}(K)$, we may fix a weakly null collection $(x_t)_{t \in \Gamma_{\xi+1,1}.D} \subset B_X$ such that

$$\inf_{t \in \mathrm{MAX}(\Gamma_{\xi+1,1}.D)} r_K \left(\sum_{s \leqslant t} \mathbb{P}_{\xi+1,1}(s) x_s \right) > \vartheta.$$

Now since $\mathbb{P}_{\xi+1,1}(s) = n^{-1} \mathbb{P}_{\xi,n}(s)$ for each $s \in \Gamma_{\xi,n}.D$,

$$\vartheta < \inf_{t \in \mathrm{MAX}(\Gamma_{\xi+1,1}.D)} r_K \left(\sum_{s \leqslant t} \mathbb{P}_{\xi+1,1}(s) x_s \right)$$

$$\leqslant \inf_{t \in \mathrm{MAX}(\Gamma_{\xi,n}.D)} r_K \left(\sum_{s \leqslant t} \mathbb{P}_{\xi+1,1}(s) x_s \right)$$

$$= \inf_{t \in \mathrm{MAX}(\Gamma_{\xi,n}.D)} r_K \left(\sum_{i=1}^n \frac{1}{n} \sum_{t \geqslant s \in \Lambda_{\xi,n,i}.D} \mathbb{P}_{\xi,n}(s) x_s \right).$$

Therefore $(x_t)_{t \in \mathrm{MAX}(\Gamma_{\xi,n}.D)}$ witnesses that $\vartheta < \theta_{\xi,n}(K)$, which yields (ii).

For (iii), we argue as in (ii), noting that there is a canonical map $j : \Gamma_{\xi+1,1}.D \to \Gamma_{\lambda,1}.D$ such that $\mathbb{P}_{\xi+1,1}(t) = \mathbb{P}_{\lambda,1}(j(t))$ for any $t \in \Gamma_{\xi+1,1}.D$. Thus any collection witnessing that $\vartheta < \theta_{\lambda,1}(K)$ has a subset witnessing that $\vartheta < \theta_{\xi+1,1}(K)$.

It follows from what we have already shown that for any ordinal ξ,

$$\theta_{\xi+1,1}(K) \leqslant \inf\{\theta_{\xi,n}(K) : n \in \mathbb{N}\}.$$

Now if $\vartheta < \inf\{\theta_{\xi,n}(K) : n \in \mathbb{N}\}$, for a fixed weak neighborhood basis D in X at 0 and for each $n \in \mathbb{N}$, we may fix $(x_t)_{t \in \Gamma_{\xi,n}.D} \subset B_X$ such that

$$\vartheta < \inf_{t \in \mathrm{MAX}(\Gamma_{\xi,n}.D)} r_K\left(\sum_{i=1}^{n} \sum_{t \geqslant s \in \Lambda_{\xi,n,i}.D} \frac{1}{n}\mathbb{P}_{\xi,n}(s)x_s\right)$$

$$= \inf_{t \in \mathrm{MAX}(\Gamma_{\xi+1,1}.D) \cap \Gamma_{\xi,n}.D} r_K\left(\sum_{s \leqslant t} \mathbb{P}_{\xi+1,1}(s)x_s\right).$$

Then $(x_t)_{t \in \Gamma_{\xi+1,1}.D} = (x_t)_{t \in \bigcup_{n=1}^{\infty} \Gamma_{\xi,n}.D}$ witnesses that $\vartheta \leqslant \theta_{\xi+1,1}(K)$. This yields that

$$\theta_{\xi+1,1}(K) = \inf\{\theta_{\xi,n}(K) : n \in \mathbb{N}\}.$$

It also follows from what we have shown that for a limit ordinal ξ,

$$\theta_{\xi,1}(K) \leqslant \inf\{\theta_{\zeta+1,1}(K) : \zeta < \xi\} = \inf\{\theta_{\zeta,n}(K) : \zeta < \xi, n \in \mathbb{N}\}.$$

We argue similarly to the previous paragraph to deduce the reverse inequality. For $\vartheta < \inf\{\theta_{\zeta+1,1}(K) : \zeta < \xi\}$, we may fix a weak neighborhood basis D at 0 in X and for each $\zeta < \xi$ a weakly null collection $(x'_t)_{t \in \Gamma_{\zeta+1,1}.D} \subset B_X$ such that

$$\vartheta < \inf_{t \in \mathrm{MAX}(\Gamma_{\zeta+1,1}.D)} r_K\left(\sum_{s \leqslant t} \mathbb{P}_{\zeta+1,1}(s)x'_s\right).$$

Now if $j_\zeta : (\omega^\zeta + \Gamma_{\zeta+1,1}).D \to \Gamma_{\zeta+1,1}.D$ is the canonical identification, $(x_{j(t)})_{t \in \Gamma_{\xi,1}.D}$ witnesses that $\vartheta \leqslant \theta_{\xi,1}(K)$. $\qquad\square$

COROLLARY 4.8. *Let ξ be an ordinal, $1 \leqslant p, q \leqslant \infty$ with $1 < p$ and $1/p + 1/q = 1$, X a Banach space, and $K \subset X^*$ weak*-compact.*

(i) $\inf_n \theta_{\xi,n}(K) = \limsup_n \alpha_{\xi,p,n}(K)/n^{1/q}$.
(ii) $Sz(K) \leqslant \omega^{\xi+1}$ *if and only if* $\inf_n \theta_{\xi,n}(K) = 0$ *if and only if* $\limsup_n \alpha_{\xi,p,n}(K)/n^{1/q} = 0$.

Proof. (i) First note that $\inf_n \theta_{\xi,n}(K) = \lim_n \theta_{\xi,n}(K)$, since $(\theta_{xi,n}(K))_{n=1}^{\infty}$ is non-increasing.

Since $\|(1/n)_{i=1}^{n}\|_{\ell_p^n} = 1/n^{1/q}$, it follows that for any weakly null $(x_t)_{t \in \Gamma_{\xi,n}} \subset B_X$,

$$\inf_{t \in \mathrm{MAX}(\Gamma_{\xi,n}.D)} r_K\left(\frac{1}{n}\sum_{i=1}^{n} z_i^t\right) \leqslant \alpha_{\xi,p,n}(K)/n^{1/q}.$$

Thus $\theta_{\xi,n}(K) \leqslant \alpha_{\xi,p,n}(K)/n^{1/q}$, and

$$\inf_n \theta_{\xi,n}(K) \leqslant \limsup_n \alpha_{\xi,p,n}(K).$$

Now to obtain a contradiction, assume

$$\inf_n \theta_{\xi,n}(K) < \vartheta < \theta < \limsup_n \alpha_{\xi,p,n}(K).$$

Let $R > 0$ be such that $K \subset RB_{X^*}$. Fix $N_1 \in \mathbb{N}$ such that $\theta_{\xi,n}(K) < \vartheta$ for all $n \geqslant N_1$ and fix N_2 so large that $N_1 \leqslant (\theta - \vartheta)N_2^{1/q}/R$. Now fix $n \geqslant N_2$ such that $\alpha_{\xi,p,n}(K)/n^{1/q} > \theta$. There exist a sequence of non-negative scalars $(a_i)_{i=1}^n \in B_{\ell_p^n}$, scalars $b_1, \ldots, b_n \in [0, R]$ such that $\theta n^{1/q} < \sum_{i=1}^n a_i b_i$, a directed set D, a weakly null $(x_t)_{t \in \Gamma_{\xi,n}.D} \subset B_X$, and a collection $(x_t^*)_{t \in \mathrm{MAX}(\Gamma_{\xi,n}.D)} \subset K$ such that for every $1 \leqslant i \leqslant n$ and each $\Lambda_{\xi,n,i}.D \ni s \leqslant t \in \mathrm{MAX}(\Gamma_{\xi,n}.D)$, $\mathrm{Re}\, x_t^*(x_s) \geqslant b_i$. Now let $S = \{i \leqslant n : b_i \geqslant \vartheta\}$ and let $T = \{1, \ldots, n\} \setminus S$. Then

$$\theta n^{1/q} \leqslant R \sum_{i \in S} a_i + \vartheta \sum_{i \in T} a_i$$
$$\leqslant R|S| + \big\|(a_i)_{i \in T}\big\|_{\ell_p^{|T|}} \vartheta |T|^{1/q}$$
$$\leqslant R|S| + \vartheta n^{1/q}.$$

From this it follows that $|S| \geqslant (\theta - \vartheta)n^{1/q}/R \geqslant N_1$.

There exists a map $d : \Gamma_{\xi,|S|}.D \to \Gamma_{\xi,n}.D$ such that, if $S = \{s_1 < \cdots < s_{|S|}\}$, $d(\Lambda_{\xi,|S|,i}.D) \subset \Lambda_{\xi,n,s_i}.D$, and such that $(x_{d(s)})_{s \in \Gamma_{\xi,|S|}.D}$ is weakly null. Let $e : \mathrm{MAX}(\Gamma_{\xi,|S|}.D) \to \mathrm{MAX}(\Gamma_{\xi,n}.D)$ be any extension of d. Then for any $(s,t) \in \Pi(\Gamma_{\xi,|S|}.D)$, $\mathrm{Re}\, x_{e(t)}^*(x_{d(t)}) \geqslant \theta$. The collection $(x_{d(t)})_{t \in \Gamma_{\xi,|S|}.D}$ yields that $\theta_{\xi,|S|}(K) \geqslant \theta$, a contradiction.

(ii) By Proposition 4.7 and Theorem 4.6, $Sz(K) \leqslant \omega^{\xi+1}$ if and only if $\lim_n \theta_{\xi,n}(K) = \inf_n \theta_{\xi,n}(K) = \theta_{\xi+1,1}(K) = 0$.

The equivalence of the last two conditions follows from (i). $\qquad \square$

REMARK 4.9. From Corollary 4.8, it follows that if $\alpha_{\xi,p}(K) = \alpha < \infty$ for some $1 < p \leqslant \infty$, $Sz(K) \leqslant \omega^{\xi+1}$. Indeed, $\lim_n \alpha_{\xi,p,n}(K)/n^{1/q} \leqslant \lim_n \alpha/n^{1/q} = 0$.

We will need the following. The first part of the proposition is contained in [7, Proposition 3.5], although not stated in precisely this way. The second part of the following proposition follows immediately from the first.

PROPOSITION 4.10. *If X is a Banach space, $K \subset X^*$ is weak*-compact, $b_1, \ldots, b_n$ are non-negative scalars, and if there exists a directed set D, a weakly null collection $(x_t)_{t \in \Gamma_{\xi,n}.D} \subset B_X$, and $(x_t^*)_{t \in \mathrm{MAX}(\Gamma_{\xi,n}.D)} \subset K$ such that for every $1 \leqslant i \leqslant n$ and each $\Lambda_{\xi,n,i}.D \ni s \leqslant t \in \mathrm{MAX}(\Gamma_{\xi,n}.D)$, $\mathrm{Re}\, x_t^*(x_s) \geqslant b_i$. Then for any $0 < \psi < 1$,*

$$s_{\psi b_1}^{\omega^\xi} \cdots s_{\psi b_n}^{\omega^\xi}(K) \neq \varnothing.$$

For $\varepsilon > 0$, $m \in \mathbb{N}$ *such that* $Sz(K, \varepsilon) \leqslant \omega^\xi m$ *and* $(x_t)_{t \in \Gamma_{\xi,n}.D} \subset B_X$, $(x_t^*)_{t \in \mathrm{MAX}(\Gamma_{\xi,n}.D)} \subset K$, *and* $b_1, \dots, b_n$ *are as above,*

$$\big|\{i \leqslant n : b_i > \varepsilon\}\big| \leqslant m.$$

THEOREM 4.11. *Let X be a Banach space, ξ an ordinal, and let $K \subset X^*$ be weak*-compact.*

(i) *If $q = \max\{\mathbf{p}_\xi(K), 1\}$ and $1/p + 1/q = 1$,*

$$p = \sup\{r \in [1, \infty] : \alpha_{\xi, r}(K) < \infty\}.$$

(ii) *If $1 \leqslant q < \infty$ and $1/p + 1/q = 1$, K has ξ-q-summable Szlenk index if and only if $\alpha_{\xi, p}(K) < \infty$.*

Proof. (i) Let $\beta = \sup\{r \in [1, \infty] : \alpha_{\xi, r}(K) < \infty\}$ and let $1/\alpha + 1/\beta = 1$. The fact that $\beta \leqslant p$ follows from the proof of the main theorem of [7]. For the reverse inequality, if $p = 1$, $p \leqslant \beta$ and we have $\beta = p$. So assume $1 < p$. Fix $1 \leqslant r < p$ and let $2s = r + p$. In the proof, r', s' denote the conjugate exponents to r, s, respectively. Let $C = \sum_{l=1}^\infty (2^{\frac{1}{r'} - \frac{1}{s'}})^l < \infty$. Fix $R \geqslant 0$ such that $K \subset RB_{X^*}$. It follows easily from the definition of $\mathbf{p}_\xi(K)$ that there exists $\varepsilon_1 \in (0, 1)$ such that for every $0 < \varepsilon \leqslant \varepsilon_1$, $Sz_\xi(K, \varepsilon) \leqslant 1/\varepsilon^{s'}$. Let $m = 1/\varepsilon_1^{s'}$ and $C_1 = Rm^{1/r'} + 2^{1/s'} m^{1/r'} C$. Now fix $n \in \mathbb{N}$, non-negative scalars $(a_i)_{i=1}^n$, $\alpha < \alpha_{\xi, p, n}(K)$, $(x_t)_{t \in \Gamma_{\xi,n}.D} \subset B_X$, functionals $(x_t^*)_{t \in \mathrm{MAX}(\Gamma_{\xi,n}.D)} \subset K$, and $b_1, \dots, b_n \subset [0, R]$ such that $\sum_{i=1}^n a_i b_i > \alpha$ and for each $1 \leqslant i \leqslant n$ and each $\Lambda_{\xi,n,i}.D \ni s \leqslant\in \mathrm{MAX}(\Gamma_{\xi,n}.D)$, $\mathrm{Re}\, x_s^*(x_s) \geqslant b_i$. For each $l \in \mathbb{N}$, let

$$S_l = \big\{i \leqslant n : b_i \in (\varepsilon_1 2^{-\frac{l}{s'}}, \varepsilon_1 2^{\frac{1-l}{s'}}]\big\}$$

and let

$$S_0 = \big\{i \leqslant n : b_i \in (\varepsilon_1, R]\big\}.$$

By Proposition 4.10, since $Sz_\xi(K, \varepsilon_1 2^{-l/s'}) \leqslant 1/(\varepsilon_1 2^{-l/s'})^{s'} = m 2^l$, it follows that $|S_l| \leqslant m 2^l$. Since $Sz(K, \varepsilon_1) \leqslant m$, $|S_0| \leqslant m$. Now let

$$y_0^* = \sum_{j \in S_0} e_j^*$$

and for each $l \in \mathbb{N}$, let

$$y_l^* = \sum_{j \in S_l} 2^{-l/r'} e_j^*.$$

Then $\|y_0^*\|_{\ell_{r'}} = |S_0|^{1/r'} \leqslant m^{1/r'}$ and for each $l \in \mathbb{N}$,

$$\big\|y_l^*\big\|_{\ell_{r'}} = 2^{-l/r'} |S_l|^{1/r'} \leqslant 2^{-l/r'} m^{1/r'} 2^{l/r'} = m^{1/r'}.$$

Let

$$y^* = Ry_0^* + 2^{1/s'} \sum_{l \in \mathbb{N}} \big(2^{\frac{1}{r'} - \frac{1}{s'}}\big)^l y_l^*$$

and note that

$$\big\|y^*\big\|_{\ell_{r'}} \leqslant Rm^{1/r'} + 2^{1/s'} m^{1/r'} C = C_1.$$

Furthermore,

$$\alpha < \sum_{i=1}^{n} a_i b_i \leqslant \sum_{i \in S_0} a_i b_i + \sum_{l=1}^{\infty} \sum_{i \in S_l} a_i b_i \leqslant y^* \left(\sum_{i=1}^{n} a_i e_i \right)$$
$$\leqslant C_1 \left\| (a_i)_{i=1}^{n} \right\|_{\ell_r}.$$

Therefore, $\alpha_{\xi,r}(K) \leqslant C_1$.

(ii) Assume that K fails to have ξ-q-summable Szlenk index. Then for any $M > 0$ there exist $\varepsilon_1, \ldots, \varepsilon_n > 0$ such that $s_{\varepsilon_1}^{\omega^\xi} \cdots s_{\varepsilon_n}^{\omega^\xi}(K) \neq \varnothing$ and $\sum_{i=1}^{n} (\varepsilon_i/5)^q > M^q$. Then arguing as in [4, Lemma 3.4], we may fix a weak neighborhood basis D at 0 in X, $(x_t)_{t \in \Gamma_{\xi,n}.D} \subset B_X$, and $(x_t^*)_{t \in \text{MAX}(\Gamma_{\xi,n}.D)} \subset K$ such that for each $1 \leqslant i \leqslant n$ and each $\Lambda_{\xi,n,i}.D \ni s \leqslant t \in \text{MAX}(\Gamma_{\xi,n}.D)$, $\text{Re}\, x_t^*(x_s) \geqslant \varepsilon_i/5$. Now fix $a_1, \ldots, a_n \subset [0,1]$ such that $\|(a_i)_{i=1}^{n}\|_{\ell_p^n} = 1$ and $\sum_{i=1}^{n} a_i \varepsilon_i/5 = (\sum_{i=1}^{n} (\varepsilon_i/5)^q)^{1/q} > M$. From this, it follows that

$$\alpha_{\xi,p,n}(K) \geqslant \inf_{t \in \text{MAX}(\Gamma_{\xi,n}.D)} \left\| \sum_{i=1}^{n} a_i z_i^t \right\| \geqslant \sum_{i=1}^{n} a_i \varepsilon_i/5 > M.$$

We therefore deduce that $\alpha_{\xi,p}(K) = \infty$.

Now if $\alpha < \alpha_{\xi,p,n}(K)$, there exist a sequence of positive scalars $(a_i)_{i=1}^{n} \in B_{\ell_p^n}$ and a weakly null collection $(x_t)_{t \in \Gamma_{\xi,n}.D} \subset B_X$ such that

$$\alpha < \inf_{t \in \text{MAX}(\Gamma_{\xi,n}.D)} r_K \left(\sum_{i=1}^{n} a_i z_i^t \right).$$

Then there exist non-negative scalars $b_1, \ldots, b_n \in \mathbb{R}$ such that $\alpha < \sum_{i=1}^{n} b_i$ and for each $1 \leqslant i \leqslant n$ and each $\Lambda_{\xi,n,i}.D \ni s \leqslant t \in \text{MAX}(\Gamma_{\xi,n}.D)$, $\text{Re}\, x_t^*(a_i x_s) \geqslant b_i$. Fix $0 < \psi < 1$ such that $\alpha < \psi \sum_{i=1}^{n} b_i$. Then by Proposition 4.10, $s_{\psi b_1/a_1}^{\omega^\xi} \cdots s_{\psi b_n/a_n}^{\omega^\xi}(K) \neq \varnothing$. Now

$$\alpha < \psi \sum_{i=1}^{n} b_i = \psi \sum_{i=1}^{n} a_i (b_i/a_i)$$
$$\leqslant \psi \left\| (a_i)_{i=1}^{n} \right\|_{\ell_p^n} \left(\sum_{i=1}^{n} (b_i/a_i)^q \right)^{1/q} \leqslant \psi \left(\sum_{i=1}^{n} (b_i/a_i)^q \right)^{1/q}.$$

From this it follows that K does not have α-ξ-q-summable Szlenk index. Therefore if $\alpha_{\xi,p}(K) = \infty$, K does not have ξ-q-summable Szlenk index. $\square$

COROLLARY 4.12. *Let X be a Banach space, ξ an ordinal, $K \subset X^*$ weak*-compact.*

(i) *For $1 \leqslant q < \infty$, K has ξ-q-summable Szlenk index if and only if $\overline{\text{abs co}}^{\text{weak}^*}(K)$ does.*

(ii) $\mathbf{p}_\xi(\overline{\text{abs co}}^{\text{weak}^*}(K)) = \max\{\mathbf{p}_\xi(K), 1\}$.

Proof. Let $1/p + 1/q = 1$. Then $\alpha_{\xi,p,n}(K) = \alpha_{\xi,p,n}(\overline{\text{abs co}}^{\text{weak}^*}(K))$ by Theorem 4.3, whence $\alpha_{\xi,p}(K) < \infty$ if and only if $\alpha_{\xi,p}(\overline{\text{abs co}}^{\text{weak}^*}(K)) < \infty$. By Theorem 4.11(ii), the former condition is equivalent to K having ξ-q-summable Szlenk index and the latter is equivalent to $\overline{\text{abs co}}^{\text{weak}^*}(K)$ having ξ-q-summable Szlenk index.

We deduce (ii) using (i) together with Theorem 4.11(i). $\square$

5. Banach ideals

In this section, we let **Ban** denote the class of all Banach spaces over $\mathbb{K}$. We let $\mathfrak{L}$ denote the class of all operators between Banach spaces and for $X, Y \in \mathbf{Ban}$, we let $\mathfrak{L}(X,Y)$ denote the set of operators from X into Y. For $\mathfrak{I} \subset \mathfrak{L}$ and $X, Y \in \mathbf{Ban}$, we let $\mathfrak{I}(X,Y) = \mathfrak{I} \cap \mathfrak{L}(X,Y)$. We recall that a class $\mathfrak{I}$ is called an ideal if

(i) for any $W, X, Y, Z \in \mathbf{Ban}$, any $C \in \mathfrak{L}(W,X)$, $B \in \mathfrak{I}(X,Y)$, and $A \in \mathfrak{L}(Y,Z)$, $ABC \in \mathfrak{I}$,
(ii) $I_{\mathbb{K}} \in \mathfrak{I}$,
(iii) for each $X, Y \in \mathbf{Ban}$, $\mathfrak{I}(X,Y)$ is a vector subspace of $\mathfrak{L}(X,Y)$.

We recall that an ideal $\mathfrak{I}$ is said to be

(i) *closed* provided that for any $X, Y \in \mathbf{Ban}$, $\mathfrak{I}(X,Y)$ is closed in $\mathfrak{L}(X,Y)$ with its norm topology,
(ii) *injective* provided that for any $X, Y, Z \in \mathbf{Ban}$, any $A : X \to Y$, and any isomorphic (equivalently, isometric) embedding $j : Y \to Z$ such that $jA \in \mathfrak{I}$, $A \in \mathfrak{I}$,
(iii) *surjective* provided that for any $W, X, Y \in \mathbf{Ban}$, any $A : X \to Y$, and any surjection (equivalently, quotient map) $q : W \to X$ such that $Aq \in \mathfrak{I}$, $A \in \mathfrak{I}$.

If $\mathfrak{I}$ is an ideal and ι assigns to each member of $\mathfrak{I}$ a non-negative real value, then we say ι is an *ideal norm* provided that

(i) for each $X, Y \in \mathbf{Ban}$, ι is a norm on $\mathfrak{I}(X,Y)$,
(ii) for any $W, X, Y, Z \in \mathbf{Ban}$ and any $C \in \mathfrak{L}(W,X)$, $B \in \mathfrak{I}(X,Y)$, $A \in \mathfrak{I}(Y,Z)$, $\iota(ABC) \leqslant \|A\|\iota(B)\|C\|$,
(iii) for any $X, Y \in \mathbf{Ban}$, any $x \in X$, and any $y \in Y$, $\iota(x \otimes y) = \|x\|\|y\|$.

If $\mathfrak{I}$ is an ideal and ι is an ideal norm on $\mathfrak{I}$, we say $(\mathfrak{I}, \iota)$ is a *Banach ideal* provided that for every $X, Y \in \mathbf{Ban}$, $(\mathfrak{I}(X,Y), \iota)$ is a Banach space.

Fix an ordinal ξ, $1 < p \leqslant \infty$, and let $1/p + 1/q = 1$. For an operator $A : X \to Y$, let $\mathfrak{a}_{\xi,p}(A) = \|A\| + \alpha_{\xi,p}(A)$. Let $\mathfrak{A}_{\xi,p}$ denote the class of all operators A with ξ-q-summable Szlenk index.

THEOREM 5.1. *For any ordinal ξ and $1 < p \leqslant \infty$, the class $(\mathfrak{A}_{\xi,p}, \mathfrak{a}_{\xi,p})$ is a Banach ideal.*

Proof. Fix Banach spaces X, Y. We first show that $\alpha_{\xi,p}$ satisfies the triangle inequality. Fix $A, B : X \to Y$, a directed set D, a weakly null collection $(x_t)_{t \in \Gamma_{\xi,n}.D} \subset B_X$, and a scalar sequence $(a_i)_{i=1}^n \in B_{\ell_p^n}$. Fix $\alpha_A > \alpha_{\xi,p,n}(A)$ and $\alpha_B > \alpha_{\xi,p,n}(B)$. Then the sets

$$\mathcal{E}_A = \left\{ t \in \mathrm{MAX}(\Gamma_{\xi,n}.D) : \left\| A \sum_{i=1}^n a_i z_i^t \right\| \leqslant \alpha_A \right\}$$

and

$$\mathcal{E}_B = \left\{ t \in \mathrm{MAX}(\Gamma_{\xi,n}.D) : \left\| B \sum_{i=1}^n a_i z_i^t \right\| \leqslant \alpha_B \right\}$$

are eventual. From this, it follows that $\mathcal{E}_A \cap \mathcal{E}_B \neq \varnothing$. Then for any $s \in \mathcal{E}_A \cap \mathcal{E}_B$,

$$\inf_{t \in \mathrm{MAX}(\Gamma_{\xi,n}.D)} \left\| (A+B) \sum_{i=1}^n z_i^t \right\| \leqslant \left\| (A+B) \sum_{i=1}^n a_i z_i^s \right\| \leqslant \alpha_A + \alpha_B.$$

From this we deduce the triangle inequality for $\alpha_{\xi,n}$, and therefore for $\mathfrak{a}_{\xi,n}$. From here it is easy to see that $\mathfrak{a}_{\xi,p}$ is a norm on $\mathfrak{L}(X, Y)$.

Now fix a Banach space Z and fix $A : X \to Y$. For $B : Y \to Z$, a directed set D, $n \in \mathbb{N}$, a scalar sequence $(a_i)_{i=1}^n \in B_{\ell_p^n}$, and a weakly null collection $(x_t)_{t \in \Gamma_{\xi,n}.D} \subset B_X$,

$$\inf_{t \in \mathrm{MAX}(\Gamma_{\xi,n}.D)} \left\| BA \sum_{i=1}^n a_i z_i^t \right\| \leqslant \|B\| \inf_{t \in \mathrm{MAX}(\Gamma_{\xi,n}.D)} \left\| A \sum_{i=1}^n z_i^t \right\| \leqslant \|B\| \alpha_{\xi,p,n}(A).$$

From this it follows that $\mathfrak{a}_{\xi,p}(BA) \leqslant \|B\| \mathfrak{a}_{\xi,p}(A)$. For $B : Z \to X$, a directed set D, $n \in \mathbb{N}$, a scalar sequence $(a_i)_{i=1}^n \in B_{\ell_p^n}$, and a weakly null collection $(x_t)_{t \in \Gamma_{\xi,n}.D} \subset B_X$,

$$\begin{aligned}
\inf_{t \in \mathrm{MAX}(\Gamma_{\xi,n}.D)} \left\| AB \sum_{i=1}^n z_i^t \right\| &\leqslant \inf_{b > \|B\|} b \inf_{t \in \mathrm{MAX}(\Gamma_{\xi,n}.D)} \left\| A \sum_{i=1}^n b^{-1} B z_i^t \right\| \\
&\leqslant \inf_{b > \|B\|} b \alpha_{\xi,p,n}(A) \\
&= \|B\| \alpha_{\xi,p,n}(A).
\end{aligned}$$

From this it follows that $\mathfrak{a}_{\xi,p}(AB) \leqslant \|B\| \mathfrak{a}_{\xi,p}$.

Since $\alpha_{\xi,p}(A) = 0$ whenever A is a compact operator, we deduce that $\mathfrak{A}_{\xi,p}$ is an operator ideal and $\mathfrak{a}_{\xi,p}$ is an ideal norm.

We last show completeness. Fix a sequence $(A_k)_{k=1}^\infty$ of $\mathfrak{a}_{\xi,p}$-Cauchy operators from X to Y. This is also a norm Cauchy sequence, which must converge in norm to some $A : X \to Y$. We note that since $\alpha_{\xi,p,n}$ is Lipschitz continuous on $\mathfrak{L}(X, Y)$,

$$\alpha_{\xi,p}(A) = \sup_n \alpha_{\xi,p,n}(A) = \sup_n \lim_k \alpha_{\xi,p,n}(A_k) \leqslant \sup_n \lim_k \alpha_{\xi,p}(A_k)$$

$$= \limsup_k \alpha_{\xi,p}(A_k) < \infty$$

and

$$\limsup_l \mathfrak{a}_{\xi,p}(A - A_l) = \limsup_l \alpha_{\xi,p}(A - A_l)$$

$$= \limsup_l \sup_n \limsup_k \alpha_{\xi,p,n}(A_k - A_l)$$

$$\leqslant \limsup_l \sup_n \limsup_k \alpha_{\xi,p}(A_k - A_l)$$

$$= \limsup_l \limsup_k \alpha_{\xi,p}(A_k - A_l) = 0. \qquad \square$$

REMARK 5.2. We have already shown that $\mathfrak{D}_\xi \subset \mathfrak{A}_{\xi,p} \subset \mathfrak{D}_{\xi+1}$. It is contained in Section 7 of [7] that each of these inclusions is proper. Furthermore, it is contained in Section 7 of [7] that for any ordinals ξ, ζ and any $1 < p, q \leqslant \infty$, $\mathfrak{A}_{\xi,p} \neq \mathfrak{A}_{\zeta,q}$ unless $(\xi,p) = (\zeta,q)$, if $\xi < \zeta$ or if $\xi = \zeta$ and $p > q$, $\mathfrak{A}_{\xi,p} \subset \mathfrak{A}_{\zeta,q}$, and $\mathfrak{A}_{\xi,q} \subset \mathfrak{T}_{\xi,p} \subset \mathfrak{A}_{\xi,p}$.

REMARK 5.3. It is easy to see that the class of ξ-asymptotically uniformly flattenable operators, denoted by $\mathfrak{T}_{\xi,\infty}$ in [6], is contained in $\mathfrak{A}_{\xi,p}$ for every $1 < p \leqslant \infty$. It was shown there in [6, Proposition 6.5] that for any ordinal ξ, there exists a Banach space X such that $\overline{\mathfrak{T}_{\xi,\infty}(X,X)}$ contains an operator which cannot be renormed to be ξ-p-AUS for any $1 < p < \infty$. This example also shows that none of the classes $\mathfrak{A}_{\xi,p}$ is closed.

Brooker [3] showed that for any ordinal ξ, the class of all operators with Szlenk index not more than ω^ξ is a closed operator ideal. An argument similar to that of Theorem 5.1 allows us to provide another proof of this fact which is dual to Brooker's proof.

THEOREM 5.4. *For any ordinal ξ, the class $\mathfrak{D}_\xi$ of operators with Szlenk index not exceeding ω^ξ is a closed operator ideal.*

Proof. Arguing as in Theorem 5.1, one proves that $\theta_{\xi,1}$ defines a seminorm on $\mathcal{L}(X,Y)$ for each pair X, Y of Banach spaces. Furthermore, $\theta_{\xi,1}(A) = 0$ for all compact operators and $\theta_{\xi,1}(ABC) \leqslant \|A\|\theta_{\xi,1}(B)\|C\|$ for any Banach spaces W, Z and any operators $C: W \to X$, $B: X \to Y$, and $A: Y \to Z$. Last, by Theorem 4.6(ii), $\theta_{\xi,1}$ is 1-Lipschitz. From these facts, it follows that the class of all operators A with $\theta_{\xi,1}(A) = 0$ is a closed operator ideal. But by Theorem 4.6(i), this is precisely the class of operators with Szlenk index not exceeding ω^ξ. $\qquad \square$

PROPOSITION 5.5. *For any ordinal ξ and any $1 \leqslant p \leqslant \infty$, $\mathfrak{A}_{\xi,p}$ and $\mathfrak{D}_\xi$ are injective and surjective.*

We need the following easy piece.

PROPOSITION 5.6. *Suppose $q : W \to X$ is a quotient map, $(x_\lambda)_{\lambda \in D_1} \subset B_X$ is weakly null, and v is a weak neighborhood of 0 in W. Then for any $b > 2$, there exist $\lambda \in D_1$ and $w \in bB_W \cap v$ such that $qw = x_\lambda$.*

Proof. By replacing v with a subset, we may assume v is convex and symmetric. Fix $b > 2$ and $\delta > 0$ such that $\delta B_W \subset \frac{1}{2}v$ and $2 + 3\delta < b$. For each λ, fix $u_\lambda \in (1+\delta)B_W$ such that $qu_\lambda = x_\lambda$. By passing to a subnet, we may assume that $(u_\lambda)_{\lambda \in D_1}$ is weak*-convergent to some $u^{**} \in (1+\delta)B_{W^{**}}$, and that $u_{\lambda_1} - u_{\lambda_2} \in \frac{1}{2}v$ for all $\lambda_1, \lambda_2 \in D_1$. Since $(x_\lambda)_{\lambda \in D_1}$ is weakly null, there exist a finite set $F \subset D_1$ and $x = \sum_{\lambda \in F} a_\lambda x_\lambda \in \mathrm{co}(x_\lambda : \lambda \in F)$ such that $\|x\| < \delta$. We may fix $u \in \delta B_W \subset \frac{1}{2}v$ such that $qu = x$. Then let $w = u_{\lambda_1} - \sum_{\lambda \in F} a_\lambda u_\lambda + u$. Then

$$\|w\| \leqslant 1 + \delta + \sum_{\lambda \in F} a_\lambda(1+\delta) + \delta = 2 + 3\delta < b,$$

$$qw = x_{\lambda_1} - x + x = x_{\lambda_1},$$

and

$$w = \sum_{\lambda \in F} a_\lambda(u_{\lambda_1} - u_\lambda) + w \in \frac{1}{2}v + \frac{1}{2}v = v. \qquad \square$$

Proof of Proposition 5.5. Injectivity is easy, since for any operator $A : X \to Y$ and any isometric isomorphism $j : Y \to Z$, clearly $\alpha_{\xi,p,n}(jA) = \alpha_{\xi,p,n}(A)$ and $\theta_{\xi,n}(jA) = \theta_{\xi,n}(A)$ for all ordinals ξ, all $1 \leqslant p \leqslant \infty$, and all $n \in \mathbb{N}$.

For surjectivity, we will show that if $A : X \to Y$ is an operator and $q : W \to X$ is a quotient map, then $\alpha_{\xi,p,n}(A) \leqslant 2\alpha_{\xi,p,n}(Aq)$, and a similar argument will yield that $\theta_{\xi,n}(A) \leqslant 2\theta_{\xi,n}(Aq)$ for any ordinal ξ, any $1 \leqslant p \leqslant \infty$, and any $n \in \mathbb{N}$. Fix $\alpha < \alpha_{\xi,p,n}(A)$, a directed set D, $(b_i)_{i=1}^n \in S_{\ell_p^n}$, and a weakly null collection $(x_t)_{t \in \Gamma_{\xi,n}.D} \subset B_X$ such that $\inf_{t \in \mathrm{MAX}(\Gamma_{\xi,n}.D)} \|A \sum_{i=1}^n b_i z_i^t\| > \alpha$. Fix $b > 2$ such that $\inf_{t \in \mathrm{MAX}(\Gamma_{\xi,n}.D)} \|A \sum_{i=1}^n b_i b^{-1} z_i^t\| > \alpha/2$. Let D_1 be any weak neighborhood basis at 0 in W. We may now recursively apply Proposition 5.6 to deduce the existence of some collection $(w_t)_{t \in \Gamma_{\xi,n}.D_1} \subset bB_W$ and a length-preserving monotone map $\phi : \Gamma_{\xi,n}.D_1 \to \Gamma_{\xi,n}.D$ such that

(i) if $\phi((\zeta_i, v_i)_{i=1}^k) = (\eta_i, u_i)_{i=1}^k$, then $\zeta_i = \eta_i$ for all $1 \leqslant i \leqslant k$,
(ii) $qw_t = x_{\phi(t)}$ for all $t \in \Gamma_{\xi,n}.D_1$.

Indeed, if $t = s \frown (\zeta, v) \in \Gamma_{\xi,n}.D_1$ and either $s = \varnothing$ or $\phi(s)$ has been defined to have property (i), we apply Proposition 5.6 to the weakly null net $(x_{\phi(s) \frown (\zeta, u)})_{u \in D}$ to deduce the existence of some $u_1 \in D$ and $w_t \in bB_W \cap v$ such that $qw_t = x_{\phi(s) \frown (\zeta, u_1)}$. We then let $\phi(t) = \phi(s) \frown (\zeta, u_1)$.

Note that for any $t \in \mathrm{MAX}(\Gamma_{\xi,n}.D_1)$ and any $1 \leqslant i \leqslant |t|$, $\phi(t|_i) = \phi(t)|_i$ and $\mathbb{P}_{\xi,n}(\phi(t|_i)) = \mathbb{P}_{\xi,n}(t|_i)$, whence

$$q \sum_{i=1}^n b_i \sum_{t \geqslant s \in \Lambda_{\xi,n,i}.D_1} \mathbb{P}_{\xi,n}(s)b^{-1}w_s = \sum_{i=1}^n b_i \sum_{\phi(t) \geqslant s \in \Lambda_{\xi,n,i}.D} \mathbb{P}_{\xi,n}(s)b^{-1}x_s$$

and

$$\inf_{t\in\mathrm{MAX}(\Gamma_{\xi,n}.D_1)}\left\|Aq\sum_{i=1}^{n}b_i\sum_{t\geqslant s\in\Lambda_{\xi,n,i}.D_1}\mathbb{P}_{\xi,n}(s)b^{-1}w_s\right\|$$

$$=\inf_{t\in\mathrm{MAX}(\Gamma_{\xi,n}.D)}\left\|A\sum_{i=1}^{n}b_i\sum_{\phi(t)\geqslant s\in\Lambda_{\xi,n,i}.D}\mathbb{P}_{\xi,n}(s)b^{-1}x_s\right\|>\alpha/2.$$

Since $(b^{-1}w_t)_{t\in\Gamma_{\xi,n}.D_1}$ is weakly null and $\alpha<\alpha_{\xi,p,n}(A)$ was arbitrary, we are done. $\square$

REMARK 5.7. Note that if u is any weak neighborhood of 0 in ℓ_1 and if $m\in\mathbb{N}$, there exist $m<i<j$, $i,j\in\mathbb{N}$ such that $\frac{1}{2}(e_i-e_j)\in u$. From this we can easily deduce that $\theta_{\xi,n}(\ell_1)=1$ for all ordinals ξ and all $n\in\mathbb{N}$, and $\alpha_{\xi,p,n}(\ell_1)=n^{1-1/p}$ for all ordinals ξ, all $1\leqslant p\leqslant\infty$, and all $n\in\mathbb{N}$. Indeed, for any weak neighborhood basis D at 0 in ℓ_1, we may recursively define for $t\in\Gamma_{\xi,n}.D$ a vector $x_t=\frac{1}{2}(e_{i_t}-e_{j_t})$ such that for all $t\in\Gamma_{\xi,n}.D$, $i_t<j_t$, if $s,t\in\Gamma_{\xi,n}.D$ and $s<t$, $j_s<i_t$, and if $t=(\zeta_i,u_i)_{i=1}^n$, $x_t\in u_n$. Then $(x_t)_{t\in\Gamma_{\xi,n}}\subset B_{\ell_1}$ is weakly null and $\|\sum_{i=1}^n a_i z_i^t\|_{\ell_1}=\sum_{i=1}^n|a_i|$ for all scalar sequences $(a_i)_{i=1}^n$.

From this, one can also deduce that if $A:X\to Y$ is an operator, Z is a subspace of X which isomorphic to ℓ_1, and if $A|_Z$ is an isomorphic embedding, then $\inf_{\xi\in\mathbf{Ord}}\theta_{\xi,1}(A)>0$. Indeed, we first fix isomorphisms $\alpha:\ell_1\to Z$, $\beta:A(Z)\to\ell_1$ such that $I_{\ell_1}=\beta A\alpha$, so that $\theta_{\xi,1}(A|_Z:Z\to A(Y))\geqslant\theta_{\xi,1}(I_{\ell_1})/\|\alpha\|\|\beta\|=1/\|\alpha\|\|\beta\|$ by the ideal property. Now if $j:A(Z)\to Y$ is the inclusion, it is clear that

$$\theta_{\xi,1}(A:X\to Y)\geqslant\theta_{\xi,1}(A|_Z:Z\to Y)=\theta_{\xi,1}\bigl(A|_Z:Z\to A(Z)\bigr).$$

6. Applications of Corollary 4.12

In this section, we prove an embedding result together with a result concerning injective tensor products.

COROLLARY 6.1. *Fix a countable ordinal ξ and $1\leqslant q<\infty$. Let X be a separable Banach space.*

(i) *If $\mathbf{p}_\xi(X)\leqslant q$, then there exists a Banach space W with a shrinking basis such that $\mathbf{p}_\xi(W)\leqslant q$ and X embeds isomorphically into W.*

(ii) *If X has ξ-q-summable Szlenk index, then there exists a Banach space W with shrinking basis having ξ-q-summable Szlenk index such that X embeds isomorphically into W.*

Proof. Schlumprecht [21] proved that if X^* is separable, after renorming X, there exist a weak*-compact set $B^*\subset B_{X^*}$, a Banach space Z with shrinking FDD which contains a subspace isometric to X, a subset $\mathbb{B}^*$ of B_{Z^*}, and a map $I^*:\mathbb{B}^*\to B^*$ such that

(i) $\overline{\mathrm{abs\ co}}^{\mathrm{weak}^*}(\mathbb{B}^*)=B_{Z^*}$,

(ii) for any $n \in \mathbb{N}$ and any $\varepsilon_1, \ldots, \varepsilon_n > 0$,

$$I^* \big(s_{\varepsilon_1}^{\omega^\xi} \cdots s_{\varepsilon_n}^{\omega^\xi} (\mathbb{B}^*) \big) \subset s_{\varepsilon_1/5}^{\omega^\xi} \cdots s_{\varepsilon_n/5}^{\omega^\xi} (B^*).$$

The construction of the space Z is the main construction of [21], while the estimate in (ii) follows from [21, Lemma 5.5]. Then for a countable ξ and $1 \leqslant q < \infty$, either property $\mathbf{p}_\xi(\cdot) \leqslant q$ or having ξ-q-summable Szlenk index passes from B_{X^*} to B^*, since $B^* \subset B_{X^*}$, from B^* to $\mathbb{B}^*$ by (ii) above and Theorem 4.11, and from $\mathbb{B}^*$ to B_{Z^*} by Corollary 4.12.

Now if $(F_n)_{n=1}^\infty$ is the FDD of Z, by a technique of Pełczyński [18], for each $n \in \mathbb{N}$ we may fix a finite dimensional space H_n such that F_n is 2-complemented in $E_n = F_n \oplus H_n$ and E_n has a basis with basis constant not exceeding 2. Then if $H = (\bigoplus_{n=1}^\infty H_n)_{c_0}$ and if $W = Z \oplus H$, W has a shrinking basis and has $\mathbf{p}_\xi(W) \leqslant q$ (resp. ξ-q-summable Szlenk index) if Z has this property. Here we are using Theorem 4.11 together with Theorem 5.1 and the fact that $\mathbf{p}_\xi(H) \leqslant \mathbf{p}_0(H) = 1 \leqslant q$ and $\alpha_{\xi,p}(H) \leqslant \alpha_{0,\infty}(H) \leqslant 1$. $\qquad \square$

In [11], the question was posed as to whether having summable Szlenk index passes to injective tensor products of Banach spaces. This question was answered in the affirmative in [8]. In Corollary 6.4, we generalize that result to ξ-q-summable Szlenk index.

Let us recall that the injective tensor product is the closed span in $\mathfrak{L}(Y^*, X)$ of the operators $x \otimes y : Y^* \to X$, where $x \otimes y(y^*) = y^*(y)x$. For $i = 0, 1$, if $A_i : X_i \to Y_i$ is an operator, we may define the operator $A_0 \otimes A_1 : X_0 \hat\otimes_\varepsilon X_1 \to Y_0 \hat\otimes_\varepsilon Y_1$. This operator is given by $A_0 \otimes A_1(u) = A_0 u A_1^* : Y_1^* \to Y_0$. Given subsets $K_0 \subset X_0^*$, $K_1 \subset X_1^*$, we let

$$[K_0, K_1] = \big\{ x_0^* \otimes x_1^* : x_0^* \in K_0, x_1^* \in K_1 \big\} \subset (X_0 \hat\otimes_\varepsilon X_1)^*.$$

We observe that the map

$$X_0^* \otimes X_1^* \supset K_0 \times K_1 \ni (x_0^*, x_1^*) \mapsto x_0^* \otimes x_1^* \in [K_0, K_1]$$

is weak*–weak* continuous if K_0, K_1 are bounded. We recall that any non-zero ordinal ξ can be uniquely written as $\xi = \omega^{\xi_1} + \cdots + \omega^{\xi_n}$ for some $\xi_1 \geqslant \cdots \geqslant \xi_n$. One obtains such a representation by first writing ξ in its Cantor normal form

$$\xi = \omega^{\varepsilon_1} m_1 + \cdots + \omega^{\varepsilon_k} m_k$$

for some $\varepsilon_1 > \cdots > \varepsilon_k$ and $m_i \in \mathbb{N}$, and then writing $\omega^{\varepsilon_i} m_i = \omega^{\varepsilon_i} + \cdots + \omega^{\varepsilon_i}$.

LEMMA 6.2. *Suppose that for $i = 0, 1$, $A_i : X_i \to Y_i$ is a non-zero operator. Let $R = \max\{\|A_0\|, \|A_1\|\}$. For any $\xi > 0$, any $\varepsilon > 0$, any finite set J, and any weak*-compact sets $K_{0,j} \subset X_0^*$, $K_{1,j} \subset X_1^*$, $j \in J$,*

$$s_\varepsilon^\xi \bigg(\bigcup_{j \in J} [K_{0,j}, K_{1,j}] \bigg) \subset \bigcup_{\substack{k_i + l_i = 1 \\ j \in J}} \big[s_{\varepsilon/4R}^{\omega^{\xi_1} k_1 + \cdots + \omega^{\xi_n} k_n} (K_{0,j}), s_{\varepsilon/4R}^{\omega^{\xi_1} l_1 + \cdots + \omega^{\xi_n} l_n} (K_{1,j}) \big],$$

where $\xi = \omega^{\xi_1} + \cdots + \omega^{\xi_n}$, $\xi_1 \geqslant \cdots \geqslant \xi_n$ and it is understood that $k_i, l_i \in \{0, 1\}$.

Proof. To obtain a contradiction, assume the lemma does not hold, and let $\xi \geq 1$ be the minimum ordinal for which the statement fails. Write $\xi = \omega^{\xi_1} + \cdots + \omega^{\xi_n}$, $\xi_1 \geq \cdots \geq \xi_n$. We consider three cases.

Case 1: $\xi_n = 0$. Then $\omega^{\xi_n} = 1$ and $\xi = \zeta + 1$, where $\zeta = \omega^{\xi_1} + \cdots + \omega^{\xi_{n-1}}$ if $n > 1$ and $\zeta = 0$ if $n = 1$. Then by the inductive hypothesis applied to ζ,

$$s_\varepsilon^\xi \left(\bigcup_{j \in J} [K_{0,j}, K_{1,j}] \right)$$

$$= s_\varepsilon \left(s_\varepsilon^\zeta \left(\bigcup_{j \in J} [K_{0,j}, K_{1,j}] \right) \right)$$

$$\subset s_\varepsilon \left(\bigcup_{\substack{k_i + l_i = 1 \\ j \in J}} \left[s_{\varepsilon/4R}^{\omega^{\xi_1} k_1 + \cdots + \omega^{\xi_{n-1}} k_{n-1}}(K_{0,j}), s_{\varepsilon/4R}^{\omega^{\xi_1} l_1 + \cdots + \omega^{\xi_{n-1}} l_{n-1}}(K_{1,j}) \right] \right).$$

By [8, Proposition 5.1],

$$s_\varepsilon \left(\bigcup_{\substack{k_i + l_i = 1 \\ j \in J}} \left[s_{\varepsilon/4R}^{\omega^{\xi_1} k_1 + \cdots + \omega^{\xi_{n-1}} k_{n-1}}(K_{0,j}), s_{\varepsilon/4R}^{\omega^{\xi_1} l_1 + \cdots + \omega^{\xi_{n-1}} l_{n-1}}(K_{1,j}) \right] \right)$$

$$\subset \bigcup_{\substack{k_i + l_i = 1 \\ j \in J}} \left[s_\varepsilon^{k_n} \left(s_{\varepsilon/4R}^{\omega^{\xi_1} k_1 + \cdots + \omega^{\xi_{n-1}} k_{n-1}}(K_{0,j}) \right), \right.$$

$$\left. s^{l_n} \left(s_{\varepsilon/4R}^{\omega^{\xi_1} l_1 + \cdots + \omega^{\xi_{n-1}} l_{n-1}}(K_{1,j}) \right) \right]$$

$$= \bigcup_{\substack{k_i + l_i = 1 \\ j \in J}} \left[s_{\varepsilon/4R}^{\omega^{\xi_1} k_1 + \cdots + \omega^{\xi_n} k_n}(K_{0,j}), s_{\varepsilon/4R}^{\omega^{\xi_1} l_1 + \cdots + \omega^{\xi_n} l_n}(K_{1,j}) \right],$$

a contradiction. Here we are using the convention that if $n = 1$, the first union is taken only over $j \in J$ and

$$s_{\varepsilon/4R}^{\omega^{\xi_1} k_1 + \cdots + \omega^{\xi_{n-1}} k_{n-1}}(K_{0,j}) = K_{0,j}$$

and

$$s_{\varepsilon/4R}^{\omega^{\xi_1} l_1 + \cdots + \omega^{\xi_{n-1}} l_{n-1}}(K_{1,j}) = K_{1,j}.$$

Case 2: ξ_n is a limit ordinal. Fix

$$u^* \in s_\varepsilon^\xi \left(\bigcup_{j \in J} [K_{0,j}, K_{1,j}] \right) = \bigcap_{\zeta < \xi_n} s^{\omega^{\xi_1} + \cdots + \omega^{\xi_{n-1}} + \omega^\zeta} \left(\bigcup_{j \in J} [K_{0,j}, K_{1,j}] \right).$$

Then for every $\zeta < \xi_n$, there exist $(k_i^\zeta)_{i=1}^n \in \{0,1\}^n$ and $j_\zeta \in J$ such that, with $l_i^\zeta = 1 - k_i^\zeta$,

$$x^* \in \left[s_\varepsilon^{\omega^{\xi_1} k_1^\zeta + \cdots + \omega^{\xi_{n-1}} k_{n-1}^\zeta + \omega^\zeta k_n^\zeta}(K_{0,j_\zeta}), s_\varepsilon^{\omega^{\xi_1} l_1^\zeta + \cdots + \omega^{\xi_{n-1}} l_{n-1}^\zeta + \omega^\zeta l_n^\zeta}(K_{1,j_\zeta}) \right].$$

Then there exist a cofinal subset M of $[0, \xi_n)$, $j \in J$, and $(k_i)_{i=1}^n, (l_i)_{i=1}^n \in \{0,1\}^n$ such that $k_i = k_i^\zeta$, $l_i = l_i^\zeta$, and $j_\zeta = j$ for all $\zeta \in M$. Then

$$u^* \in \bigcap_{\zeta \in M} \left[s_\varepsilon^{\omega^{\xi_1} k_1 + \cdots + \omega^{\xi_{n-1}} k_{n-1} + \omega^\zeta k_n}(K_{0,j}), s_\varepsilon^{\omega^{\xi_1} l_1 + \cdots + \omega^{\xi_{n-1}} l_{n-1} + \omega^\zeta l_n}(K_{1,j}) \right].$$

Now for each $\zeta \in M$, we may fix

$$x_{0,\zeta}^* \in s_\varepsilon^{\omega^{\xi_1} k_1 + \cdots + \omega^{\xi_{n-1}} k_{n-1} + \omega^\zeta k_n}(K_{0,j})$$

and

$$x_{1,\zeta}^* \in s_\varepsilon^{\omega^{\xi_1} l_1 + \cdots + \omega^{\xi_{n-1}} l_{n-1} + \omega^\zeta l_n}(K_{1,j})$$

such that $x_{0,\zeta}^* \otimes x_{1,\zeta}^* = u^*$. Then if

$$(x_0^*, x_1^*) \in \bigcap_{\mu < \xi_n} \overline{\left\{ (x_{0,\zeta}^*, x_{1,\zeta}^*) : \mu < \zeta \in M \right\}}^{\text{weak}^*} \subset (X_0 \oplus X_1)^*,$$

$$u^* = x_0^* \otimes x_1^* \in \left[s_\varepsilon^{\omega^{\xi_1} k_1 + \cdots + \omega^{\xi_n} k_n}(K_{0,j}), s_\varepsilon^{\omega^{\xi_1} l_1 + \cdots + \omega^{\xi_n} l_n}(K_{1,j}) \right].$$

Since this holds for any $u^* \in s_\varepsilon^\xi(\bigcup_{j \in J}[K_{0,j}, K_{1,j}])$, we reach a contradiction.

Case 3: ξ_n is a successor ordinal, say $\xi_n = \zeta + 1$. Fix

$$u^* \in s_\varepsilon^\xi \left(\bigcup_{j \in J}[K_{0,j}, K_{1,j}] \right) = \bigcap_{m \in \mathbb{N}} s^{\omega^{\xi_1} + \cdots + \omega^{\xi_{n-1}} + \omega^\zeta m} \left(\bigcup_{j \in J}[K_{0,j}, K_{1,j}] \right).$$

Then for every $m \in \mathbb{N}$, there exist $j_m \in J$, $(k_i^m)_{i=1}^n, (l_i^m)_{i=1}^n \in \{0,1\}^{n-1}$ and $p_m, q_m \subset \{0, \ldots, m\}$ with $p_m + q_m = m$ and $k_i^m + l_i^m = 1$ such that

$$x^* \in \left[s_\varepsilon^{\omega^{\xi_1} k_1^m + \cdots + \omega^{\xi_{n-1}} k_{n-1}^m + \omega^\zeta p_m}(K_{0,j_\zeta}), s_\varepsilon^{\omega^{\xi_1} l_1^m + \cdots + \omega^{\xi_{n-1}} l_{n-1}^m + \omega^\zeta q_m}(K_{1,j_\zeta}) \right].$$

Then there exist a cofinal subset M of $\mathbb{N}$, $j \in J$, and $(k_i)_{i=1}^{n-1} \in \{0,1\}^{n-1}$ such that $k_i = k_i^m$ and $j_m = j$ for all $m \in M$. Let $k_n = 1$ and $l_n = 0$ if $\{p_m : m \in M\}$ is unbounded, and otherwise let $k_n = 0$ and $l_n = 1$. Then

$$u^* \in \bigcap_{m \in M} \left[s_\varepsilon^{\omega^{\xi_1} k_1 + \cdots + \omega^{\xi_{n-1}} k_{n-1} + \omega^\zeta p_m}(K_{0,j}), s_\varepsilon^{\omega^{\xi_1} l_1 + \cdots + \omega^{\xi_{n-1}} l_{n-1} + \omega^\zeta q_m}(K_{1,j}) \right].$$

Now for each $m \in M$, we may fix

$$x_{0,m}^* \in s_\varepsilon^{\omega^{\xi_1} k_1 + \cdots + \omega^{\xi_{n-1}} k_{n-1} + \omega^\zeta p_m}(K_{0,j})$$

and

$$x_{1,m}^* \in s_\varepsilon^{\omega^{\xi_1} l_1 + \cdots + \omega^{\xi_{n-1}} l_{n-1} + \omega^\zeta q_m}(K_{1,j})$$

such that $x_{0,m}^* \otimes x_{1,m}^* = u^*$. Then if

$$(x_0^*, x_1^*) \in \bigcap_{l \in \mathbb{N}} \overline{\left\{ (x_{0,m}^*, x_{1,m}^*) : l < m \in M \right\}}^{\text{weak}^*} \subset (X_0 \oplus X_1)^*,$$

$$u^* = x_0^* \otimes x_1^* \in \left[s_\varepsilon^{\omega^{\xi_1} k_1 + \cdots + \omega^{\xi_n} k_n}(K_{0,j}), s_\varepsilon^{\omega^{\xi_1} l_1 + \cdots + \omega^{\xi_n} l_n}(K_{1,j}) \right].$$

Since this holds for any $u^* \in s_\varepsilon^\xi(\bigcup_{j \in J}[K_{0,j}, K_{1,j}])$, we reach a contradiction. $\square$

Iterating the previous result immediately yields the following.

COROLLARY 6.3. *Suppose that for* $i = 0, 1$, $A_i : X_i \to Y_i$ *is a non-zero operator. Let* $R = \max\{\|A_0\|, \|A_1\|\}$. *For any ordinals* $\xi_1, \ldots, \xi_n$, *any* $\varepsilon_1, \ldots, \varepsilon_n > 0$, *any finite set* J, *and any weak*-compact sets* $K_{0,j} \subset X_0^*$, $K_{1,j} \subset X_1^*$, $j \in J$,

$$s_{\varepsilon_1}^{\omega^{\xi_1}} \cdots s_{\varepsilon_n}^{\omega^{\xi_n}}\left(\bigcup_{j \in J}[K_{0,j}, K_{1,j}]\right)$$

$$\subset \bigcup_{\substack{k_i + l_i = 1 \\ j \in J}} \left[s_{\varepsilon_1/4R}^{\omega^{\xi_1} k_1} \cdots s_{\varepsilon_n/4R}^{\omega^{\xi_n} k_n}(K_{0,j}), s_{\varepsilon_1/4R}^{\omega^{\xi_1} l_1} \cdots s_{\varepsilon_n/4R}^{\omega^{\xi_n} l_n}(K_{1,j})\right].$$

COROLLARY 6.4. *Fix an ordinal* ξ *and* $1 \leqslant q < \infty$. *Let* $A_0 : X_0 \to Y_0$, $A_1 : X_1 \to Y_1$ *be operators and let* $A_0 \otimes A_1 : X_0 \hat\otimes_\varepsilon X_1 \to Y_0 \hat\otimes_\varepsilon Y_1$ *be the induced operator. Then if* A_0, A_1 *have* ξ-q-*summable Szlenk index, so does* $A_0 \otimes A_1$.

Proof. If either operator is the zero operator, the result is trivial. Assume $A_0, A_1 \neq 0$ and let $R = \max\{\|A_0\|, \|A_1\|\}$. Suppose A_0 (resp. A_1) has M_0 (resp. M_1)-ξ-q-summable Szlenk index. Let $K = [A_0^* B_{Y_0^*}, A_1^* B_{Y_1^*}]$. Suppose $\varepsilon_1, \ldots, \varepsilon_n > 0$ are such that $s_{\varepsilon_1}^{\omega^\xi} \cdots s_{\varepsilon_n}^{\omega^\xi}(K) \neq \varnothing$. Then by Corollary 6.3, there exist $(k_i)_{i=1}^n \in \{0,1\}^n$ such that $s_{\varepsilon_1/4R}^{\omega^\xi k_1} \cdots s_{\varepsilon_n/4R}^{\omega^\xi k_n}(A_0^* B_{Y_0^*}) \neq \varnothing$ and $s_{\varepsilon_1/4R}^{\omega^\xi (1-k_1)} \cdots s_{\varepsilon_n/4R}^{\omega^\xi (1-k_n)}(A_1^* B_{Y_0^*}) \neq \varnothing$. Then

$$\sum_{i=1}^n k_i(\varepsilon_i/4R)^q \leqslant M_0^q$$

and

$$\sum_{i=1}^n (1-k_i)(\varepsilon_i/4R)^q \leqslant M_1^q,$$

whence

$$\sum_{i=1}^n \varepsilon_i^q \leqslant 4^q R^q\big(M_0^q + M_1^q\big),$$

and K has $4R(M_0^q + M_1^q)^{1/q}$-summable Szlenk index. Since $\overline{\text{co}}^{\text{weak}^*}(K) = (A_0 \otimes A_1)^* B_{(Y_0 \hat\otimes_\varepsilon Y_1)^*}$ by the Hahn–Banach theorem, Corollary 4.12 yields that $A_0 \otimes A_1$ has ξ-q-summable Szlenk index. $\square$

In light of Theorem 4.11, Corollary 6.4 also offers another proof that having ξ-Szlenk power type not exceeding q also passes to injective tensor products, which was shown in [11] in the case $\xi = 0$ and in the general case in [6].

7. Direct sums

In the final section, we are concerned with the behavior of the Szlenk index and $\alpha_{\xi,p}$ under ℓ_p and c_0 direct sums. More specifically, we will have some set Λ and some collection $(A_\lambda : X_\lambda \to Y_\lambda)_{\lambda \in \Lambda}$ of Asplund operators such that $\sup_{\lambda \in \Lambda} \|A_\lambda\| < \infty$. To avoid incredible inconvenience, for a subset Υ of Λ, we agree that $\ell_0(\Upsilon)$ shall mean $c_0(\Upsilon)$. We may then define $E_p = (\bigoplus_{\lambda \in \Lambda} X_\lambda)_{\ell_p(\Lambda)}$ and $F_p = (\bigoplus_{\lambda \in \Lambda} Y_\lambda)_{\ell_p(\Lambda)}$ for $p \in \{0\} \cup [1,\infty]$. We also define for each subset $\Upsilon \subset \Lambda$ the projections $P_\Upsilon : E_p \to E_p$, $Q_\Upsilon : F_p \to F_p$ by $P_\Upsilon(x_\lambda)_{\lambda \in \Lambda} = (1_\Upsilon(\lambda)x_\lambda)_{\lambda \in \Lambda}$ and $Q_\Upsilon(y_\lambda)_{\lambda \in \Lambda} = (1_\Upsilon(\lambda)y_\lambda)_{\lambda \in \Lambda}$. For $\lambda \in \Lambda$, we let $P_\lambda = P_{\{\lambda\}}$ and $Q_\lambda = Q_{\{\lambda\}}$. For each $p \in \{0\} \cup [1,\infty]$, we then identify X_λ with $P_\lambda(E_p)$ and Y_λ with $Q_\lambda(F_p)$.

Given $p, r \in \{0\} \cup [1,\infty]$ such that either $1 \leqslant p \leqslant r \leqslant \infty$ or $r = 0$ and $p < \infty$, and $\Upsilon \subset \Lambda$, we wish to study the behavior of the Szlenk index and $\alpha_{\xi,\gamma}$ seminorms of the operators $D_\Upsilon^{p,r} : E_p \to F_r$ given by $D_\Upsilon^{p,r}(x_\lambda)_{\lambda \in \Lambda} = (1_\Upsilon(\lambda)A_\lambda x_\lambda)_{\lambda \in \Lambda}$. For $\lambda \in \Lambda$, we let $D_\lambda^{p,r} = D_{\{\lambda\}}^{p,r}$. Note that $\alpha_{\xi,s,n}(D_\lambda^{p,r}) = \alpha_{\xi,s,n}(A_\lambda)$ for any p, r, s in the appropriate ranges. Of course, we are interested in the cases in which the operator $D_\Lambda^{p,r}$ is Asplund. We first delineate the cases in which the operator $D_\Lambda^{p,r}$ is Asplund.

(1) $p = r = 1$ and $(\|A_\lambda\|)_{\lambda \in \Lambda} \in c_0(\Lambda)$.
(2) $p = r = \infty$ and $(\|A_\lambda\|)_{\lambda \in \Lambda} \in c_0(\Lambda)$.
(3) $r = 0$ and $p < \infty$.
(4) $1 \leqslant p \leqslant r \leqslant \infty$, $1 < r$, and $p < \infty$.

We first remark that for any finite subset Υ of Λ,

$$\alpha_{\xi,\gamma,n}\left(D_\Upsilon^{p,r}\right) = \alpha_{\xi,\gamma,n}\left(\bigoplus_{\lambda \in \Upsilon} A_\lambda : \left(\bigoplus_{\lambda \in \Upsilon} X_\lambda\right)_{\ell_p(\Upsilon)} \to \left(\bigoplus_{\lambda \in \Upsilon} Y_\lambda\right)_{\ell_r(\Upsilon)}\right)$$

and

$$\theta_{\xi,n}\left(D_\Upsilon^{p,r}\right) = \theta_{\xi,n}\left(\bigoplus_{\lambda \in \Upsilon} A_\lambda : \left(\bigoplus_{\lambda \in \Upsilon} X_\lambda\right)_{\ell_p(\Upsilon)} \to \left(\bigoplus_{\lambda \in \Upsilon} Y_\lambda\right)_{\ell_r(\Upsilon)}\right)$$

for any $1 \leqslant \gamma \leqslant \infty$, any $n \in \mathbb{N}$, and any ordinal ξ. Therefore, we may freely identify $D_\Upsilon^{p,r}$ with the operators between finite direct sums.

We isolate the following easy consequence of Hölder's inequality.

FACT 7.1. *Suppose $1 \leqslant r < \gamma \leqslant \infty$, and $1/\gamma + 1/s = 1/r$. Then for any $n \in \mathbb{N}$ and any scalar sequence $(\alpha_i)_{i=1}^n$,*

$$\left\|(\alpha_i)_{i=1}^n\right\|_{\ell_s^n} = \sup\left\{\left\|(\alpha_i\beta_i)_{i=1}^n\right\|_{\ell_r^n} : (\beta_i)_{i=1}^n \in B_{\ell_\gamma^n}\right\}.$$

FACT 7.2. *Fix $1 \leqslant p, r, \gamma \leqslant \infty$ with $p \leqslant r$. For any natural numbers m, n, any scalar sequence $(\alpha_i)_{i=1}^m$, any $(b_j)_{j=1}^n \in B_{\ell_\gamma^n}$, and any $(a_{ij})_{i=1,j=1}^{mn} \in$*

$B_{\ell_\infty^n(\ell_p^m)}$,

$$\left\|\left(\left\|(\alpha_i b_j a_{ij})_{j=1}^n\right\|_{\ell_\gamma^n}\right)_{i=1}^m\right\|_{\ell_r^m} \leqslant \begin{cases} \|(\alpha_i)_{i=1}^m\|_{\ell_\infty^m} : & \gamma \leqslant r, \\ \|(\alpha_i)_{i=1}^m\|_{\ell_s^m} : & \gamma > r \ and \ \frac{1}{\gamma} + \frac{1}{s} = \frac{1}{r}. \end{cases}$$

Proof. Suppose $\gamma \leqslant r$. Let $\alpha = \|(\alpha_i)_{i=1}^m\|_{\ell_\infty^m}$. Then using Minkowski's inequality,

$$\begin{aligned}
\left\|\left(\left\|(\alpha_i b_j a_{ij})_{j=1}^n\right\|_{\ell_\gamma^n}\right)_{i=1}^m\right\|_{\ell_r^m} &\leqslant \alpha \left\|\left(\left\|(b_j a_{ij})_{j=1}^n\right\|_{\ell_\gamma^n}\right)_{i=1}^m\right\|_{\ell_r^m} \\
&\leqslant \alpha \left\|\left(|b_j|\left\|(a_{ij})_{i=1}^m\right\|_{\ell_\gamma^n}\right)_{j=1}^n\right\|_{\ell_r^m} \\
&\leqslant \alpha \left\|(b_j)_{j=1}^n\right\|_{\ell_\gamma^n} \leqslant \alpha.
\end{aligned}$$

Now assume that $\gamma > r$. Then

$$\begin{aligned}
\left\|\left(\left\|(b_j a_{ij})_{j=1}^n\right\|_{\ell_\gamma^n}\right)_{i=1}^m\right\|_{\ell_\gamma^m} &= \left\|\left(b_j\left\|(a_{ij})_{i=1}^m\right\|_{\ell_\gamma^m}\right)_{j=1}^n\right\|_{\ell_\gamma^n} \\
&\leqslant \left\|\left(b_j\left\|(a_{ij})_{i=1}^m\right\|_{\ell_p^m}\right)_{j=1}^n\right\|_{\ell_\gamma^n} \\
&\leqslant \left\|(b_j)_{j=1}^n\right\|_{\ell_\gamma^n} \leqslant 1.
\end{aligned}$$

Since $\frac{1}{\gamma} + \frac{1}{s} = \frac{1}{r}$, Fact 7.1 yields that

$$\left\|\left(\left\|(\alpha_i b_j a_{ij})_{j=1}^n\right\|_{\ell_\gamma^n}\right)_{i=1}^m\right\|_{\ell_r^m} \leqslant \|(\alpha_i)_{i=1}^m\|_{\ell_s^m}. \qquad \square$$

Before we state the next theorem, we recall that, given a Banach space X, $(x_t)_{t\in\Gamma_{\zeta,n}.D} \subset X$, and $t \in \mathrm{MAX}(\Gamma_{\xi,n})$, the sequence $(z_i^t)_{i=1}^n$ is the convex block sequence defined by

$$z_i^t = \sum_{\Lambda_{\zeta,n,i} \ni s \leqslant t} \mathbb{P}_{\xi,n}(s) x_s.$$

THEOREM 7.3. *Suppose* $\Upsilon \subset \Lambda$ *is a finite set and* $R \geqslant 0$ *is such that* $\|A_\lambda\| \leqslant R$ *for all* $\lambda \in \Lambda$. *Fix* $1 \leqslant p, r, \gamma \leqslant \infty$ *with* $p \leqslant r$ *and ordinals* ξ, ζ.

(i) *Let* $1/r + 1/s = 1$. *If* $\lambda \in \Lambda$ *is such that* $\theta_{\zeta,m}(D_\lambda^{p,r}) \leqslant \varepsilon$, *then for any directed set* D, *any* $n \in \mathbb{N}$, *and sequence* $(b_i)_{i=1}^n$ *of non-negative scalars, any weakly null* $(x_t)_{t\in\Gamma_{\zeta,n}.D} \subset B_{E_p}$, *any sequence of positive scalars* $(a_i)_{i=1}^n \in B_{\ell_p^n}$ *such that* $\|P_\lambda x_t\| \leqslant a_i$ *for all* $1 \leqslant i \leqslant n$, *and any* $\varepsilon' > \varepsilon\|(b_i a_i)_{i=1}^n\|_{\ell_1^n} + Rm^{1/s}\|(b_i a_i)_{i=1}^n\|_{\ell_r^n}$,

$$\left\{ t \in \mathrm{MAX}(\Gamma_{\zeta,n}.D) : \left\| D_\lambda^{p,r} \sum_{i=1}^n b_i z_i^t \right\| \leqslant \varepsilon' \right\}$$

is eventual.

(ii) *Let s be such that $1/r + 1/s = 1$. If $m \in \mathbb{N}$ is such that $\theta_{\zeta,m}(A_\lambda) \leqslant \varepsilon$ for all $\lambda \in \Upsilon$, then for any directed set D, any $n \in \mathbb{N}$, any weakly null $(x_t)_{t \in \Gamma_{\zeta,m}.D} \subset B_{E_p}$, and any non-negative scalars $(b_i)_{i=1}^n$,*

$$\inf_{t \in \mathrm{MAX}(\Gamma_{\xi,m}.D)} \left\| D_\Upsilon^{p,r} \sum_{i=1}^n b_i z_i^t \right\| \leqslant \varepsilon \|(b_i)_{i=1}^n\|_{\ell_1^n} + Rm^{1/s} \|(b_i)_{i=1}^n\|_{\ell_r^n}$$

and and for any $\varepsilon_1 > \varepsilon \|(b_i)_{i=1}^n\|_{\ell_1^n} + Rm^{1/s} \|(b_i)_{i=1}^n\|_{\ell_r^n}$,

$$\left\{ t \in \mathrm{MAX}(\Gamma_{\xi,m}.D) : \left\| D_\Upsilon^{p,r} \sum_{i=1}^m b_i z_i^t \right\| \leqslant \varepsilon_1 \right\}$$

is eventual.

(iii)

$$\alpha_{\xi,\gamma}(D_\Upsilon^{p,r}) = \begin{cases} \|(\alpha_{\xi,\gamma}(A_\lambda))_{\lambda \in \Upsilon}\|_{\ell_\infty(\Upsilon)} : & \gamma \leqslant r, \\ \|(\alpha_{\xi,\gamma}(A_\lambda))_{\lambda \in \Upsilon}\|_{\ell_s(\Upsilon)} : & \gamma > r \text{ and } \frac{1}{\gamma} + \frac{1}{s} = \frac{1}{r}. \end{cases}$$

Before we prove the theorem, we prove a single instance of a stabilization technique, variants of which will be used several times in the remainder of this work.

LEMMA 7.4. *Suppose $\alpha < \alpha_{\xi,\gamma,n}(D_\Upsilon^{p,r})$. Then there exist a directed set D, a sequence of non-negative scalars $(b_i)_{i=1}^n \in S_{\ell_\gamma^n}$, real numbers $c_1, \ldots, c_n$, vectors $(a_\lambda^1)_{\lambda \in \Upsilon}, \ldots, (a_\lambda^n)_{\lambda \in \Upsilon} \in (0,\infty)^\Upsilon$, a weakly null collection $(x_t)_{t \in \Gamma_{\xi,n}.D} \subset B_{E_p}$, and functionals $(y_t^*)_{t \in \mathrm{MAX}(\Gamma_{\xi,n}.D)} \subset B_{F_r^*}$ with the following properties:*

(i) *$\sum_{i=1}^n b_i c_i > \alpha$*
(ii) *for each $1 \leqslant i \leqslant n$, $\|(a_\lambda^i)_{\lambda \in \Upsilon}\|_{\ell_p(\Upsilon)} \leqslant 1$,*
(iii) *for each $\lambda \in \Upsilon$, each $1 \leqslant i \leqslant n$, and each $t \in \Lambda_{\xi,n,i}.D$, $\|P_\lambda x_t\| \leqslant a_\lambda^i$,*
(iv) *for each $1 \leqslant i \leqslant n$ and each $\Lambda_{\xi,n,i}.D \ni s \leqslant t \in \mathrm{MAX}(\Gamma_{\xi,n}.D)$, $\mathrm{Re}\, y_t^*(D_\Upsilon^{p,r} x_s) \geqslant c_i$,*
(v) *for any $\varepsilon_\lambda > \alpha_{\xi,\gamma,n}(A_\lambda) \|(a_\lambda^i b_i)_{i=1}^n\|_{\ell_\gamma^n}$,*

$$\left\{ t \in \mathrm{MAX}(\Gamma_{\xi,\gamma,n}.D) : \left\| D_\lambda^{p,r} \sum_{j=1}^n b_j z_j^t \right\| \leqslant \varepsilon_\lambda \right\}$$

is eventual.

Proof. Fix $0 < \phi < 1$ such that $\alpha < \phi \alpha_{\xi,\gamma,n}(D_\Upsilon^{p,r})$. Fix $(x_t')_{t \in \Gamma_{\xi,n}.D} \subset B_{E_p}$, $(z_t^*)_{t \in \mathrm{MAX}(\Gamma_{\xi,n}.D)} \subset B_{F_r^*}$, $(b_i)_{i=1}^n \in S_{\ell_\gamma^n}$, non-negative scalars $(c_i)_{i=1}^n$ such that $\sum_{i=1}^n b_i c_i > \alpha/\phi$ and for any $1 \leqslant i \leqslant n$ and $\Lambda_{\xi,n,i}.D \ni s \leqslant t \in \mathrm{MAX}(\Gamma_{\xi,n}.D)$, $\mathrm{Re}\, z_t^*(x_s') \geqslant c_i$. Now fix a finite subset F of $B_{\ell_p(\Upsilon)}$ such that for every $(a_\lambda')_{\lambda \in \Upsilon} \in \phi B_{\ell_p(\Upsilon)}$, there exists $(a_\lambda)_{\lambda \in \Upsilon} \in F$ such that $|a_\lambda'| \leqslant a_\lambda$, and such that for each $(a_\lambda)_{\lambda \in \Upsilon} \in F$ and $\lambda \in \Upsilon$, $a_\lambda > 0$. Now define $f : \Gamma_{\xi,n}.D \to F$ by letting $f(t)$ be any member $(a_\lambda)_{\lambda \in \Upsilon}$ of F such that for each $\lambda \in \Lambda$, $\phi\|P_\lambda x_t'\| \leqslant a_\lambda$. By Lemma 2.1, there exist a level map $d : \Gamma_{\xi,n}.D \to \Gamma_{\xi,n}.D$, an extension e of d,

and for each $1 \leqslant i \leqslant n$ some $(a_\lambda^i)_{\lambda \in \Upsilon} \in F$ such that $\phi \|P_\lambda x'_{d(t)}\| \leqslant a_\lambda^i$ for all $t \in \Lambda_{\xi,n,i}.D$ and $\lambda \in \Upsilon$. Now let $x_t = \phi x'_{d(t)}$ and $y_t^* = z_{e(t)}^*$ for $t \in \mathrm{MAX}(\Gamma_{\xi,n}.D)$. We now note that for any $\varepsilon_\lambda > \alpha_{\xi,\gamma,n}(A_\lambda) \|(a_\lambda^i b_i)_{i=1}^n\|_{\ell_\gamma^n}$,

$$\left\{ t \in \mathrm{MAX}(\Gamma_{\xi,\gamma,n}.D) : \left\| D_\lambda^{p,r} \sum_{j=1}^n b_j z_j^t \right\| \leqslant \varepsilon_\lambda \right\}$$

$$= \left\{ t \in \mathrm{MAX}(\Gamma_{\xi,\gamma,n}.D) : \left\| D_\lambda^{p,r} \sum_{j=1}^n \sum_{t \geqslant s \in \Lambda_{\xi,n,j}.D} b_j a_\lambda^j \mathbb{P}_{\xi,n}(s) \frac{P_\lambda x_s}{a_\lambda^j} \right\| \leqslant \varepsilon_\lambda \right\}$$

must be eventual. $\qquad \square$

Proof of Theorem 7.3. (i) If it were not true, after relabeling, we can assume there exist $(y_t^*)_{t \in \mathrm{MAX}(\Gamma_{\zeta,n}.D)} \subset B_{F_r^*}$, $c_1, \ldots, c_n \in [0, \infty)$ such that $\sum_{i=1}^n b_i c_i > \varepsilon \|(b_i a_i)_{i=1}^n\|_{\ell_1^n} + R m^{1/s} \|(b_i a_i)_{i=1}^n\|_{\ell_r^n}$ and for each $1 \leqslant i \leqslant n$ and $\Lambda_{\zeta,n,i}.D \ni s \leqslant t \in \mathrm{MAX}(\Gamma_{\zeta,n}.D)$, $\mathrm{Re}\, y_t^*(P_\lambda x_s) \geqslant c_i$. Now let $T = \{i \leqslant n : c_i/a_i > \varepsilon\}$ and $S = \{1, \ldots, n\} \setminus T$. We note that $|T| < m$. If it were not so, we could find some $l_1 < \cdots < l_m$, $l_i \in T$, and a level map $d : \Gamma_{\zeta,m}.D \to \Gamma_{\zeta,n}.D$ such that $d(\Lambda_{\zeta,m,i}.D) \subset \Lambda_{\zeta,n,l_i}.D$ and such that $(x_{d(t)})_{t \in \Gamma_{\zeta,m}.D}$ is weakly null. Now if $x_t' = a_{l_i}^{-1} x_{d(t)}$ for $t \in \Lambda_{\xi,m,i}.D$, $(x_t')_{t \in \Gamma_{\zeta,m}.D}$ is weakly null and if e is any extension of d, it

$$\varepsilon \geqslant \theta_{\zeta,m}(D_\lambda^{p,r}) > \inf_{t \in \mathrm{MAX}(\Gamma_{\zeta,m}.D)} m^{-1} \mathrm{Re}\, y_{e(t)}^* \sum_{i=1}^m \sum_{t \geqslant s \in \Lambda_{\zeta,m,i}.D} \mathbb{P}_{\zeta,m}(s) x_s$$

$$\geqslant \frac{\min\{c_i/a_i : i \in T\}}{m} > \varepsilon,$$

a contradiction. Note also that for any $1 \leqslant i \leqslant n$, if we fix $\Lambda_{\zeta,n,i}.D \ni s \leqslant t \in \mathrm{MAX}(\Gamma_{\zeta,n}.D)$,

$$c_i/a_i \leqslant \mathrm{Re}\, y_t^*(D_\lambda^{p,r} x_s/a_i) \leqslant \|y_t^*\| \|A_\lambda\| \|P_\lambda x_s\|/a_i \leqslant R.$$

Then

$$\varepsilon \|(b_i a_i)_{i=1}^n\|_{\ell_1^n} + R m^{1/s} \|(b_i a_i)_{i=1}^n\|_{\ell_r^n} < \sum_{i=1}^n b_i c_i = \sum_{i=1}^n a_i b_i \frac{c_i}{a_i}$$

$$= \sum_{i \in S} a_i b_i \frac{c_i}{a_i} + \sum_{i \in T} a_i b_i \frac{c_i}{a_i}$$

$$\leqslant \varepsilon \sum_{i=1}^n a_i b_i + R \sum_{i \in T} a_i b_i$$

$$\leqslant \varepsilon \|(a_i b_i)_{i=1}^n\|_{\ell_1^n} + R \|(a_i b_i)_{i \in T}\|_{\ell_1(T)}$$

$$\leqslant \varepsilon \|(a_i b_i)_{i=1}^n\|_{\ell_1^n} + R m^{1/s} \|(a_i b_i)_{i=1}^n\|_{\ell_r^n},$$

a contradiction.

(ii) Fix a sequence $(b_i)_{i=1}^n$ of non-negative scalars and $\varepsilon_1 > \varepsilon \|(b_i)_{i=1}^n\|_{\ell_1^n} + Rm^{1/s}\|(b_i)_{i=1}^n\|_{\ell_r^n}$. To obtain a contradiction, we assume that there exist a directed set D and a weakly null collection $(x_t)_{t \in \Gamma_{\zeta,n}.D} \subset B_{E_p}$ such that

$$\left\{ t \in \mathrm{MAX}(\Gamma_{\zeta,n}.D) : \left\| D_\Upsilon^{p,r} \sum_{i=1}^n b_i z_i^t \right\| > \varepsilon_1 \right\}$$

is cofinal. By replacing ε_1 with a strictly smaller value which still exceeds $\varepsilon\|(b_i)_{i=1}^n\|_{\ell_1^n} + Rm^{1/s}\|(b_i)_{i=1}^n\|_{\ell_r^n}$, applying Lemma 2.1 and the remark following it, and relabeling, we may assume that

$$\inf_{t \in \mathrm{MAX}(\Gamma_{\zeta,n}.D)} \left\| D_\Upsilon^{p,r} \sum_{i=1}^n b_i z_i^t \right\| \geqslant \varepsilon_1$$

and there exist $c_1, \ldots, c_n \in [0, R]$, $(y_t^*)_{t \in \mathrm{MAX}(\Gamma_{\zeta,n}.D)} \subset B_{F_r^*}$ such that $\varepsilon_1 < \sum_{i=1}^n b_i c_i$ and for any $1 \leqslant i \leqslant n$ and $\Lambda_{\zeta,n,i}.D \ni s \leqslant t \in \mathrm{MAX}(\Gamma_{\zeta,n}.D)$, $\mathrm{Re}\, y_t^*(x_s) = \mathrm{Re}\, y_t^*(P_\Upsilon x_s) \geqslant c_i$. Fix $0 < \phi < 1$ such that $\varepsilon_1 < \sum_{i=1}^n \phi b_i c_i$. Arguing as in Lemma 7.4, we may stabilize, relabel, and assume that there exists $(a_{i\lambda})_{i=1,\lambda \in \Upsilon}^n \in B_{\ell_\infty^n(\ell_p(\Upsilon))}$ such that for each $1 \leqslant i \leqslant n$, each $t \in \Lambda_{\zeta,n,i}.D$, and each $\lambda \in \Upsilon$, $\phi\|P_\lambda x_t\| \leqslant a_{i\lambda}$. Now fix $\varepsilon' > 0$ such that

$$\varepsilon_1 - n\varepsilon' > \varepsilon\|(b_i)_{i=1}^n\|_{\ell_1^n} + Rm^{1/s}\|(b_i)_{i=1}^n\|_{\ell_r^n}.$$

For each $\lambda \in \Upsilon$,

$$\mathcal{E}_\lambda := \left\{ t \in \mathrm{MAX}(\Gamma_{\zeta,n}.D) : \left\| D_\lambda^{p,r} \sum_{i=1}^n b_i \phi z_i^t \right\| \leqslant \varepsilon' + \varepsilon\|(b_i a_{ij})_{i=1}^n\|_{\ell_1^n} \right.$$

$$\left. + Rm^{1/s}\|(b_i a_{ij})_{i=1}^n\|_{\ell_r^n} \right\}$$

is eventual by (i). Then if $t \in \bigcap_{\lambda \in \Upsilon} \mathcal{E}_\lambda$, by Minkowski's inequality,

$$\varepsilon_1 \leqslant \left\| D_\Upsilon^{p,r} \sum_{i=1}^n b_i z_i^t \right\|$$

$$\leqslant n\varepsilon' + \varepsilon\left\| \left(\|(b_i a_{i\lambda})_{i=1}^n\|_{\ell_1^n}\right)_{\lambda \in \Upsilon} \right\|_{\ell_r(\Upsilon)} + Rm^{1/s}\left\| \left(\|(b_i a_{i\lambda})_{i=1}^n\|_{\ell_r^n}\right)_{\lambda \in \Upsilon} \right\|_{\ell_r(\Upsilon)}$$

$$\leqslant n\varepsilon' + \varepsilon\left\| \left(b_i \|(a_{i\lambda})_{\lambda \in \Upsilon}\|_{\ell_r(\Upsilon)}\right)_{i=1}^n \right\|_{\ell_1^n} + Rm^{1/s}\left\| \left(b_i \|(a_{i\lambda})_{\lambda \in \Upsilon}\|_{\ell_r(\Upsilon)}\right)_{i=1}^n \right\|_{\ell_r^n}$$

$$\leqslant n\varepsilon' + \varepsilon\|(b_i)_{i=1}^n\|_{\ell_1^n} + Rm^{1/s}\|(b_i)_{i=1}^n\|_{\ell_r^n},$$

a contradiction.

(iii) If $\alpha < \alpha_{\xi,\gamma,n}(D_\Upsilon^{p,r})$, we may fix $(x_t)_{t \in \Gamma_{\xi,n}.D} \subset B_{E_p}$, $(a_\lambda^i)_{\lambda \in \Upsilon} \in B_{\ell_p^n}$, $i = 1, \ldots, n$, $(x_t^*)_{t \in \mathrm{MAX}(\Gamma_{\xi,n}.D)} \subset B_{F^*}$ to satisfy the conclusions of Lemma 7.4. Now for each $\lambda \in \Upsilon$, fix $\varepsilon_\lambda > \alpha_{\xi,\gamma,n}(A_\lambda)\|(a_\lambda^i b_i)_{i=1}^n\|_{\ell_\gamma^n}$ and $t \in \mathrm{MAX}(\Gamma_{\xi,n}.D)$

such that for each $\lambda \in \Upsilon$, $\|D_\lambda^{p,r} \sum_{i=1}^n b_i z_i^t\| \leqslant \varepsilon_\lambda$. For each $\lambda \in \Upsilon$, the set $\mathcal{E}_\lambda$ of $t \in \mathrm{MAX}(\Gamma_{\xi,n}.D)$ satisfying this inequality is eventual, so $\bigcap_{\lambda \in \Upsilon} \mathcal{E}_\lambda \neq \varnothing$. Then

$$\alpha < \inf_{s \in \mathrm{MAX}(\Gamma_{\xi,n}.D)} \left\| D_\Upsilon^{p,r} \sum_{i=1}^n b_i z_i^s \right\| \leqslant \left\| D_\Upsilon^{p,r} \sum_{i=1}^n b_i z_i^t \right\|$$

$$= \left\| \left(\|D_\lambda^{p,r}\| \left\| \sum_{i=1}^n b_i z_i^t \right\| \right)_{\lambda \in \Upsilon} \right\|_{\ell_r^\Upsilon}$$

$$\leqslant \left\| (\varepsilon_\lambda)_{\lambda \in \Upsilon} \right\|_{\ell_r(\Upsilon)}.$$

Since we may do this for any $\varepsilon_\lambda > \alpha_{\xi,\gamma,n}(A_\lambda) \|(a_\lambda^i b_i)_{i=1}^n\|_{\ell_\gamma^n}$, we deduce that

$$\alpha \leqslant \left\| \left(\left\| \left(\alpha_{\xi,\gamma,n}(A_\lambda) a_\lambda^i b_i \right)_{i=1}^n \right\|_{\ell_\gamma^n} \right)_{\lambda \in \Upsilon} \right\|_{\ell_r(\Upsilon)}.$$

By Fact 7.2, the last quantity cannot exceed $\|(\alpha_{\xi,\gamma,n}(A_\lambda))_{\lambda \in \Upsilon}\|_{\ell_\infty(\Upsilon)}$ if either $r = 0$ or $\gamma \leqslant r$, and cannot exceed $\|(\alpha_{\xi,\gamma,n}(A_\lambda))_{\lambda \in \Upsilon}\|_{\ell_s(\Upsilon)}$ if $1 \leqslant r < \gamma$ and $1/\gamma + 1/s = 1/r$. Here we are using the fact that $\ell_0(\Upsilon) = \ell_\infty(\Upsilon)$ since Υ is finite.

For convenience, let us assume $\Upsilon = \{1, \ldots, m\}$. Fix $0 < \phi < \varphi < 1$ and $k \in \mathbb{N}$ such that for each $i \in \Upsilon$, $\alpha_{\xi,\gamma,k}(A_i) \geqslant \varphi \alpha_{\xi,\gamma}(A_i)$. Let D be a weak neighborhood basis at 0 in X. For each $i \in \Upsilon$, we may fix $(b_j^i)_{j=1}^k \in S_{\ell_\gamma^k}$ and $(x_t^i)_{t \in \Gamma_{\xi,k}.D} \subset B_{E_p}$ such that $x_t^i = P_i x_t^i$ for all $t \in \Gamma_{\xi,k}.D$ and

$$\inf_{t \in \mathrm{MAX}(\Gamma_{\xi,k}.D)} \left\| D_i^{p,r} \sum_{j=1}^k \sum_{t \geqslant s \in \Lambda_{\xi,k,j}.D} b_j^i \mathbb{P}_{\xi,k}(s) x_s^i \right\| \geqslant \phi \alpha_{\xi,\gamma}(A_i).$$

By Fact 7.1, there exists $(c_i)_{i \in \Upsilon} \in S_{\ell_\gamma^m}$ such that

$$\left\| (\alpha_{\xi,\gamma}(A_i) c_i)_{i=1}^m \right\|_{\ell_r^m} = \left\| (\alpha_{\xi,\gamma}(A_i))_{i=1}^m \right\|_{\ell_s^m}.$$

Now for $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant k$, let $d_{(i-1)k+j} = c_i b_j^i$ and note that $\|(d_i)_{i=1}^{mk}\|_{\ell_\gamma^{mk}} = 1$. For $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant k$, and $t \in \Lambda_{\xi,mk,(i-1)k+j}.D$, we may write $t = s_1 \frown (\omega^\xi(k(m-i)) + s)$, where $s_1 \in \mathrm{MAX}(\Lambda_{\xi,mk,(i-1)k}.D)$ if $i > 1$ and $s_1 = \varnothing$ if $i = 1$, and where $s \in \Gamma_{\xi,k}.D$. We then let $x_t = u_s^i$. Then $(x_t)_{t \in \Gamma_{\xi,mk}.D} \subset B_{E_p}$ is weakly null and

$$\inf_{t \in \mathrm{MAX}(\Gamma_{\xi,mk}.D)} \left\| D_\Upsilon^{p,r} \sum_{i=1}^{mk} d_i z_i^t \right\| \geqslant \phi \left\| (c_i \alpha_{\xi,\gamma}(A_i))_{i=1}^m \right\|_{\ell_r^m}$$

$$= \phi \left\| (\alpha_{\xi,\gamma}(A_i))_{i=1}^m \right\|_{\ell_m^s}.$$

This yields that $\alpha_{\xi,\gamma}(D_\Upsilon^{p,r}) \geqslant \phi \|(\alpha_{\xi,\gamma}(A_i))_{i=1}^m\|_{\ell_m^s}$. Since this holds for any $0 < \phi < 1$, we deduce that $\alpha_{\xi,\gamma}(D_\Upsilon^{p,r}) \geqslant \|(\alpha_{\xi,\gamma}(A_i))_{i=1}^m\|_{\ell_m^s}$. $\square$

In the following, if for each $\lambda \in \Lambda$, $\alpha_\lambda \in [0, \infty]$, we let $\|(\alpha_\lambda)_{\lambda \in \Lambda}\|_{\ell_s(\Lambda)} = \infty$ if either $\alpha_\lambda = \infty$ for some $\lambda \in \Lambda$ or if $\alpha_\lambda \in [0, \infty)$ for all λ and $(\alpha_\lambda)_{\lambda \in \Lambda} \notin \ell_s(\Lambda)$.

COROLLARY 7.5. (i) $D_\Lambda^{1,1}$ is Asplund if and only if $(A_\lambda)_{\lambda \in \Lambda} \in c_0(\Lambda)$, in which case

$$Sz(D_\Lambda^{1,1}) = \sup_{\lambda \in \Lambda} Sz(A_\lambda)$$

and for any $1 \leqslant \gamma \leqslant \infty$, $\alpha_{\xi,\gamma}(D_\Lambda^{1,1}) = \|(\alpha_{\xi,\gamma}(A_\lambda))_{\lambda \in \Lambda}\|_{\ell_s(\Lambda)}$, where $1/\gamma + 1/s = 1$.

(ii) $D_\Lambda^{\infty,\infty}$ is Asplund if and only if $(A_\lambda)_{\lambda \in \Lambda} \in c_0(\Lambda)$, in which case

$$Sz(D_\Lambda^{\infty,\infty}) = \sup_{\lambda \in \Lambda} Sz(A_\lambda)$$

and for any $1 \leqslant \gamma \leqslant \infty$, $\alpha_{\xi,\gamma}(D_\Lambda^{\infty,\infty}) = \|(\alpha_{\xi,\gamma}(A_\lambda))_{\lambda \in \Lambda}\|_{\ell_\infty(\Lambda)}$.

Proof. (i) If $(\|A_\lambda\|)_{\lambda \in \Lambda} \in \ell_\infty(\Lambda) \setminus c_0(\Lambda)$, then $D_\Lambda^{1,1}$ preserves an isomorph of ℓ_1 and $\sup_{\xi \in \mathbf{Ord}} \theta_{\xi,1}(D_\Lambda^{1,1}) > 0$. Now if $(\|A_\lambda\|)_{\lambda \in \Lambda} \in c_0(\Lambda)$, then $D_\Lambda^{1,1}$ lies in the norm closure of $\{D_\Upsilon^{1,1} : \Upsilon \subset \Lambda \text{ finite}\}$. By the norm closedness of $\{B : E_p \to F_r : Sz(B) \leqslant \sup_{\lambda \in \Lambda} Sz(A_\lambda)\}$ and the fact that for any finite $\Upsilon \subset \Lambda$, $Sz(D_\Upsilon^{1,1}) = \max_{\lambda \in \Upsilon} Sz(A_\lambda)$, $Sz(D_\Lambda^{1,1}) \leqslant \sup_{\lambda \in \Lambda} Sz(A_\lambda)$. By the ideal property, the reverse inequality holds. Furthermore, if $1/\gamma + 1/s = 1$, Theorem 7.3 yields that for any finite subset Υ of Λ,

$$\alpha_{\xi,\gamma}(D_\Upsilon^{1,1}) = \|(\alpha_{\xi,\gamma}(A_\lambda))_{\lambda \in \Upsilon}\|_{\ell_s(\Upsilon)}.$$

By the norm continuity of $\alpha_{\xi,\gamma,n}$, and since $D_\Lambda^{1,1}$ lies in the norm closure of $\{D_\Upsilon^{1,1} : \Upsilon \subset \Lambda \text{ finite}\}$,

$$\begin{aligned} \alpha_{\xi,\gamma,n}(D_\Lambda^{1,1}) &\leqslant \sup\{\alpha_{\xi,\gamma,n}(D_\Upsilon^{1,1}) : \Upsilon \subset \Lambda \text{ finite}\} \\ &\leqslant \sup\{\alpha_{\xi,\gamma}(D_\Upsilon^{1,1}) : \Upsilon \subset \Lambda \text{ finite}\} = \|(\alpha_{\xi,\gamma}(A_\lambda))_{\lambda \in \Lambda}\|_{\ell_s(\Lambda)}. \end{aligned}$$

Taking the supremum over $n \in \mathbb{N}$, we deduce that

$$\alpha_{\xi,\gamma}(D_\Lambda^{1,1}) \leqslant \|(\alpha_{\xi,\gamma}(A_\lambda))_{\lambda \in \Lambda}\|_{\ell_s(\Lambda)}.$$

By the ideal property,

$$\alpha_{\xi,\gamma}(D_\Lambda^{1,1}) \geqslant \sup\{\alpha_{\xi,\gamma}(D_\Upsilon^{1,1}) : \Upsilon \subset \Lambda \text{ finite}\} = \|(\alpha_{\xi,\gamma}(A_\lambda))_{\lambda \in \Lambda}\|_{\ell_s(\Lambda)},$$

giving equality.

(ii) We argue similarly, except Theorem 7.3 gives

$$\alpha_{\xi,\gamma}(D_\Upsilon^{\infty,\infty}) = \|(\alpha_{\xi,\gamma}(A_\lambda))_{\lambda \in \Lambda}\|_{\ell_\infty(\Upsilon)}$$

for any finite subset of Υ. Here, we are using that either $(A_\lambda)_{\lambda \in \Lambda} \in c_0(\Lambda)$, or $D_\Lambda^{\infty,\infty}$ preserves an isomorph of ℓ_1. $\square$

THEOREM 7.6. *Suppose that p, r are as in one of the cases (3) or (4). If $r = 0$, suppose $1 \leqslant \gamma \leqslant \infty$ and if $1 \leqslant r \leqslant \infty$, suppose $1 \leqslant \gamma \leqslant r$. Then for any ordinal ζ,*

$$\alpha_{\zeta,\gamma}(D_\Lambda^{p,r}) \leqslant \sup_{\lambda \in \Lambda} \|A_\lambda\| + \sup_{\lambda \in \Lambda} \alpha_{\zeta,\gamma}(A_\lambda).$$

Proof. We prove that

$$\alpha_{\zeta,\gamma,n}\big(D_\Lambda^{p,r}\big) \leqslant \sup_{\lambda\in\Lambda}\|A_\lambda\| + \sup_{\lambda\in\Lambda}\alpha_{\zeta,\gamma}(A_\lambda)$$

by induction on $n\in\mathbb{N}$. Let $\beta = \sup_{\lambda\in\Lambda}\|A_\lambda\|$, $\alpha = \sup_{\lambda\in\Lambda}\alpha_{\xi,\gamma}(A_\lambda)$.

The $n=1$ case is trivial, since $\alpha_{\zeta,\gamma,1}(D_\Lambda^{p,r}) \leqslant \beta \leqslant \beta+\alpha$.

Assume the result holds for $n\in\mathbb{N}$ and fix $\vartheta < \alpha_{\zeta,\gamma,n+1}(D_\Lambda^{p,r})$. Fix $(b_i)_{i=1}^{n+1} \in B_{\ell_\gamma^{n+1}}$ and a weakly null collection $(x_t)_{t\in\Gamma_{\zeta,n+1}.D} \subset B_{E_p}$ such that $\vartheta < \inf_{t\in\mathrm{MAX}(\Gamma_{\zeta,n+1}.D)}\|D_\Lambda^{p,r}\sum_{i=1}^{n+1}b_i z_i^t\|$. Fix $0<\phi<1$ such that $\vartheta < \inf_{t\in\mathrm{MAX}(\Gamma_{\zeta,n+1}.D)}\|D_\Lambda^{p,r}\sum_{i=1}^{n+1}b_i\phi z_i^t\|$. Fix $\delta>0$. Fix $t\in\mathrm{MAX}(\Lambda_{\zeta,n+1,1}.D)$ and a finite subset Υ of Λ such that $\|D_{\Lambda\setminus\Upsilon}^{p,r}\sum_{s\leqslant t}\mathbb{P}_{\zeta,n+1}(s)x_s\| < \delta$. Arguing as in Lemma 7.4, we may stabilize, relabel, and assume that there exist $(u_i,v_i)_{i=2}^{n+1} \in B_{\ell_\infty^n(\ell_p^2)}$ such that for any $2\leqslant i\leqslant n+1$ and any $t\in\Lambda_{\xi,n+1,i}.D$, $\phi\|P_\Upsilon x_t\| < u_i$ and $\phi\|P_{\Lambda\setminus\Upsilon}x_t\| < v_i$. This may be done so that the property

$$\vartheta < \inf_{t\in\mathrm{MAX}(\Gamma_{\zeta,n+1}.D)}\left\|D_\Lambda^{p,r}\sum_{i=1}^{n+1}b_i\phi z_i^t\right\|$$

is retained. Let $u_1=1$ and $v_1=0$, so that $(u_i,v_i)_{i=1}^{n+1} \in B_{\ell_\infty^{n+1}(\ell_p^2)}$.

Now using the identification of $\Gamma_{\zeta,n}.D$ with $\{s\in\Gamma_{\zeta,n+1}.D : t<s\}$ and the fact that

$$\alpha_{\zeta,\gamma}\big(D_\Upsilon^{p,r}\big) = \big\|(\alpha_{\zeta,\gamma}(A_\lambda))_{\lambda\in\Upsilon}\big\|_{\ell_\infty(\Upsilon)} \leqslant \alpha$$

by Theorem 7.3, we deduce using the inductive hypothesis that each of the subsets of $\mathrm{MAX}(\Gamma_{\zeta,n+1}.D)$

$$\mathcal{E}_1 := \left\{s : t<s, \left\|D_\Upsilon^{p,r}\sum_{i=1}^{n+1}b_i\phi z_i^s\right\| \leqslant \delta + (\beta+\alpha)\big\|(b_iu_i)_{i=1}^{n+1}\big\|_{\ell_\gamma^{n+1}}\right\}$$

$$\supset \left\{s : t<s, \left\|D_\Upsilon^{p,r}\sum_{i=1}^{n+1}b_i\phi z_i^s\right\| \leqslant \delta + \beta u_1 + \alpha\big\|(b_iu_i)_{i=2}^{n+1}\big\|_{\ell_\gamma^{n}}\right\}$$

$$\supset \left\{s : t<s, \left\|D_\Upsilon^{p,r}\sum_{i=2}^{n+1}b_i\phi z_i^s\right\| \leqslant \delta + \alpha\big\|(b_iu_i)_{i=2}^{n+1}\big\|_{\ell_\gamma^{n}}\right\}$$

$$\supset \left\{s : t<s, \left\|D_\Upsilon^{p,r}\sum_{i=2}^{n+1}b_iu_i\phi\frac{P_\Upsilon z_i^s}{u_i}\right\| \leqslant \delta + \alpha\big\|(b_iu_i)_{i=2}^{n+1}\big\|_{\ell_\gamma^{n}}\right\},$$

$$\mathcal{E}_2 := \left\{s : t<s, \left\|D_{\Lambda\setminus\Upsilon}^{p,r}\sum_{i=1}^{n+1}b_i\phi z_i^s\right\| \leqslant 2\delta + (\beta+\alpha)\big\|(b_iv_i)_{i=1}^{n+1}\big\|_{\ell_\gamma^{n+1}}\right\}$$

$$\supset \left\{s : t<s, \left\|D_{\Lambda\setminus\Upsilon}^{p,r}\sum_{i=2}^{n+1}b_i\phi z_i^s\right\| \leqslant \delta + (\beta+\alpha)\big\|(b_iv_i)_{i=1}^{n+1}\big\|_{\ell_\gamma^{n+1}}\right\}$$

$$= \left\{ s : t < s, \left\| D_\Lambda^{p,r} \sum_{i=2}^{n+1} b_i v_i \frac{P_{\Lambda\backslash\Upsilon} z_i^s}{v_i} \right\| \leqslant \delta + (\beta + \alpha) \left\| (b_i v_i)_{i=1}^{n+1} \right\|_{\ell_\gamma^{n+1}} \right\}$$

is eventual. Here we are using the fact that for any $t < s \in \mathrm{MAX}(\Gamma_{\zeta,n+1}.D)$, $z_1^s = \sum_{u \leqslant t} \mathbb{P}_{\zeta,n+1}(u) x_u$, $\|D_\Upsilon^{p,r} z_1^s\| \leqslant \beta$, and $\|D_{\Lambda\backslash\Upsilon}^{p,r} z_1^s\| < \delta$.

If $r = 0$, let $r_1 = \infty$, and otherwise let $r_1 = r$. Now for any $s \in \mathcal{E}_1 \cap \mathcal{E}_2$, Minkowski's inequality and the fact that $p \leqslant r_1$ give that

$$\vartheta < \left\| D_\Lambda^{p,r} \sum_{i=1}^{n+1} b_i \phi z_i^s \right\| = \left\| \left(\left\| D_\Upsilon^{p,r} \sum_{i=1}^{n+1} b_i \phi z_i^s \right\|, \left\| D_{\Lambda\backslash\Upsilon}^{p,r} \sum_{i=1}^{n+1} b_i \phi z_i^s \right\| \right) \right\|_{\ell_r^2}$$

$$\leqslant 3\delta + (\beta + \alpha) \left\| \left(\left\| (b_i u_i)_{i=1}^{n+1} \right\|_{\ell_\gamma^{n+1}}, \left\| (b_i v_i)_{i=1}^{n+1} \right\|_{\ell_\gamma^{n+1}} \right) \right\|_{\ell_{r_1}^2}$$

$$\leqslant 3\delta + (\beta + \alpha) \left\| \left(b_i \left\| (u_i, v_i) \right\|_{\ell_{r_1}^2} \right)_{i=1}^{n+1} \right\|_{\ell_\gamma^{n+1}}$$

$$\leqslant 3\delta + (\beta + \alpha) \left\| \left(b_i \left\| (u_i, v_i) \right\|_{\ell_p^2} \right)_{i=1}^{n+1} \right\|_{\ell_\gamma^{n+1}} \leqslant 3\delta + \beta + \alpha.$$

Since $\delta > 0$ and $\vartheta < \alpha_{\zeta,\gamma,n+1}(D_\Lambda^{p,r})$ were arbitrary, we are done. $\square$

COROLLARY 7.7. *In either of cases* (3), (4),

$$Sz(D_\Lambda^{p,r}) \leqslant \left(\sup_{\lambda \in \Lambda} Sz(A_\lambda) \right) \omega.$$

Proof. If $r = 0$, let $\gamma = \infty$, and otherwise let $\gamma = r$. Let $\sup_{\lambda \in \Lambda} Sz(A_\lambda) = \omega^\xi$. Then for every $\lambda \in \Lambda$, $\alpha_{\xi,\gamma,1}(A_\lambda) = \alpha_{\xi,\gamma}(A_\lambda) = 0$, whence

$$\alpha_{\xi,\gamma}(D_\Lambda^{p,r}) \leqslant \sup_{\lambda \in \Lambda} \|A_\lambda\| < \infty$$

by Theorem 7.6. By Remark 4.9, $Sz(D_\Lambda^{p,r}) \leqslant \omega^{\xi+1} = \omega^\xi \omega$. $\square$

COROLLARY 7.8. *Suppose p, r are as in either case* (3) *or case* (4). *For any ordinal ζ,*

$$\inf_{m \in \mathbb{N}} \sup_{\lambda \in \Lambda} \theta_{\zeta,m}(A_\lambda) = \inf_{m \in \mathbb{N}} \theta_{\zeta,m}(D_\Lambda^{p,r}) = \theta_{\zeta+1,1}(D_\Lambda^{p,r}).$$

In particular, for an ordinal $\xi > 0$,

$$\inf \left\{ \theta_{\xi,1}(D_{\Lambda\backslash\Upsilon}^{p,r}) : \Upsilon \subset \Lambda \text{ finite} \right\} = 0$$

if and only if for every $\varepsilon > 0$, there exist a finite subset Υ of Λ, an ordinal $\zeta < \xi$, and $m \in \mathbb{N}$ such that $\sup_{\lambda \in \Upsilon} \theta_{\zeta,m}(A_\lambda) \leqslant \varepsilon$ if and only if for every $\varepsilon > 0$, there exists a finite subset Υ of Λ such that $\sup_{\lambda \in \Lambda\backslash\Upsilon} Sz(A_\lambda, \varepsilon) < \omega^\xi$.

Proof. If $r = 0$, let $\gamma = \infty$, and otherwise let $\gamma = r$. Let $1/\gamma + 1/s = 1$. Note that since $r \neq 1$, $\gamma \neq 1$, so $s < \infty$. It follows from Proposition 4.7 that $\inf_{m \in \mathbb{N}} \theta_{\zeta,m}(D_\Lambda^{p,r}) = \theta_{\zeta+1,1}(D_\Lambda^{p,r})$, and it follows from the fact that $\theta_{\zeta,m}$ is an ideal seminorm that $\inf_{m \in \mathbb{N}} \sup_{\lambda \in \Lambda} \theta_{\zeta,m}(A_\lambda) \leqslant \inf_{m \in \mathbb{N}} \theta_{\zeta,m}(D_\Lambda^{p,r})$. Fix $\varepsilon_1 > \inf_m \sup_{\lambda \in \Lambda} \theta_{\zeta,m}(A_\lambda)$. Let $m \in \mathbb{N}$ and $\varepsilon_2, \varepsilon$ be such that $\varepsilon_1 > \varepsilon_2 > \varepsilon >$

$\theta_{\zeta,m}(A_\lambda)$ for all $\lambda \in \Lambda$. Let $R = \sup_{\lambda \in \Lambda} \|A_\lambda\|$ and fix $n \in \mathbb{N}$, $\delta > 0$ such that $Rm^{1/s}/n^{1/s} < \varepsilon_2 - \varepsilon$, $\delta + R/n^{1/s} < \varepsilon_1 - \varepsilon_2$, and $\delta + \varepsilon + Rm^{1/s}/n^{1/s} < \varepsilon_2$.

Fix a weakly null collection $(x_t)_{t \in \Gamma_{\zeta,n^2}.D} \subset B_{E_p}$. Let $\Upsilon_0 = \varnothing$, $t_0 = \varnothing$, and fix any $t_1 \in \mathrm{MAX}(\Lambda_{\zeta,n^2,n}.D)$. Now suppose we have chosen $t_1 < \cdots < t_{k-1}$ and finite sets $\Upsilon_1 \subset \cdots \subset \Upsilon_{k-1} \subset \Lambda$ such that for each $1 \leqslant i < k$, $t_i \in \mathrm{MAX}(\Lambda_{\zeta,n^2,in}.D)$,

$$\left\| D_{\Lambda \setminus \Upsilon_i}^{p,r} \sum_{j=(i-1)n+1}^{in} \sum_{t_i \geqslant s \in \Lambda_{\zeta,n^2,j}.D} \mathbb{P}_{\zeta,n^2}(s)x_s \right\| < \delta$$

and

$$\left\| D_{\Upsilon_{i-1}}^{p,r} n^{-1} \sum_{j=(i-1)n+1}^{in} \sum_{t_i \geqslant s \in \Lambda_{\zeta,n^2,j}.D} \mathbb{P}_{\zeta,n^2}(s)x_s \right\| < \varepsilon_2.$$

Now since we may identify

$$\left\{ s \in \bigcup_{j=(k-1)n+1}^{kn} \Lambda_{\zeta,n^2,j}.D : t_{k-1} < s \right\}$$

with $\Gamma_{\zeta,n}.D$, by Theorem 7.3, there exists $t_k \in \mathrm{MAX}(\Lambda_{\zeta,n^2,kn}.D)$ such that

$$\left\| D_{\Upsilon_{k-1}}^{p,r} \sum_{j=(k-1)n+1}^{kn} \sum_{t_k \geqslant s \in \Lambda_{\zeta,n^2,j}.D} n^{-1} \mathbb{P}_{\zeta,n^2}(s)x_s \right\|$$

$$\leqslant \delta + \varepsilon \left\| (1/n)_{j=1}^n \right\|_{\ell_1^n} + Rm^{1/s} \left\| (1/n)_{j=1}^n \right\|_{\ell_\gamma^n}$$

$$= \delta + \varepsilon + R \frac{m^{1/s}}{n^{1/s}} < \varepsilon_2.$$

Here we are using that $\ell_r^n = \ell_\gamma^n$. Fix $\Upsilon_k \subset \Lambda$ finite such that $\Upsilon_{k-1} \subset \Upsilon_k$ and

$$\left\| D_{\Lambda \setminus \Upsilon_k}^{p,r} \sum_{j=(k-1)n+1}^{kn} \sum_{t_k \geqslant s \in \Lambda_{\zeta,n^2,j}} n^{-1} \mathbb{P}_{\zeta,n^2}(s)x_s \right\| < \delta.$$

These completes the recursive construction. Let $t = t_n$ and for each $1 \leqslant i \leqslant n$, let $F_i = \Upsilon_i \setminus \Upsilon_{i-1}$. For each $1 \leqslant i \leqslant n$, let $z_i = n^{-1} \sum_{t_{i-1} < s \leqslant t_i} \mathbb{P}_{\zeta,n^2}(s)x_s$. Then

$$\left\| D_\Lambda^{p,r} \sum_{j=1}^{n^2} n^{-2} z_i^t \right\| = \left\| D_\Lambda^{p,r} n^{-1} \sum_{i=1}^n z_i \right\|$$

$$= \left\| D_\Lambda^{p,r} n^{-1} \sum_{i=1}^n [P_{\Upsilon_{i-1}} + P_{F_i} + P_{\Lambda \setminus \Upsilon_i}] z_i \right\|$$

$$\leqslant \varepsilon_2 + \left\| n^{-1} \sum_{i=1}^n P_{F_i} z_i \right\| + \delta$$

$$\leqslant \varepsilon_2 + R/n^{1/s} + \delta < \varepsilon_1.$$

From this it follows that

$$\inf\{\theta_{\zeta,k}(D_\Lambda^{p,r}) : k \in \mathbb{N}\} \leqslant \theta_{\zeta,n^2}(D_\Lambda^{p,r}) < \varepsilon_1.$$

Since $\varepsilon_1 > \inf_{k \in \mathbb{N}} \sup_{\lambda \in \Lambda} \theta_{\zeta,k}(A_\lambda)$ was arbitrary,

$$\inf_{m \in \mathbb{N}} \sup_{\lambda \in \Lambda} \theta_{\zeta,m}(A_\lambda) \geqslant \inf_{m \in \mathbb{N}} \theta_{\zeta,m}(D_\Lambda^{p,r}).$$

The last statement follows immediately from the previous statements.   □

We now eliminate a trivial case. The proof is obvious, so we omit it.

PROPOSITION 7.9. *Suppose that* $\sup_{\lambda \in \Lambda} Sz(A_\lambda) = 1$. *Assume that* p, r *are as in one of cases* (3), (4) *above. Then* $D_\Lambda^{p,r}$ *is compact,* $\theta_{0,1}(D_\Lambda^{p,r}) = \alpha_{0,\gamma}(D_\Lambda^{p,r}) = 0$ *for all* $1 \leqslant \gamma \leqslant \infty$, *and* $Sz(D_\Lambda^{p,r}) = 1$ *if and only if* $(\|A_\lambda\|)_{\lambda \in \Lambda} \in c_0(\Lambda)$. *If* $(\|A_\lambda\|)_{\lambda \in \Lambda} \in \ell_\infty(\Lambda) \setminus c_0(\Lambda)$, *then* $Sz(D_\Lambda^{p,r}) = \omega$,

$$\theta_{0,1}(D_\Lambda^{p,r}) = \inf\{\|D_{\Lambda \setminus \Upsilon}^{p,r}\| : \Upsilon \subset \Lambda \text{ finite}\} > 0$$

and

(i) *if* $r = 0$ *or* $r = \infty$, $\alpha_{0,\gamma}(D_\Lambda^{p,r}) = \theta_{0,1}(D_\Lambda^{p,r}) < \infty$ *for any* $1 \leqslant \gamma \leqslant \infty$,
(ii) *if* $1 \leqslant p \leqslant r < \infty$ *and* $1 < r$, $\alpha_{0,\gamma}(D_\Lambda^{p,r}) = \theta_{0,1}(D_\Lambda^{p,r})$ *for all* $1 \leqslant \gamma \leqslant r$ *and* $\alpha_{0,\gamma}(D_\Lambda^{p,r}) = \infty$ *for all* $r < \gamma \leqslant \infty$.

The next theorem incorporates Brooker's result about the Szlenk index, but also includes new information regarding the behavior of the $\theta_{\zeta,m}(A_\lambda)$ quantities, which are only indirectly related to $Sz(A_\lambda, \varepsilon)$.

THEOREM 7.10. *Suppose* p, r *are as in either case* (3) *or case* (4) *and*

$$\sup_{\lambda \in \Lambda} Sz(A_\lambda) = \omega^\xi > 1.$$

The following are equivalent.

(i) $Sz(D_\Lambda^{p,r}) = \omega^\xi$.
(ii) *For every* $\varepsilon > 0$, $\sup_{\lambda \in \Lambda} Sz(A_\lambda, \varepsilon) < \omega^\xi$.
(iii) $\inf\{\sup_{\lambda \in \Lambda} \theta_{\zeta,m}(A_\lambda) : \zeta < \xi, m \in \mathbb{N}\} = 0$.

Proof. (i)$\Rightarrow$(iii) If $Sz(D_\Lambda^{p,r}) = \omega^\xi$, then by Proposition 4.7 and the fact that $\theta_{\zeta,m}$ is an ideal seminorm,

$$0 = \theta_{\xi,1}(D_\Lambda^{p,r}) = \inf\{\theta_{\zeta,m}(D_\Lambda^{p,r}) : \zeta < \xi, m \in \mathbb{N}\}$$

$$\geqslant \inf\{\sup_{\lambda \in \Lambda} \theta_{\zeta,m}(A_\lambda) : \zeta < \xi, m \in \mathbb{N}\} \geqslant 0.$$

(iii)$\Rightarrow$(i) We first remark that $Sz(D_\Lambda^{p,r}) \geqslant \omega^\xi$. Indeed, for any $\zeta < \xi$, there exists $\lambda \in \Lambda$ such that $Sz(A_\lambda) > \omega^\zeta$, so $\theta_{\zeta,1}(D_\Lambda^{p,r}) \geqslant \theta_{\zeta,1}(A_\lambda) > 0$, and $Sz(D_\Lambda^{p,r}) > \omega^\zeta$. Since $Sz(D_\Lambda^{p,r})$ must be of the form ω^γ for some γ, it follows that $Sz(D_\Lambda^{p,r}) \geqslant \omega^\xi$.

If ξ is a successor, say $\xi = \eta + 1$, then by Proposition 4.7 and Corollary 7.8,

$$0 = \inf\left\{\sup_{\lambda \in \Lambda} \theta_{\zeta,m}(A_\lambda) : \zeta < \xi, m \in \mathbb{N}\right\} = \inf\left\{\sup_{\lambda \in \Lambda} \theta_{\eta,m}(A_\lambda) : m \in \mathbb{N}\right\}$$
$$= \theta_{\eta+1,1}\left(D_\Lambda^{p,r}\right) = \theta_{\xi,1}\left(D_\Lambda^{p,r}\right),$$

so $Sz(D_\Lambda^{p,r}) \leqslant \omega^\xi$.

Now if ξ is a limit, then by Proposition 4.7, for any $\varepsilon > 0$, there exist $\zeta < \xi, m \in \mathbb{N}$ such that $\sup_{\lambda \in \Lambda} \theta_{\zeta,m}(A_\lambda) < \varepsilon$. Then by Corollary 7.8, and Proposition 4.7, $\theta_{\xi,1}(D_\Lambda^{p,r}) \leqslant \theta_{\zeta+1,1}(D_\Lambda^{p,r}) \leqslant \varepsilon$. Since this holds for any $\varepsilon > 0$, $\theta_{\xi,1}(D_\Lambda^{p,r}) = 0$.

(ii)$\Rightarrow$(iii) Fix $\varepsilon > 0$ and $\zeta < \xi$, $m \in \mathbb{N}$ such that $Sz(A_\lambda, \varepsilon/3) \leqslant \omega^\zeta m$ for all $\lambda \in \Lambda$. Fix $n \in \mathbb{N}$ such that $Rm/n < 2\varepsilon/3$, where $R = \sup_{\lambda \in \Lambda} \|A_\lambda\|$. Now if $\lambda \in \Lambda$ is such that $\theta_{\zeta,n}(A_\lambda) > \varepsilon$, there exist a directed set D, a weakly null collection $(x_t)_{t \in \Gamma_{\zeta,n}.D} \subset B_{E_p}$, functionals $(y_t^*)_{t \in \mathrm{MAX}(\Gamma_{\zeta,n}.D)} \subset B_{Y_\lambda^*}$, and real numbers $b_1, \ldots, b_n$ such that $\sum_{i=1}^n \frac{b_i}{n} > \varepsilon$ and for each $1 \leqslant i \leqslant n$ and $\Lambda_{\zeta,n,i}.D \ni s \leqslant t \in \mathrm{MAX}(\Gamma_{\zeta,n}.D)$, $\mathrm{Re}\, y_t^*(x_s) \geqslant b_i$. Let $T = \{i \leqslant n : b_i > 2\varepsilon/3\}$ and fix $1/2 < \psi < 1$. We claim that $|T| < m$. If it were not so, we could find a level map $d : \Gamma_{\zeta,m}.D \to \Gamma_{\zeta,n}.D$ such that $(x_{d(t)})_{t \in \Gamma_{\zeta,m}.D}$ is weakly null and $d(\Lambda_{\zeta,m,i}) \subset \Lambda_{\zeta,n,l_i}.D$, where $l_1 < \cdots < l_m$, $l_i \in T$. Then if e is any extension of d, the collections $(x_{d(t)})_{t \in \Gamma_{\zeta,m}.D}$, $(y_{e(t)}^*)_{t \in \mathrm{MAX}(\Gamma_{\zeta,m}.D)}$ witness that

$$\varnothing \neq s_{\psi\varepsilon_{l_m}}^{\omega^\zeta} \cdots s_{\psi\varepsilon_{l_1}}^{\omega^\zeta}\left(A_\lambda^* B_{Y_\lambda^*}\right) \supset s_{\psi 2\varepsilon/3}^{\omega^\zeta}\left(A_\lambda^* B_{Y_\lambda^*}\right) \supset s_{\varepsilon/3}^{\omega^\zeta m}\left(A_\lambda^* B_{Y_\lambda^*}\right),$$

a contradiction. Thus $\sup_{\lambda \in \Lambda} \theta_{\zeta,n}(A_\lambda) \leqslant \varepsilon$, whence

$$\inf\left\{\sup_{\lambda \in \Lambda} \theta_{\zeta,m}(A_\lambda) : \zeta < \xi, m \in \mathbb{N}\right\} = 0.$$

(iii)$\Rightarrow$(ii) We recall that if for some $\lambda \in \Lambda$, $\varepsilon > 0$, $\zeta \in \mathbf{Ord}$, and $m \in \mathbb{N}$, $Sz(A_\lambda, \varepsilon) > \omega^\zeta m$, then $\theta_{\zeta,m}(A_\lambda) \geqslant \varepsilon/4$. From this, it follows that since $\inf\{\sup_{\lambda \in \Lambda}(A_\lambda) : \zeta < \xi, m \in \mathbb{N}\} < \varepsilon/4$, there exist $\zeta < \xi$ and $m \in \mathbb{N}$ such that $Sz(A_\lambda, \varepsilon) \leqslant \omega^\zeta m < \omega^\xi$ for all $\lambda \in \Lambda$. $\qquad\square$

We next elucidate the behavior of $\alpha_{\xi,\gamma}(D_\Lambda^{p,r})$ for cases (3) and (4).

THEOREM 7.11. *Suppose that p, r are as in either case (3) or case (4). Suppose that $Sz(D_\Lambda^{p,r}) = \omega^{\xi+1}$ and $\sup_{\lambda \in \Lambda} Sz(A_\lambda) = \omega^\zeta$.*

If $\zeta = \xi$, $\alpha_{\xi,\gamma}(D_\Lambda^{p,r}) < \infty$ if and only if either

(i) *$r = 0$, or*
(ii) *$1 \leqslant \gamma \leqslant r$.*

If $\zeta = \xi + 1$, $\alpha_{\xi,\gamma}(D_\Lambda^{p,r}) < \infty$ if and only if either

(i) *$r = 0$ and $(\alpha_{\xi,\gamma}(A_\lambda))_{\lambda \in \Lambda} \in \ell_\infty(\Lambda)$,*
(ii) *$1 \leqslant \gamma \leqslant r \leqslant \infty$ and $(\alpha_{\xi,\gamma}(A_\lambda))_{\lambda \in \Lambda} \in \ell_\infty(\Lambda)$, or*

(iii) $1 \leqslant r < \gamma \leqslant \infty$, $\inf\{\theta_{\xi,1}(D_{\Lambda\setminus\Upsilon}^{p,r}) : \Upsilon \subset \Lambda \text{ finite}\} = 0$, and

$$\left(\alpha_{\xi,\gamma}(A_\lambda)\right)_{\lambda\in\Lambda} \in \ell_s(\Lambda),$$

where $1/\gamma + 1/s = 1/r$.

Proof. First suppose that $Sz(D_\Lambda^{p,r}) = \omega^{\xi+1}$ and $\sup_{\lambda\in\Lambda} Sz(A_\lambda) = \omega^\xi$. By Theorem 7.6, if $r = 0$ or $1 \leqslant p \leqslant r \leqslant \infty$ and $1 \leqslant \gamma \leqslant r$,

$$\alpha_{\xi,\gamma}\left(D_\Lambda^{p,r}\right) \leqslant \sup_{\lambda\in\Lambda} \|A_\lambda\| + \sup_{\lambda\in\Lambda} \alpha_{\xi,\gamma}(A_\lambda) = \sup_{\lambda\in\Lambda} \|A_\lambda\| < \infty,$$

since $\alpha_{\xi,\gamma}(A_\lambda) = 0$ for each $\lambda \in \Lambda$. Now suppose that $1 \leqslant p \leqslant r < \gamma$ and let $1/\gamma + 1/s = 1/r$. By Theorem 7.10, since $Sz(D_\Lambda^{p,r}) > \sup_{\lambda\in\Lambda} Sz(A_\lambda)$, this means

$$\inf\left\{\sup_{\lambda\in\Lambda} \theta_{\eta,m}(A_\lambda) : \eta < \xi, m \in \mathbb{N}\right\} > \varepsilon > 0.$$

Then for any $n \in \mathbb{N}$, we may partition Λ into sets $\Lambda_1, \ldots, \Lambda_n$ such that for each $1 \leqslant i \leqslant n$, $\inf\{\sup_{\lambda\in\Lambda_i} \theta_{\eta,m}(A_\lambda) : \eta < \xi, m \in \mathbb{N}\} > \varepsilon$, whence $\theta_{\xi,1}(D_{\Lambda_i}^{p,r}) > \varepsilon$ for each $1 \leqslant i \leqslant n$. Then we may find $(x_t)_{t\in\Gamma_{\xi,n}.D} \subset B_{E_p}$ weakly null such that for any $1 \leqslant i \leqslant n$, $\inf_{t\in\mathrm{MAX}(\Gamma_{\xi,n}.D)} \|D_{\Lambda_i}^{p,r} \sum_{t\geqslant s\in\Lambda_{\xi,n,i}.D} \mathbb{P}_{\xi,n}(s)x_s\| > \varepsilon$ and for any $t \in \Lambda_{\xi,n,i}.D$, $x_t = P_{\Lambda_i}x_t$. Then

$$\alpha_{\xi,\gamma,n}\left(D_\Lambda^{p,r}\right) \geqslant \inf_{t\in\mathrm{MAX}(\Gamma_{\xi,n}.D)}\left\|D_\Lambda^{p,r}\sum_{i=1}^n z_i^t\right\| \geqslant \varepsilon n^{1/s}.$$

This yields that $\alpha_{\xi,\gamma}(D_\Lambda^{p,r}) = \infty$ if $1 \leqslant p \leqslant r < \gamma$.

Now suppose $Sz(D_\Lambda^{p,r}) = \sup_{\lambda\in\Lambda} Sz(A_\lambda) = \omega^{\xi+1}$. Then if either $r = 0$ or $1 \leqslant p \leqslant r \leqslant \infty$ and $1 \leqslant \gamma \leqslant r$, by Theorem 7.6,

$$\sup_{\lambda\in\Lambda} \alpha_{\xi,\gamma}(A_\lambda) \leqslant \alpha_{\xi,\gamma}\left(D_\Lambda^{p,r}\right) \leqslant \sup_{\lambda\in\Lambda} \|A_\lambda\| + \sup_{\lambda\in\Lambda} \alpha_{\xi,\gamma}(A_\lambda),$$

so that $\alpha_{\xi,\gamma}(D_\Lambda^{p,r}) < \infty$ if and only if $\sup_{\lambda\in\Lambda} \alpha_{\xi,\gamma}(A_\lambda) < \infty$.

We now assume that $1 \leqslant p \leqslant r < \gamma \leqslant \infty$ and $1/\gamma + 1/s = 1/r$. We will show that $\alpha_{\xi,\gamma}(D_\Lambda^{p,r}) < \infty$ if and only if $(\alpha_{\xi,\gamma}(A_\lambda))_{\lambda\in\Lambda} \in \ell_s(\Lambda)$ and $\inf\{\theta_{\xi,1}(D_{\Lambda\setminus\Upsilon}^{p,r}) : \Upsilon \subset \Lambda \text{ finite}\} = 0$. First suppose that $(\alpha_{\xi,\gamma}(A_\lambda))_{\lambda\in\Lambda} \in \ell_s(\Lambda)$ and $\inf\{\theta_{\xi,1}(D_{\Lambda\setminus\Upsilon}^{p,r} : \Upsilon \subset \Lambda \text{ finite}\} = 0$. Then for any finite subset Υ of Λ and $n \in \mathbb{N}$, by Theorem 7.3,

$$\begin{aligned}
\alpha_{\xi,\gamma}\left(D_\Lambda^{p,r}\right) &\leqslant \alpha_{\xi,\gamma,n}\left(D_\Upsilon^{p,r}\right) + \alpha_{\xi,\gamma,n}\left(D_{\Lambda\setminus\Upsilon}^{p,r}\right) \\
&\leqslant \left\|(\alpha_{\xi,\gamma}(A_\lambda))_{\lambda\in\Upsilon}\right\|_{\ell_s(\Upsilon)} + n\alpha_{\xi,\gamma,1}\left(D_{\Lambda\setminus\Upsilon}^{p,r}\right) \\
&\leqslant \left\|(\alpha_{\xi,\gamma}(A_\lambda))_{\lambda\in\Lambda}\right\|_{\ell_s(\Lambda)} + n\theta_{\xi,\gamma,1}\left(D_{\Lambda\setminus\Upsilon}^{p,r}\right).
\end{aligned}$$

Taking the infimum over $\Upsilon \subset \Lambda$ and then the supremum over $n \in \mathbb{N}$ yields that $\alpha_{\xi,\gamma,}(D_\Lambda^{p,r}) \leqslant \|(\alpha_{\xi,\gamma}(A_\lambda))_{\lambda\in\Lambda}\|_{\ell_s(\Lambda)} < \infty$.

Now assume that one of the conditions

$$\left(\alpha_{\xi,\gamma}(A_\lambda)\right)_{\lambda\in\Lambda} \in \ell_s(\Lambda),$$

$$\inf\{\theta_{\xi,1}(D^{p,r}_{\Lambda\setminus\Upsilon}) : \Upsilon \subset \Lambda \text{ finite}\} = 0$$

fails. If $(\alpha_{\xi,\gamma}(A_\lambda))_{\lambda\in\Lambda} \notin \ell_s(\Lambda)$, then by Theorem 7.3,

$$\alpha_{\xi,\gamma}(D^{p,r}_\Lambda) \geqslant \sup\{\alpha_{\xi,\gamma}(D^{p,r}_\Upsilon) : \Upsilon \subset \Lambda \text{ finite}\} = \|(\alpha_{\xi,\gamma}(A_\lambda))_{\lambda\in\Lambda}\|_{\ell_s(\Lambda)} = \infty.$$

Now assume that $\inf\{\theta_{\xi,1}(D^{p,r}_{\Lambda\setminus\Upsilon}) : \Upsilon \subset \Lambda \text{ finite}\} > 2\varepsilon > 0$. Fix $n \in \mathbb{N}$, a weak neighborhood basis D at 0 in E_p, and $\delta > 0$ such that $n\delta < \varepsilon n^{1/r}$, where $1/r + 1/r' = 1$. Let $\Upsilon_\varnothing = \varnothing$. Since $\theta_{\xi,1}(D^{p,r}_\Lambda) > 2\varepsilon$, we may, by identifying $\Lambda_{\xi,n,1}.D$ with $\Gamma_{\xi,1}.D$, fix a weakly null collection $(x_t)_{t\in\Lambda_{\xi,n,1}.D} \subset B_{E_p}$ such that $\inf_{t\in\mathrm{MAX}(\Lambda_{\xi,n,1}.D)} \|D^{p,r}_\Lambda \sum_{s\leqslant t} \mathbb{P}_{\xi,n}(s)x_s\| > 2\varepsilon$. For each $t \in \mathrm{MAX}(\Lambda_{\xi,n,1}.D)$, fix a finite set Υ_t such that $\|D^{p,r}_{\Lambda\setminus\Upsilon_t} \sum_{s\leqslant t} \mathbb{P}_{\xi,n}(s)x_s\| < \varepsilon$. Now assume that for some $i < n$, a weakly null collection $(x_t)_{t\in\bigcup_{j=1}^i \Lambda_{\xi,n,j}.D}$ has been chosen. Suppose also that for each $1 \leqslant j \leqslant i$ and each $t \in \mathrm{MAX}(\Lambda_{\xi,n,j}.D)$, a finite subset Υ_t has been chosen such that if $t > s \in \mathrm{MAX}(\Lambda_{\xi,n,k}.D)$ for some $1 \leqslant k < j$, $\Upsilon_s \subset \Upsilon_t$, and

$$\left\| D^{p,r}_{\Lambda\setminus\Upsilon_t} \sum_{\Lambda_{\xi,n,j}.D\ni u\leqslant t} \mathbb{P}_{\xi,n}(u)x_u \right\| < \varepsilon.$$

For each $t \in \mathrm{MAX}(\Lambda_{\xi,n,i}.D)$, since $\theta_{\xi,1}(D^{p,r}_{\Lambda\setminus\Upsilon_t}) > 2\varepsilon$, by identifying $\Gamma_{\xi,1}.D$ with $\{s \in \Lambda_{\xi,n,i+1}.D : t < s\}$, we may fix a collection $(x_s)_{t<s\in\Lambda_{\xi,n,i+1}.D} \subset B_{E_p}$ such that $P_{\Lambda\setminus\Upsilon_t}x_s = x_s$ and

$$\inf\left\{ \left\| D^{p,r}_\Lambda \sum_{t<u\leqslant s} \mathbb{P}_{\xi,n}(u)x_u \right\| : t < s \in \mathrm{MAX}(\Lambda_{\xi,n,i+1}) \right\} > 2\varepsilon.$$

Now for each $s \in \mathrm{MAX}(\Lambda_{\xi,n,i+1})$ such that $t < s$, fix a finite set $\Upsilon_s \subset \Lambda$ such that $\Upsilon_t \subset \Upsilon_s$ and

$$\left\| D^{p,r}_{\Lambda\setminus\Upsilon_s} \sum_{t<u\leqslant s} \mathbb{P}_{\xi,n}(u)x_u \right\| < \varepsilon.$$

This completes the recursive choice of a weakly null collection $(x_t)_{t\in\Gamma_{\xi,n}.D} \subset B_{E_p}$. Now for any $t \in \mathrm{MAX}(\Gamma_{\xi,n}.D)$, let $\varnothing = t_0 < t_1 < \cdots < t_n$ be such that $t_i \in \mathrm{MAX}(\Lambda_{\xi,n,i}.D)$ for each $1 \leqslant i \leqslant n$ and let $\varnothing \subset \Upsilon_{t_1} \subset \cdots \subset \Upsilon_{t_n}$ be as in the choice of $(x_s)_{s\in\Gamma_{\xi,n}.D}$. Now let $F_i = \Upsilon_{t_i} \setminus \Upsilon_{t_{i-1}}$, $z_i = \sum_{t\geqslant s\in\Lambda_{\xi,n,i}.D} \mathbb{P}_{\xi,n}(s)x_s$, and note that

$$\left\| D^{p,r}_\Lambda \sum_{i=1}^n z_i \right\| \geqslant \left\| D^{p,r}_\Lambda \sum_{i=1}^n P_{F_i} z_i \right\| - \sum_{i=1}^n \|D^{p,r}_{\Lambda\setminus\Upsilon_{t_i}} z_i\| \geqslant 2\varepsilon n^{1/r} - n\delta > \varepsilon n^{1/r}.$$

From this it follows that

$$\alpha_{\xi,\gamma}(D^{p,r}_\Lambda) \geqslant \sup_n \varepsilon n^{\frac{1}{r}-\frac{1}{\gamma}} = \infty. \qquad \square$$

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